Weyl's semisimplicity taken to infinity

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Talk dedicated to the memory of Ivan Todorov (1933-2025)

The talk is based on joint work with E.Dan-Cohen (Fiebig), K.Styrkas, V.Serganova, P.Zadunaisky.
The main reference is the monograph I.Penkov, C.Hoyt **Classical Lie Algebras at infinity** Springer Verlag, 2022. For the Pieri rule see I.Penkov, P.Zudanaisky, "The Pieri rule at infinity", arXiv:2601.14879

Semisimplicity, understood naively as being a direct sum of simples, is a basic property in mathermatics but it does not hold in many natural situations.

For instance, a vector space is always semisimple and it is a direct sum of 1-dimensional subspaces (existence of basis). A vector bundle on a complex manifold is **locally semisimple** as it is a direct sum of vector bundles of rank 1 on suitable neighbourhoods of points, but not necessarily globally. A well known theorem of Grothendieck claims that a vector bundle on $\mathbb{CP}^1$ is semisimple.

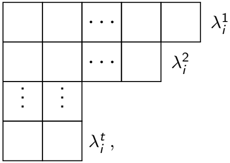
The tautological bundles S_i for $i \geqslant 2$ on a complex flag variety $\mathrm{Fl}(1, 2, \cdots, n-1, \mathbb{C}^n)$ are examples on non-semisimple vector bundles.

And, of course representations (modules) over the polynomial ring $\mathbb{C}[x]$ are not semisimple in general: for instance, x may act on $\mathbb{C}^2$ by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (Jordan normal form).

A celebrated theorem of H.Weyl from 1925 claims that any finite-dimensional representation of the Lie algebra $\mathrm{sl}(n,\mathbb{C})$ is semisimple. Concretely, if W is a finite-dimensional representation of $\mathrm{sl}(n,\mathbb{C})$, then

$$W \simeq \bigoplus_i c_i V_{\lambda_i} = \bigoplus_i c_i \mathbb{S}_{\lambda_i}(V_n)$$

where $\lambda_i = (\lambda_i^1 \geqslant \lambda_i^2 \geqslant \cdots \geqslant \lambda_i^t)$ are partitions, or equivalently Young diagrams:



and V_{λ_i} stands for the simple $\mathrm{sl}(n)$-representation with highest weight λ_i.
Examples

$$V_n = V_{\square},\, V_{\square}^* = V_{(1,1,\cdots,1,0)}$$

$$V_n \otimes V_n \simeq S^2 V_n \oplus \Lambda^2 V_n = V_{(2)} \oplus V_{(1,1)} = V_{\square\square} \oplus V_{\substack{\square\\ \square}}$$

$$V_n \otimes V_n^* = \mathrm{adj}_n \oplus \mathbb{C} = V_{(2,1,\cdots,1,0)} \oplus V_{\varnothing}.$$

The subrepresentations V_λ of W in Weyl's Theorem are canonical. So is there something non-canonical in this theorem?

$$\text{The answer is YES: this is the choice of } n.$$

When we put all $n \geqslant 2$ together we get sl$(\infty, \mathbb{C})$ instead of sl$(n, \mathbb{C})$. In fact,

$$\mathrm{sl}(2) \subset \mathrm{sl}(3) \subset \ \dots \ \subset \ \bigcup_{n \geqslant 2} \mathrm{sl}(n) = \mathrm{sl}(\infty).$$

The representation

$$V_\square \otimes V_\square^*$$

is nothing but $n \times n$-matrices as an sl(n)-representation. We have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{adj}_2 \arrow[r] \arrow[d] & V_\square \otimes V_\square^* = \mathrm{adj}_2 \oplus \mathbb{C} \arrow[r, shift left, "\mathrm{tr}"] \arrow[d] & \mathbb{C} \arrow[r] \arrow[l, shift left] \arrow[d] & 0 \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathrm{adj}_n \arrow[r] \arrow[d] & V_\square \otimes V_\square^* = \mathrm{adj}_n \oplus \mathbb{C} \arrow[r, shift left, "\mathrm{tr}"] \arrow[d] & \mathbb{C} \arrow[r] \arrow[l, shift left] \arrow[d] & 0 \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathrm{adj}_\infty \arrow[r] & \mathbf{V}_\square \otimes \mathbf{V}_{*\square} \arrow[r, shift left, "\mathrm{tr}"] & \mathbb{C} \arrow[r] \arrow[l, shift left, "/" marking] & 0
\end{tikzcd}$$

where $\mathbf{V} = \varinjlim V_n$, $\mathbf{V}_{*\square} = \varinjlim V_n^*$.

In other words, the sl(∞)-representation

$$\mathbf{V}_{\Box} \otimes \mathbf{V}_{*\Box} = \bigcup_{n} (V_{\Box} \otimes V_{\Box}^{*})$$

is not semisimple. The one-line proof of this fact is that the space $\mathbf{V}_{\Box} \otimes \mathbf{V}_{*\Box}$ of infinite finitary matrices has no sl(∞)-invariant.

The choice of n here reminds us of the choice of a neighborhood over which a vector bundle is semisimple. Indeed the representation $\mathbf{V}_{\Box} \otimes \mathbf{V}_{*\Box}$ is a semisimple $\mathrm{sl}(n)$-representation for each particular n but is not a semisimple $\mathrm{sl}(\infty)$-representation.

By taking Weyl's semisimplicity to infinity we mean discovering a structure on tensor spaces like $\mathbf{V}_{\Box} \otimes \mathbf{V}_{*\Box}$ which is valid for all n simulatenously, i.e. applies to these spaces considered as sl(∞)-representations.

Definition

$\mathbb{T}_{\mathrm{sl}(\infty)}$ is the category of sl(∞)-representations which have finite length (a finite Jordan-Hölder series) and are isomorphic to subquotients of suitable direct sums of the tensor algebra $T(V_{\square} \oplus V_{*\square})$ with itself.

Set $\mathbf{V}_{\lambda} = \lim\limits_{\rightarrow}(V_n)_{\lambda}$, $\mathbf{V}_{*\mu} = \lim\limits_{\rightarrow}(V_n^*)_{\mu}$ for arbitraty partitions λ and μ.The sl(∞)-representation $\mathbf{V}_{\lambda} \otimes \mathbf{V}_{*\mu}$ is indecomposable and has a simple socle, i.e. a unique simple submodule. Denote this simple module by $\mathbf{V}_{\lambda,\mu}$. The sl(∞)-representations $\mathbf{V}_{\lambda,\mu}$ are pairwise non-isomorphic, and any simple object in the categeory $\mathbb{T}_{\mathrm{sl}(\infty)}$ is isomorphic to $\mathbf{V}_{\lambda,\mu}$ for some partitions λ and μ.

Indecomposable injective objects:
The representation $\mathbf{V}_{\lambda} \otimes \mathbf{V}_{*\mu}$ is injective in $\mathbb{T}_{\mathrm{sl}(\infty)}$, and hence $\mathbf{V}_{\lambda} \otimes \mathbf{V}_{*\mu}$ is an injective hull of its socle $\mathbf{V}_{\lambda,\mu}$.

General Formula for socle multiplicity:
$\mathbf{V}_{\tilde{\lambda},\tilde{\mu}}$ enters the $(|\lambda| - |\tilde{\lambda}|)$-th layer of the socle filtration of $\mathbf{V}_{\lambda} \otimes \mathbf{V}_{*\mu}$ with multiplicity

$$m_{\lambda,\mu;\tilde{\lambda},\tilde{\mu}} = \sum_{|\gamma|=|\tilde{\lambda}|-|\tilde{\mu}|} N^{\lambda}_{\tilde{\lambda},\gamma} \, N^{\mu}_{\tilde{\mu},\gamma}.$$

where $N^{\alpha}_{\beta,\delta}$ are the Littlewood-Richardson coefficients.

Examples of socle filtrations of Indecomposable injectives

$$\mathbf{V}_\square \otimes \mathbf{V}_\square \otimes \mathbf{V}_{*\square} : \quad \boxed{\frac{V_{(1);(0)}}{V_{(2);(1)}}}, \quad \boxed{\frac{V_{(1);(0)}}{V_{(1,1);(1)}}}$$

$$\mathbf{V}_\square \otimes \mathbf{V}_\square \otimes \mathbf{V}_{*\square} \otimes \mathbf{V}_{*\square} : \quad \boxed{\begin{array}{c} V_{(0);(0)} \\ \hline V_{(1);(1)} \\ \hline V_{(2);(2)} \end{array}}, \quad \boxed{\frac{V_{(1);(1)}}{V_{(2);(1,1)}}}, \quad \boxed{\frac{V_{(1);(1)}}{V_{(1,1);(2)}}}, \quad \boxed{\begin{array}{c} V_{(0);(0)} \\ \hline V_{(1);(1)} \\ \hline V_{(1,1);(1,1)} \end{array}}$$

$$\boxed{\begin{array}{c} \mathrm{V}_{(1);(0)} \\ \hline \mathrm{V}_{(1,1);(1)} \oplus 2\mathrm{V}_{(2);(1)} \\ \hline \mathrm{V}_{(1,1,1);(1,1)} \oplus 2\mathrm{V}_{(2,1);(1,1)} \oplus \mathrm{V}_{(2,1);(2)} \oplus \mathrm{V}_{(3);(1,1)} \oplus \mathrm{V}_{(3);(2)} \\ \hline \mathrm{V}_{(2,1,1);(1,1,1)} \oplus \mathrm{V}_{(2,1,1);(2,1)} \oplus \mathrm{V}_{(3,1);(1,1,1)} \oplus \mathrm{V}_{(3,1);(2,1)} \\ \hline \mathrm{V}_{(3,1,1);(2,1,1)} \end{array}}$$

The category $\mathbb{T}_{\mathrm{sl}(\infty)}$ is of Koszul type. Informally this means that the Ext's between simple modules are compatible with a $\mathbb{Z}_{\geqslant 0}$-grading. Fix $|\lambda| - |\mu| \in \mathbb{Z}$. This determines a block in $\mathbb{T}_{\mathrm{sl}(\infty)}$. Then for $|\lambda| - |\mu| = |\tilde{\lambda}| - |\tilde{\mu}|$

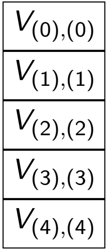
$$\mathrm{Ext}^{i}(\mathbf{V}_{\lambda,\mu}\mathbf{V}_{\widetilde{\lambda},\widetilde{\mu}}) \neq 0 \implies |\lambda| - |\tilde{\lambda}| = i.$$

Koszul self-duality

The situation is even better: $\mathbb{T}_{\mathrm{sl}(\infty)}$ is Koszul self-dual. This implies

$$\dim \mathrm{Ext}^{i}(\mathbf{V}_{\lambda,\mu}, \mathbf{V}_{\widetilde{\lambda},\widetilde{\mu}}) = m_{\lambda,\mu^{\perp},\lambda',\mu'^{\perp}}.$$

Examples of socle filtrations vs Exts



$V_{(0),(0)}$

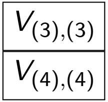
$V_{(1),(1)}$

$V_{(2),(2)}$

$V_{(3),(3)}$

$V_{(4),(4)}$

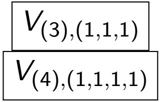
socle filtration



$V_{(3),(3)}$

$V_{(4),(4)}$

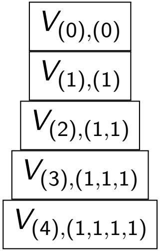
Exts



$V_{(3),(1,1,1)}$

$V_{(4),(1,1,1,1)}$

socle filtration



$V_{(0),(0)}$

$V_{(1),(1)}$

$V_{(2),(1,1)}$

$V_{(3),(1,1,1)}$

$V_{(4),(1,1,1,1)}$

Exts

On algebraic duals of $\mathbf{V}_{\lambda,\mu}$

Fact: $\mathbf{V}^*_{\lambda,\mu}$ has finitely many non-isomorphic simple subquotients (!), all except $\mathbf{V}_{\mu,\lambda}$ have uncountable multiplicities, and $|\mu| - |\tilde{\mu}| \geqslant 0$, $|\lambda| - |\tilde{\lambda}| \geqslant 0$ for any simple subquotient $\mathbf{V}_{\tilde{\mu},\tilde{\lambda}}$ of $\mathbf{V}^*_{\lambda,\mu}$.
Examples:

$$\mathbf{V}^*_{\square} \;=\; \frac{2^{\aleph_0}\,\mathbb{C}}{\mathbf{V}_{*\square}}$$

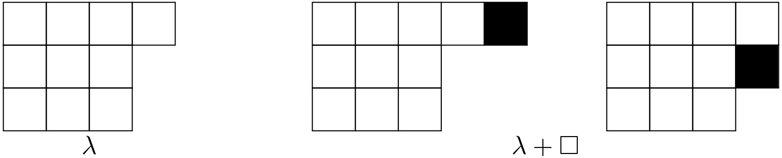
$$\mathbf{V}^*_{\square,\square} = (\mathrm{adj}_\infty)^* \;=\; \cfrac{2^{\aleph_0}\,\mathbb{C}}{\cfrac{2^{\aleph_0}\mathbf{V}_{\square} \;\oplus\; 2^{\aleph_0}\mathbf{V}_{*\square} \;\oplus\; 2^{\aleph_0}\mathbb{C}}{\mathrm{adj}_\infty = \mathbf{V}_{\square,\square}}}$$

Recall that $\mathbf{V}_{*\square} = \mathbf{V}_{\varnothing,\square}$.

Classical Pieri Rule

Let $V = V_n$. Then

$$V_\lambda \otimes V_\Box \;=\; \bigoplus_{\lambda+\Box} V_{\lambda+\Box},$$



At infinity, i.e. for sl(∞) the representation $\mathbf{V}_{\lambda,\mu} \otimes \mathbf{V}_{\Box,\varnothing}$ is indecomposable whenever $\mu \neq \varnothing$ and has the following socle filtration:

$$\frac{\displaystyle\bigoplus_{\mu-\Box} \mathbf{V}_{\lambda,\mu-\Box}}{\displaystyle\bigoplus_{\mu+\Box} \mathbf{V}_{\lambda+\Box,\mu}}.$$

If $\mu = \varnothing$, then the representation $\mathbf{V}_{\lambda,\mu} \otimes \mathbf{V}_{\Box,\varnothing}$ is semisimple, and the classical Pieri rule holds.

More generally, Zadunaisky and I studied the structure of $M_\delta \otimes F$, where M_δ is any simple integrable sl(∞)-representation with highest weight δ and F is any weight-multiplicity-free integrable representation of sl(∞). In this situation, $M_\delta \otimes F$ is semisimple or indecomposable and we construct an explicit (linkage) filtration of $M_\delta \otimes F$. This filtration is a $\mathbb{Z}$-filtration, and it may be two-sided in interesting examples.