# VIRASORO COADJOINT orbits & Jackiw – TEITELBOIM GRAVITY

# COADJOINT ORBITS

$G$ = CONNECTED LIE GROUP

$G \curvearrowright \mathfrak{g} = \mathrm{Lie}(G), \quad \mathfrak{g}^*$

ADJOINT COADJOINT

$\lambda \in \mathfrak{g}^*, \quad \mathcal{O}_\lambda = \{ \mathrm{Ad}_g^* \lambda \; ; \; g \in G \}$

COADJOINT ORBIT

[KIRILLOV - KOSTANT - SOURIAU]

$\mathcal{O}_\lambda$ SYMPLECTIC

$$\omega_\lambda = \frac{1}{2} \langle \lambda, [g^{-1}dg, g^{-1}dg] \rangle$$

$$\xi = g \lambda g^{-1} \quad ( = \mathrm{Ad}_g^* \lambda )$$

# Poisson brackets

[ Todorov - AA '93, Lu ]

$G$ = semisimple

$\lambda_\alpha = \langle \lambda, \alpha \rangle$

$$\begin{array}{c} \Pi_\lambda \\ \| \\ \omega_\lambda^{-1} \end{array} = \underbrace{\sum_{\alpha>0} \frac{1}{\lambda_\alpha} e_\alpha \wedge e_{-\alpha}}_{= r(\lambda)}$$

classical dynamical

$r$ - matrix

[ Etingof - Shiffmann ]

# APPLICATIONS :

- GEOMETRIC ACTIONS

$$S_\lambda(g, A) = \int_\gamma \langle \lambda, g^{-1}dg + g^{-1}Ag \rangle$$

EXTERNAL

GAUGE FIELD $\quad A \in \Omega^1(\gamma, \mathfrak{g})$

HAMILTONIAN $\quad H = \langle \xi, A \rangle$

$$\boxed{\int e^{iS_\lambda(g, A)} \mathcal{D}g = \chi_\lambda\left(P\exp \int A\right)}$$

- $G = SU(2)$

$$\omega_j = \frac{j}{2}\, \xi \cdot (d\xi \times d\xi)$$

$$\|\xi\| = 1, \quad \xi \in S^2$$

$j =$ SPIN

# VIRASORO COADJOINT ORBITS

- $0 \longrightarrow \mathbb{R} \longrightarrow Vir \longrightarrow \mathcal{X}(S^1) \longrightarrow 0$

$$[u,v]_{Vir} = [u,v] + \int_{S^1} u v''' \, dx$$

$u, v \in \mathcal{X}(S^1)$

- $$Vir^* = \left\{ c \frac{d^2}{dx^2} + T(x) \right\}$$

Hill operators

[G. W. Hill, Annals 1 no 1, 5-10]

$$\boxed{c = 1}$$

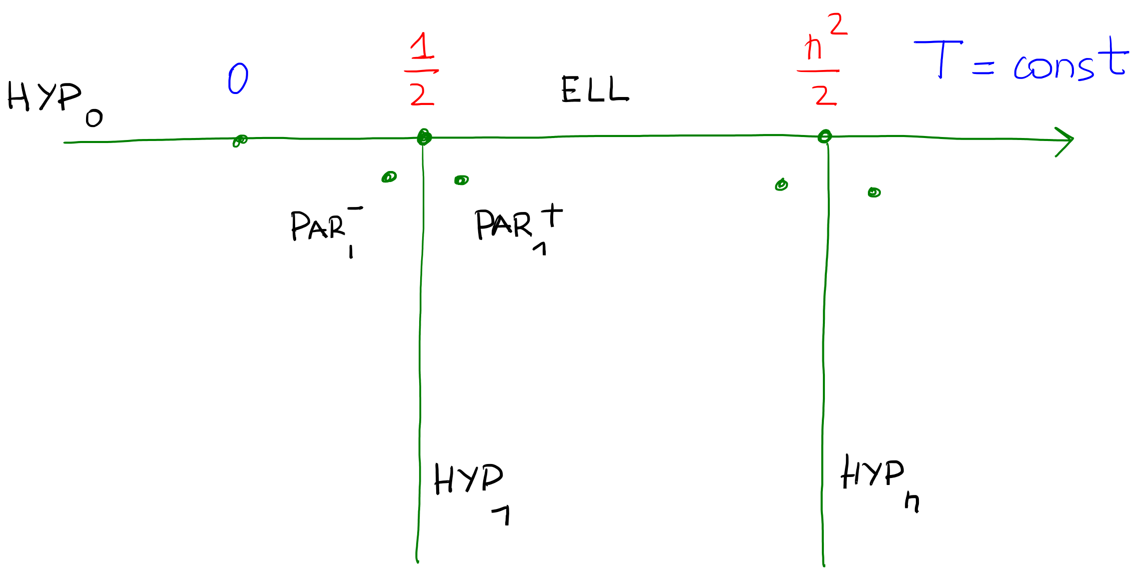
$f \in \mathrm{Diff}^+(S^1)$

$$T^f(x) = T(f(x))\, f'(x)^2 + \frac{1}{2} \mathcal{S}(f)$$

Schwarzian : $\boxed{\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2}$

# Classificaction

[Lazutkin-Pankratova, Segal, Kirillov, Witten, ...]




HYP$_0$ & ELL $\qquad \mathcal{O}_T \cong \mathrm{Diff}^+(S^1)/S^1$

$$\mathcal{O}_{\frac{n^2}{2}} \cong \mathrm{Diff}^+(S^1)/PSL^{(n)}(2,\mathbb{R})$$

# Symplectic forms

$$T = \text{const}$$

$$\omega_T = \frac{1}{2} \int \delta(\log f') \wedge \delta(\log f')' +$$
$$+ T \int \delta f' \wedge \delta f$$

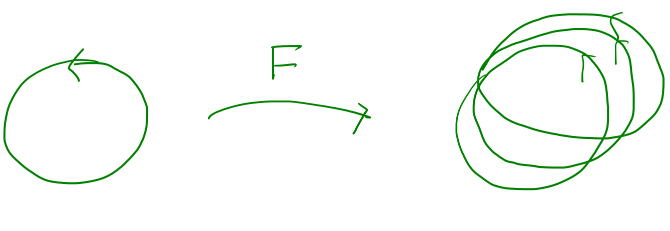
$\delta$ = de Rham differential

on $\mathrm{Diff}^+(S^1)$

REMARK: $\quad T = \frac{n^2}{2}$

$$F(x) = n f(x)$$

$$\omega_T = \frac{1}{2} \int \delta(\log F') \wedge \delta(\log F')' +$$
$$+ \frac{1}{2} \int \delta F' \wedge \delta F$$




$(n = 3)$

## HAMILTONIAN

$$v = \frac{\partial}{\partial x}$$

$$H = \int \left( T(f(x)) f'^2(x) + \frac{1}{2} \mathcal{S}(f) \right) dx$$

TEICHMÜLLER ORBIT:

$$T = \frac{1}{2}$$

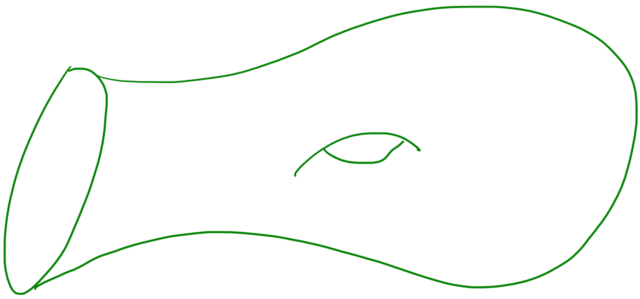
$$\Rightarrow \quad H(f) = \frac{1}{2} \int \left( \mathcal{S}(f) + f'^2(x) \right) dx$$

SCHWARZIAN ACTION

$$T = \frac{n^2}{2}$$

$$\Rightarrow \quad H(F) = \frac{1}{2} \int \left( \mathcal{S}(F) + F'^2(x) \right) dx$$

# JACKIW – TEITELBOIM GRAVITY



$\Sigma = $ 2-mfld

$\partial\Sigma \cong S^1$

$$S(h,\phi) = \int_\Sigma \phi\,(R_h + 2)\, dvol_g$$

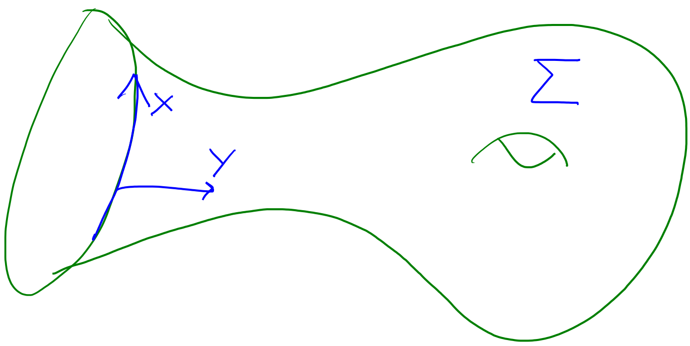
$+$ bdry terms

$\Rightarrow \quad R_h = -2 \quad \Rightarrow \quad h$ hyperbolic

[STANFORD – Witten, SAAD – Shenker – Stanford]

- Moduli of hyperbolic metrics is symplectic
- Schwarzian action generates rotations of $\partial\Sigma$

# TEICHMŪLLER SPACE




$$\mathrm{TEICH}(\Sigma) = \frac{\left\{ h = \text{hyperbolic}\ ;\ y^2 h \text{ extends to } \partial\Sigma \right\}}{\mathrm{Diff}_0^+(\Sigma, \partial\Sigma)}$$

e.g.

$$\mathrm{TEICH}(\mathbb{D}) = {}_{PSL(2,\mathbb{R})}\backslash^{\mathrm{Diff}^+(\mathbb{D})} \Big/ \mathrm{Diff}^+(\mathbb{D}, \partial\mathbb{D})$$

$$\cong PSL(2,\mathbb{R}) \backslash \mathrm{Diff}^+(S^1) \cong \mathcal{O}_{\frac{1}{2}}$$

**Theorem** [MEINRENKEN, AA]

Fix $(\cdot,\cdot)$ invariant scalar product on $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$.

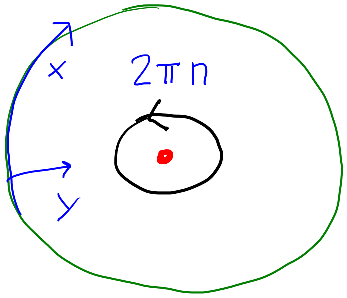
- $\mathrm{Teich}(\Sigma)$ is SYMPLECTIC
- ACTION OF $\widetilde{\mathrm{Diff}^+(\partial\Sigma)}$ HAMILTONIAN WITH Virasoro central extension

e.g. • $\mathrm{Teich}(\mathbb{D}) \cong \mathcal{O}_{\frac{1}{2}}$

as symplectic spaces

- Schwarzian action generates rotations of the bdry
- $\mathrm{genus}(\Sigma) \geq 1 \Rightarrow$ explicit DARBOUX coordinates

# EXCEPTIONAL ORBITS




- $\mathrm{Teich}^{(n)}(\mathbb{D}_*)$

  $\|$

  $\mathrm{Hyp}^{(n)}(\mathbb{D}_*)/\mathrm{Diff}^+(\mathbb{D};\partial\mathbb{D},*)$

- $\mathrm{Hyp}^{(n)}(\mathbb{D}_*) = \{ h = \text{hyperbolic};$
  $y^2 h \text{ extends to } \partial\mathbb{D},$
  $h \sim r^{2(n-1)} h_0 \text{ near } * \}$

**Theorem** [Dalipi - Shatashvili - AA]

- $\mathrm{Teich}^{(n)}(\mathbb{D}_*) \cong \mathrm{Diff}^+(S^1)/S^1$

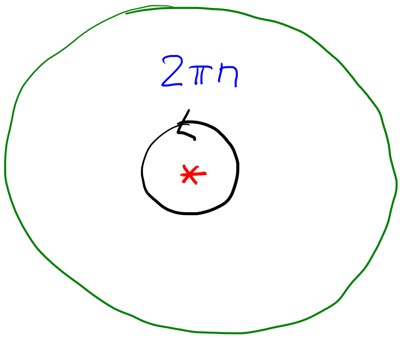
- presymplectic, 2-form descends to

  $\mathrm{Teich}^{(n)}(\mathbb{D}_*) \longrightarrow \mathrm{Diff}^+(S^1)/PSL^{(n)}(2,\mathbb{R})$

  $\cong$

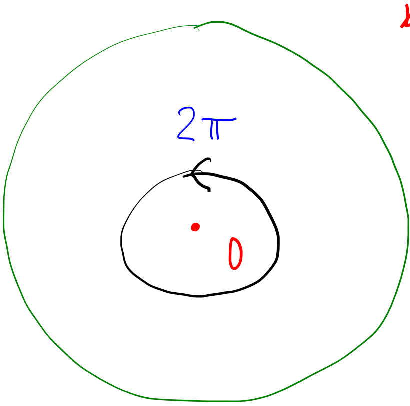
  $\mathcal{O}_{\frac{n^2}{2}}$

# GEOMETRIC INTERPRETATION




$$h \sim r^{2(n-1)} h_0$$

$\varphi$ = n-fold cover ramified over *




$$h_0 = \frac{4\,|dz|^2}{(1-|z^2|)^2}$$

Poincaré metric

$$\varphi\big|_{\partial\mathbb{D}} = \text{n-fold cover}$$

$$(f \longmapsto F)$$

Thank you !