Unification of Gravities with Internal Interactions

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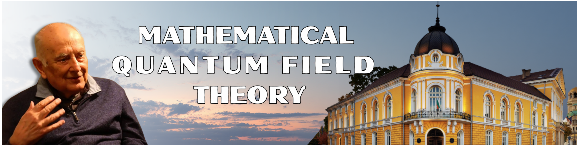

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Table of contents

1. *Gravity as a gauge theory in 4 dimensions*
2. *Conformal Gravity as a gauge theory*
3. *Noncommutativity and gauge theories*
4. *Noncommutative four-dimensional gravity: Fuzzy Gravity*
5. *Fermions and Unification with Internal Interactions*

Preliminaries

Second order formulation (Einstein gravity):

- metric tensor $g_{\mu\nu}$
- curvature parametrized by Riemann tensor:
  $R^{\rho}_{\mu\nu\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$
- torsion: $T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 0$
- Christoffel Symbols: $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$
- action: $S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \rightarrow$ Einstein Field Equations

First order formulation:

- vierbein and spin connection $e_\mu{}^a, \omega_\mu{}^{ab}$
- curvature parametrized by the curvature 2-form:
  $R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} - \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b$
- torsion: $T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^c - \omega_\nu{}^a{}_b e_\mu{}^c$
- action: $S = \frac{1}{16\pi G}\int \frac{1}{2}\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}$ (Palatini action)
- $\rightarrow$ Einstein Field Equations + Torsionless condition

Einstein 4d Gravity as a Gauge Theory

The algebra

see for details:
Utiyama '56, Kibble '61, Kaku-Townsend-Nieu/zen '77, McDowell-Mansuri '77, Chamseddine-West '77, Stelle-West, '80, Ivanov-Niederle '82, Kibble-Stelle '85, Witten '88, Wilczek '98, Ortin '04, Roumelioti-Stefas-Z '24

- Employ the first order formulation of GR
- Gauge theory of Poincaré group ISO(1,3)
- Ten generators (Translations P_a & LT M_{ab})

Generators satisfy the commutation relations:

$$[M_{ab}, M_{cd}] = \eta_{ac} M_{db} - \eta_{bc} M_{da} - \eta_{ad} M_{cb} + \eta_{bd} M_{ca}$$
$$[P_a, M_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b \,, \qquad [P_a, P_b] = 0$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ and $a, b, c, d = 1, \ldots, 4$.

The gauging procedure

- Introduction of a gauge vector field for each generator. For the Poincaré group, $ISO(1,3)$:
  $\rightarrow$ 4 fields $e_\mu{}^a$ for the translation operators P_a
  $\rightarrow$ 6 fields $\omega_\mu{}^{ab}$ for the local $SO(1,3)$ (LT)
- The gauge connection is:

$$A_\mu(x) = e_\mu{}^a(x) P_a + \tfrac{1}{2}\omega_\mu{}^{ab}(x) M_{ab}$$

- Transforms in the adjoint rep, according to the rule:

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]$$

- The gauge transformation parameter, $\epsilon(x)$ is expanded as:

$$\epsilon(x) = \xi^a(x) P_a + \tfrac{1}{2}\lambda^{ab}(x) M_{ab}$$

- *Combining* the above $\rightarrow$ transformations of the fields:

$$\delta e_\mu{}^a = \partial_\mu \xi^a - e_\mu{}^b \lambda^a{}_b + \omega_\mu{}^{ab}\xi_b$$
$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} - \lambda^a{}_c \omega_\mu{}^{cb} + \lambda^b{}_c \omega_\mu{}^{ca}$$

- Gauge transf $\leftrightarrow$ diffeo transf (imposing torsionless and on shell conditions)

Curvature and Torsion

- Field Strength:

$$R_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- Tensor $R_{\mu\nu}$ is also valued in Poincaré algebra:

$$R_{\mu\nu}(A) = T_{\mu\nu}{}^{a} P_a + \tfrac{1}{2} R_{\mu\nu}{}^{ab} M_{ab}$$

- *Combining* the above $\rightarrow$ component tensor curvatures:

$$\begin{aligned} T_{\mu\nu}{}^{a} &= \partial_\mu e_\nu{}^{a} - \partial_\nu e_\mu{}^{a} + e_\mu{}^{b} \omega_{\nu b}{}^{a} - e_\nu{}^{b} \omega_{\mu b}{}^{a} \\ R_{\mu\nu}{}^{ab} &= \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{cb} \omega_\nu{}^{a}{}_{c} + \omega_\mu{}^{ac} \omega_{\nu c}{}^{b} \end{aligned}$$

- Palatini action is considered
- Torsionless condition + Field equations

Gauge theory of SO(2,3)

- Instead of the Poincaré group - Anti-de Sitter group: SO(2,3)
- Same amount of generators BUT they can be written on equal footing (semisimple group):

$$[\hat{M}_{AB}, \hat{M}_{CD}] = \eta_{AC}\hat{M}_{DB} - \eta_{BC}\hat{M}_{DA} - \eta_{AD}\hat{M}_{CB} + \eta_{BD}\hat{M}_{CA}$$

- η_{AB} is the 5-dim Minkowski metric with two timelike coefficients (1st and 5th) and $A, \ldots, D = 1 \ldots 5$
- Perform a splitting of the indices $A = (a, 5)$
- Define $\hat{M}_{ab} = M_{ab}$ and $\hat{M}_{a5} = \frac{1}{m}P_a$, $[m] = L^{-1}$
- Gauge connection: $A_\mu = \frac{1}{2}\hat{\omega}_\mu{}^{AB}\hat{M}_{AB} = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a$
- where $\hat{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab}$ and $\hat{\omega}_\mu{}^{a5} = m e_\mu{}^a$
- The same for the field strength tensor $\hat{R}_{\mu\nu}{}^{AB}$:

$$\hat{R}_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + 2m^2 e_\mu{}^{[a} e_\mu{}^{b]}\,, \quad \hat{R}_{\mu\nu}{}^{a5} = m T_{\mu\nu}{}^a$$

- Consider the following $SO(2,3)$ invariant quadratic action:

$$S = a_{AdS} \int d^4x \Big(m y^E \epsilon_{ABCDE} \frac{1}{4} \hat{R}_{\mu\nu}{}^{AB} \hat{R}_{\rho\sigma}{}^{CD} \epsilon^{\mu\nu\rho\sigma} + \\ + \lambda \left(y^E y_E + m^{-2} \right) \Big)$$

- y^E an auxiliary scalar field in the vector rep
- vector taken to be gauge fixed towards the 5-th direction:

$$y = y^0 = \left(0, 0, 0, 0, m^{-1}\right).$$

- the non-vanishing value $y^5(x)$ is responsible for the symmetry breaking of $SO(2,3)$ to the $SO(1,3)$

$$S = \frac{a_{AdS}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu}{}^{ab} \hat{R}_{\rho\sigma}{}^{cd} \epsilon_{abcd}$$
$$= \frac{a_{AdS}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \left(\mathcal{L}_{RR} + m^2 \mathcal{L}_{eeR} + m^4 \mathcal{L}_{eeee} \right)$$

- $\mathcal{L}_{RR}$: Gauss-Bonnet - no contribution to the e.o.m.
- $\mathcal{L}_{eeR}$: Palatini action (torsionless + Einstein Field Equations)
- $\mathcal{L}_{eeee}$: Plays the role of cosmological constant
- Solution of Einstein Field Equations is the Anti-de Sitter space
- If $m \to 0$: Minkowski spacetime (flat solution).

Conformal 4d Gravity as a Gauge Theory

- Group parametrizing the symmetry: $SO(2,4)$
- 15 generators: 6 LT M_{ab}, 4 translations, P_a, 4 conformal boosts K_a and the dilatation D
- Group generators satisfy the following algebra:

$$
\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} + \eta_{ad} M_{bc} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}, \\
[M_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b, \\
[M_{ab}, K_c] &= \eta_{bc} K_a - \eta_{ac} K_b, \\
[P_a, D] &= P_a, \\
[K_a, D] &= -K_a, \\
[K_a, P_b] &= -2(\eta_{ab} D + M_{ab}),
\end{aligned}
$$

- Following the same procedure one calculates transf of the gauge fields and tensors after defining the gauge connection
- Action is taken of $SO(2,4)$ invariant quadratic form
- Initial symmetry breaks under certain constraints resulting to the *Weyl action* *Kaku,Townsend,Nieu/zen '77, Fradkin, Tseytlin '85*
- Initial symmetry breaks spontaneously by introducing a scalar in the adjoint rep fixed in the dilatation direction, or by two scalars in vector reps.

Roumelioti, Stefas, Z '24

SSB by using a scalar in the adjoint representation

Gauge connection:

$$A_\mu = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a + b_\mu{}^a K_a + \tilde{a}_\mu D,$$

Field strength tensor:

$$F_{\mu\nu} = \frac{1}{2}R_{\mu\nu}{}^{ab}M_{ab} + \tilde{R}_{\mu\nu}{}^a P_a + R_{\mu\nu}{}^a K_a + R_{\mu\nu} D,$$

where

$$\begin{aligned}
R_{\mu\nu}{}^{ab} &= \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b - 8e_{[\mu}{}^{[a}b_{\nu]}{}^{b]} \\
&= R^{(0)ab}_{\mu\nu} - 8e_{[\mu}{}^a b_{\nu]}{}^{b]}, \\
\tilde{R}_{\mu\nu}{}^a &= \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^{ab}e_{\nu b} - \omega_\nu{}^{ab}e_{\mu b} - 2\tilde{a}_{[\mu}e_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu}(e) - 2\tilde{a}_{[\mu}e_{\nu]}{}^a, \\
R_{\mu\nu}{}^a &= \partial_\mu b_\nu{}^a - \partial_\nu b_\mu{}^a + \omega_\mu{}^{ab}b_{\nu b} - \omega_\nu{}^{ab}b_{\mu b} + 2\tilde{a}_{[\mu}b_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu}(b) + 2\tilde{a}_{[\mu}b_{\nu]}{}^a, \\
R_{\mu\nu} &= \partial_\mu\tilde{a}_\nu - \partial_\nu\tilde{a}_\mu + 4e_{[\mu}{}^a b_{\nu]a},
\end{aligned}$$

We start with the parity conserving action, which is quadratic in terms of the field strength tensor and introduce a scalar in the rep 15

$$S_{SO(2,4)} = a_{CG} \int d^4x \left[\mathrm{tr}\, \epsilon^{\mu\nu\rho\sigma} m \phi F_{\mu\nu} F_{\rho\sigma} + \lambda \left(\phi^2 - m^{-2} \mathbb{1}_4 \right) \right],$$

The scalar expanded on the generators is:

$$\phi = \phi^{ab} M_{ab} + \tilde{\phi}^a P_a + \phi^a K_a + \tilde{\phi} D,$$

We pick the specific gauge in which ϕ is diagonal of the form $diag(1, 1, -1, -1)$. Specifically we choose ϕ to be only in the direction of the dilatation generator D:

$$\phi = \phi^0 = \tilde{\phi} D \xrightarrow{\phi^2 = m^{-2} \mathbb{1}_4} \phi = -2m^{-1} D.$$

The resulting broken action is (after employing anticommutator relations and the traces over the generators):

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd}$$

The $\tilde{a}_\mu$ is not present in the action, so we can set it equal to zero.

$R_{\mu\nu}$ is also absent so we can also set it equal to zero

$$R_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu + 4 e_{[\mu}{}^a b_{\nu]a} = 0 \xrightarrow{\tilde{a}_\mu = 0}$$

$$e_\mu{}^a b_{\nu a} - e_\nu{}^a b_{\mu a} = 0$$

We examine two possible solutions of the above equation:

- $b_\mu{}^a = a e_\mu{}^a$, *Chamseddine '03*
- $b_\mu{}^a = -\frac{1}{4}\left(R_\mu{}^a + \frac{1}{6} R e_\mu{}^a\right)$ *Kaku, Townsend, Nieu/zen, '78*

Freedman, Van Proyen 'Supergravity' '12

The first choice leads to the Einstein-Hilbert action, while the second leads to Weyl action.

→ Similar results are obtained using two scalars in the vector rep.

Einstein-Hilbert action

- When $b_\mu{}^a = a e_\mu{}^a$, the broken action becomes:

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} \implies$$

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \Big[R^{(0)ab}_{\mu\nu} R^{(0)cd}_{\rho\sigma} - 16 m^2 a R^{(0)ab}_{\mu\nu} e_\rho{}^c e_\sigma{}^d + \\ + 64 m^4 a^2 e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d \Big]$$

This action consists of three terms: one G-B topological term, the E-H action, and a cosmological constant. For $a < 0$ describes GR in AdS space.

Weyl action

- When $b_\mu{}^a = -\frac{1}{4}(R_\mu{}^a + \frac{1}{6}Re_\mu{}^a)$, the broken action becomes

$$S = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}$$

$$\left[R^{(0)ab}_{\mu\nu} - \frac{1}{2}\left(\tilde{e}_\mu{}^{[a}R_\nu{}^{b]} - \tilde{e}_\nu{}^{[a}R_\mu{}^{b]}\right) + \frac{1}{3}R\tilde{e}_\mu{}^{[a}\tilde{e}_\nu{}^{b]}\right]$$
$$\left[R^{(0)cd}_{\rho\sigma} - \frac{1}{2}\left(\tilde{e}_\rho{}^{[c}R_\sigma{}^{d]} - \tilde{e}_\sigma{}^{[c}R_\rho{}^{d]}\right) + \frac{1}{3}R\tilde{e}_\rho{}^{[c}\tilde{e}_\sigma{}^{d]}\right],$$

where $\tilde{e}_\mu{}^a = me_\mu{}^a$ is the rescaled vierbein. The above action is equal to

$$S = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}C_{\mu\nu}{}^{ab}C_{\rho\sigma}{}^{cd} \Rightarrow 2a_{CG}\int \mathrm{d}^4x\left(R_{\mu\nu}R^{\nu\mu} - \frac{1}{3}R^2\right)$$

where $C_{\mu\nu}{}^{ab}$ is the Weyl conformal tensor.

- Weyl action can be broken further to E-H by <**6**> which breaks the dilaton, D, and gives additional contributions to generators K_a.

The NC framework & gauge theories

- Quantization of phase space of $x^i, p_j \rightarrow$ replaced by Herm operators: $\hat{x}^i, \hat{p}_j$ satisfying:$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j$
- Noncommutative space $\rightarrow$ quantization of space: $x^i \rightarrow$ replace with operators $\hat{x}^i$ ($\in \mathcal{A}$) satisfying: $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x})$

Connes '94, Madore '99

- Antisymmetric tensor $\theta^{ij}(\hat{x})$ - defines the NC of the space
  - Canonical case: $\theta^{ij}(\hat{x}) = \theta^{ij}, i, j = 1, \dots, N$
    For $N = 2 \rightarrow$ *Moyal plane*
  - Lie-type case: $\theta^{ij}(\hat{x}) = C^{ij}{}_k \hat{x}^k, i, j = 1, \dots, N$
    For $N = 3 \rightarrow$ Noncommutative *(fuzzy) sphere* (SU(2))
- NC framework admits a matrix representation (operators)
  - Derivation: $e_i(A) = [d_i, A]$, $d_i \in \mathcal{A}$
  - Integration $\rightarrow$ Trace

For Reviews:
Szabo '01, Douglas-Nekrasov '01

The NC Gauge Fields & transformations

Madore-Wess et al. '00

Consider a field $\phi(X_a)$ on a fuzzy space described by NC coordinates X_a. An infinitesimal gauge transformation

$$\delta\phi(X_a) = \lambda(X_a)\phi(X_a)\,,$$

where $\lambda(X_a)$ is a gauge transf parameter:

- $U(1)$ if $\lambda(X_a)$ is antihermitian function of X_a
- $U(P)$ if $\lambda(X_a)$ is valued in Lie algebra of $P \times P$ matrices

Coordinates X_a are invariant under gauge transformation, i.e. $\delta(X_a) = 0$. Therefore:

- $\delta(X_a\phi) = X_a\lambda(X_a)\phi \neq \lambda(X_a)X_a\phi$
- $\delta(\mathcal{X}_a\phi) = \lambda(X_a)\mathcal{X}_a\phi$,
  which holds if: $\delta(\mathcal{X}_a) = [\lambda(X_a), \mathcal{X}_a]$
- where $\mathcal{X}_a = X_a + A_a$ the covariant coordinate $\rightarrow$ NC analogue of cov. der. and A_a are interpreted as gauge fields

The NC Gauge Fields & transformations (2)

Note that the transformation of A_a is:

$$\delta A_a = -[X_a, \lambda] + [\lambda, A_a],$$

supporting the interpretation of A_a as gauge field.

Correspondingly, define:

$$\begin{aligned} F_{ab} =& [X_a, A_b] - [X_b, A_a] + [A_a, A_b] = -C^c{}_{ab} A_c \\ =& [\mathcal{X}_a, \mathcal{X}_b] - C^c{}_{ab} \mathcal{X}_c, \end{aligned}$$

an analogue of the field strength tensor whose transformation is given by:

$$\delta F_{ab} = [\lambda, F_{ab}]$$

Non-Abelian case

▷ *In nonabelian case, where are the gauge fields valued?*

- Let us consider the CR of two elements of an algebra:

$$[\epsilon, A] = [\epsilon^A T^A, A^B T^B] = \frac{1}{2}\{\epsilon^A, A^B\}[T^A, T^B] + \frac{1}{2}[\epsilon^A, A^B]\{T^A, T^B\}$$

- *Not* possible to restrict to a matrix algebra:
  last term neither *vanishes* in NC nor is an *algebra element*

- There are two options to overpass the difficulty:

Aschieri, Castellani '09

Ćirić-Gočanin-Konjik-Radovanović '18

  - Consider the universal enveloping algebra
  - Extend the generators and/or fix the rep so that the anticommutators close

▷ *We employ the second option*

The 4d covariant noncommutative space

Motivation for a 4d covariant NC space

- Constructing field theories on NC spaces is non-trivial: NC deformations break Lorentz invariance
- such an example is the fuzzy sphere (2d space) - coords are identified as rescaled SU(2) generators

Madore '92
Hammou-Lagraa-Sheikh Jabbari '02
Vitale-Wallet '13, Vitale '14
Jurman-Steinacker '14
Chatzistavrakidis-Jonke-Jurman-Manolakos-Manousselis-GZ '18

- Previous work on 3d NC gravity on the covariant spaces $R^3_\lambda (R^{1,2}_\lambda)$
- Need of 4d covariant NC space to construct a gravity gauge theory

Construction of the 4d covariant NC space

- dS_4: homogeneous spacetime with constant curvature (positive)
- Described by the embedding $\eta^{AB} X_A X_B = R^2$ into M_5
- Aim for a NC version of dS_4

- Introduce a natural minimal length
- Assign the spacetime coordinates to elements of the 4-d dS group, $SO(1,4)$

Snyder '47

- The SO(1,4) generators, $J_{mn}, m, n = 0, \ldots, 4$, satisfy the commutation relation:

$$[J_{mn}, J_{rs}] = i(\eta_{mr}J_{ns} + \eta_{ns}J_{mr} - \eta_{nr}J_{ms} - \eta_{ms}J_{nr})$$

- Consider decomposition of $SO(1,4)$ to max subgroup, $SO(1,3)$
- Convert the generators to physical quantities by setting $\Theta_{ij} = \hbar J_{ij}\,, X_i = \lambda J_{i4}\,;\ \lambda$ a length parameter
- Thus, the commutation relations regarding the operators $\Theta_{\mu\nu}$ and X_μ are:

$$[\Theta_{ij}, \Theta_{kl}] = i\hbar\left(\eta_{ik}\Theta_{jl} + \eta_{jl}\Theta_{ik} - \eta_{jk}\Theta_{il} - \eta_{il}\Theta_{jk}\right),$$
$$[\Theta_{ij}, X_k] = i\hbar\left(\eta_{ik}X_j - \eta_{jk}X_i\right),$$
$$[X_i, X_j] = \frac{i\lambda^2}{\hbar}\Theta_{ij}$$

- The noncommutativity of coordinates becomes manifest

Yang's Model '47

- Extending covariance to include also momenta generators → use a group with larger symmetry → min extension: $SO(1,5)$

Yang '47
Kimura '02, Heckman-Verlinde '15
Steinacker '16
Sperling-Steinacker '17,'19
Burić-Madore '14,'15
Manolakos-Manousselis-GZ '19,'21

- The SO(1,5) generators, $J_{MN}, M, N = 0, \ldots, 5$, satisfy the commutation relation:

$$[J_{MN}, J_{P\Sigma}] = i(\eta_{MP}J_{N\Sigma} + \eta_{N\Sigma}J_{MP} - \eta_{NP}J_{M\Sigma} - \eta_{M\Sigma}J_{NP})$$

- Employ a *2-step* decomposition $SO(1,5) \supset SO(1,4) \supset SO(1,3)$

Yang's Model '47 (Continued)

- Convert the generators to physical quantities by identifying $\Theta_{ij} = \hbar J_{ij}, \;\; X_i = \lambda J_{i5}, \;\; P_i = \frac{\hbar}{\lambda} J_{i4}, \;\; h = J_{45}$
- Thus, the commutation relations regarding all the operators $\Theta_{\mu\nu}, X_\mu, P_\mu, h$ are:

$$
\begin{aligned}
[\Theta_{\mu\nu}, \Theta_{\rho\sigma}] &= i\hbar(\eta_{\mu\rho}\Theta_{\nu\sigma} + \eta_{\nu\sigma}\Theta_{\mu\rho} - \eta_{\nu\rho}\Theta_{\mu\sigma} - \eta_{\mu\sigma}\Theta_{\nu\rho}) \, , \\
[\Theta_{\mu\nu}, X_\rho] &= i\hbar(\eta_{\mu\rho}X_\nu - \eta_{\nu\rho}X_\mu) \\
[\Theta_{\mu\nu}, P_\rho] &= i\hbar(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[P_\mu, P_\nu] &= i\frac{\hbar}{\lambda^2}\Theta_{\mu\nu} \, , \qquad [X_\mu, X_\nu] = i\frac{\lambda^2}{\hbar}\Theta_{\mu\nu} \, , \\
[P_\mu, h] &= -i\frac{\hbar}{\lambda^2}X_\mu \, , \qquad [X_\mu, h] = i\frac{\lambda^2}{\hbar}P_\mu \, , \\
[P_\mu, X_\nu] &= i\hbar\eta_{\mu\nu}h \, , \qquad [\Theta_{\mu\nu}, h] = 0
\end{aligned}
$$

- The above relations describe the noncommutative space

Noncommutative gauge theory of 4d gravity

- Formulation of gravity on the above space
- Noncommutative gauge theory construction + the procedure described in the Einstein gravity case

Kimura '02, Heckman-Verlinde '15

- Gauge the isometry group of the space, $\mathrm{SO}(1,4)$ as seen as a subgroup of the $\mathrm{SO}(1,5)$ we ended up
- Anticommutators do not close $\rightarrow$ enlargement of the algebra + fix the representation

Aschieri-Castellani '09

Chatzistavrakidis-Jonke-Jurman-Manolakos-Manousselis-Z '18

- Noncommutative gauge theory of $\mathrm{SO}(2,4) \times \mathrm{U}(1)$

Manolakos-Manousselis-Z '19, '21

Roumelioti-Stefas-Z '24

- The generators of the group are represented by combinations of the 4×4 gamma matrices
- Specifically, the generators are expressed by:
  - six Lorentz rotation generators: $M_{ab} = -\frac{i}{4}\left[\gamma_a, \gamma_b\right]$
  - four generators for conformal boosts: $K_a = \frac{1}{2}\gamma_a(1+\gamma_5)$
  - four generators for translations: $P_a = -\frac{1}{2}\gamma_a(1-\gamma_5)$
  - one generator for special conformal transformations: $\mathrm{D} = -\frac{1}{2}\gamma_5$
  - one U(1) generator: $\mathbb{1}$
- The above expressions of the generators allow the calculation of the algebra they satisfy:

$$
\begin{aligned}
&[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}, \\
&[K_a, P_b] = -2\left(\eta_{ab}D + M_{ab}\right), \ [P_a, D] = P_a, \ [K_a, D] = -K_a, \\
&[M_{ab}, K_c] = \eta_{bc}K_a - \eta_{ac}K_b, \ [M_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b
\end{aligned}
$$

- Generators satisfy the following anticommutation relations:

Smolin '03

$$\begin{aligned}
\{M_{ab}, M_{cd}\} &= \frac{1}{2}\left(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}\right) - i\epsilon_{abcd}D, \\
\{M_{ab}, P_c\} &= +i\epsilon_{abcd}P^d, \\
\{M_{ab}, K_c\} &= -i\epsilon_{abcd}K^d, \\
\{M_{ab}, D\} &= 2M_{ab}D, \\
\{P_a, K_b\} &= 4M_{ab}D + \eta_{ab}, \\
\{K_a, K_b\} &= \{P_a, P_b\} = -\eta_{ab}, \\
\{P_a, D\} &= \{K_a, D\} = 0.
\end{aligned}$$

- We will introduce gauge fields in a motivated way
- Use the general treatment of NC gauge theories

NC gauge theory

Manolakos-Manousselis-Z '21

- Since the gauge group is determined to be $SO(2,4) \times U(1)$, we can move on with the gauging procedure.
- Consider the *covariant coordinate* $\mathcal{X}_\mu = X_\mu + A_\mu$
- Determine appropriate *covariant field strength tensor* $\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\frac{\lambda^2}{\hbar}\hat{\Theta}_{\mu\nu}$,
  where $\hat{\Theta}_{\mu\nu} = \Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}$, the *covariant noncommutative tensor*
- For the SSB to take place we:
    - Introduce scalar field $\Phi(X)$ belonging in the 2nd rank antisym. of $SO(4)$, *charged* under $U(1) \rightarrow U(1)$ breaks and doesn't appear in final action
    - Gauge fix $\Phi(X)$ in the direction that leads to Lorentz group

Gauge connection and field strength tensor decompose as:

$$A_\mu(X) = e_\mu{}^a \otimes P_a + \omega_\mu{}^{ab} \otimes M_{ab} + b_\mu{}^a \otimes K_a + \tilde{a}_\mu \otimes D + a_\mu \otimes \mathbf{I}_4 \,.$$

$$\mathcal{R}_{\mu\nu}(X) = \tilde{R}_{\mu\nu}{}^a \otimes P_a + R_{\mu\nu}{}^{ab} \otimes M_{ab} + R_{\mu\nu}{}^a \otimes K_a + \tilde{R}_{\mu\nu} \otimes D + R_{\mu\nu} \otimes \mathbf{I}_4 \,.$$

The component curvatures:

$$R_{\mu\nu} = [X_\mu, a_\nu] - [X_\nu, a_\mu] + [a_\mu, a_\nu] + [b_\mu{}^a, b_{\nu a}] + [\tilde{a}_\mu, \tilde{a}_\nu] + \frac{1}{2}[\omega_\mu{}^{ab}, \omega_{\nu ab}]$$

$$+ [e_{\mu a}, e_\nu{}^a] - \frac{i\hbar}{\lambda^2} B_{\mu\nu}$$

$$\tilde{R}_{\mu\nu} = [X_\mu, \tilde{a}_\nu] + [a_\mu, \tilde{a}_\nu] - [X_\nu, \tilde{a}_\mu] - [a_\nu, \tilde{a}_\mu] - i\{b_{\mu a}, e_\nu{}^a\} + i\{b_{\nu a}, e_\mu{}^a\}$$

$$+ \frac{1}{2}\epsilon_{abcd}[\omega_\mu{}^{ab}, \omega_\nu{}^{cd}] - \frac{i\hbar}{\lambda^2}\tilde{B}_{\mu\nu}$$

$$R_{\mu\nu}{}^a = [X_\mu, b_\nu{}^a] + [a_\mu, b_\nu{}^a] - [X_\nu, b_\mu{}^a] - [a_\nu, b_\mu{}^a] + i\{b_{\mu b}, \omega_\mu{}^{ab}\} - i\{b_{\nu b}, \omega_\mu{}^{ab}\}$$

$$+ i\{\tilde{a}_\mu, e_\nu{}^a\} - i\{\tilde{a}_\nu, e_\mu{}^a\} + \epsilon_{abcd}([e_\mu{}^b, \omega_\nu{}^{cd}] - [e_\nu{}^b, \omega_\mu{}^{cd}]) - \frac{i\hbar}{\lambda^2} B_{\mu\nu}{}^a$$

$$\tilde{R}_{\mu\nu}{}^a = [X_\mu, e_\nu{}^a] + [a_\mu, e_\nu{}^a] - [X_\nu, e_\mu{}^a] - [a_\nu, e_\mu{}^a] + i\{b_\mu{}^a, \tilde{a}_\nu\} - i\{b_\nu{}^a, \tilde{a}_\mu\}$$

$$- ([b_\mu{}^b, \omega_\nu{}^{cd}] - [b_\nu{}^b, \omega_\mu{}^{cd}])\epsilon_{abcd} - i\{\omega_\mu{}^{ab}, e_{\nu b}\} + i\{\omega_\nu{}^{ab}, e_{\mu b}\} - \frac{i\hbar}{\lambda^2}\tilde{B}_{\mu\nu}{}^a$$

$$R_{\mu\nu}{}^{ab} = [X_\mu, \omega_n{}^{ab}] + [a_\mu, \omega_\nu{}^{ab}] - [X_\nu, \omega_\mu{}^{ab}] - [a_\nu, \omega_m{}^{ab}] + 2i\{b_\mu{}^a, b_\nu{}^b\} + ([b_\mu{}^c, e_\nu{}^d]$$

$$- [b_\nu{}^c, e_\mu{}^d])\epsilon_{abcd} + \frac{1}{2}([\tilde{a}_\mu, \omega_\nu{}^{cd}] - [\tilde{a}_\nu, \omega_\mu{}^{cd}])\epsilon_{abcd} + 2i\{\omega_\mu{}^{ac}, \omega_\nu{}^b{}_c\}$$

$$+ 2i\{e_\mu{}^a, e_\nu{}^b\} - \frac{i\hbar}{\lambda^2} B_{\mu\nu}{}^{ab}$$

Symmetry breaking

Introduction of auxiliary field $\Phi(X)$ charged under $U(1)$:

$$\Phi = \tilde{\phi}^a \otimes P_a + \phi^{ab} \otimes M_{ab} + \phi^a \otimes K_a + \phi \otimes \mathbf{I}_4 + \tilde{\phi} \otimes D$$

into the action:

$$\mathcal{S} = \mathrm{Trtr}_G \, m\Phi(X)\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} + \lambda(\Phi(X)^2 - m^{-2}\mathbf{I}_N \otimes \mathbf{I}_4)\,,$$

induces a symmetry breaking:

$$\boxed{\mathcal{S}_{br} = \mathrm{Tr}\left(\frac{\sqrt{2}}{4}\varepsilon_{abcd}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd} - 4R_{\mu\nu}\tilde{R}_{\rho\sigma}\right)\varepsilon^{\mu\nu\rho\sigma}}$$

when the auxiliary field is gauge fixed as:

$$\Phi(X) = \tilde{\phi}(X) \otimes D|_{\tilde{\phi}=-2m^{-1}} = -2m^{-1}\mathbf{I}_N \otimes D$$

Residual symmetry: $SO(1,3) \times U(1)$

The constraints that correspond to the above breaking are:

Chamseddine '02

$R_{\mu\nu}{}^a = \frac{i}{2}\tilde{R}_{\mu\nu}{}^a = 0$ leading to $\tilde{a}_\mu = 0$, $b_\mu{}^a = \frac{i}{2}e_\mu{}^a$ and $B_{\mu\nu}{}^a = \frac{i}{2}\tilde{B}_{\mu\nu}{}^a$

The commutative limit

- The 2-form field, $\mathcal{B}_{\mu\nu}$ and a_μ decouple
- The commutators of functions vanish: $[f(x), g(x)] \to 0$
- The anticommutators of functions reduce to product: $\{f(x), g(x)\} \to 2f(x)g(x)$
- The inner derivation becomes: $[X_\mu, f] \to \partial_\mu f$
- Trace reduces to integration: $\frac{\sqrt{2}}{4}\mathrm{Tr} \to \int d^4x$
- We also regard the following reparametrizations:

- $e_\mu{}^a \to ime_\mu{}^a\,, \quad P_a \to -\frac{i}{m}P_a\,, \quad \tilde{R}_{\mu\nu}{}^a \to imT_{\mu\nu}{}^a$
- $\omega_\mu{}^{ab} \to -\frac{i}{2}\omega_\mu{}^{ab}\,, \quad M_{ab} \to 2iM_{ab}\,, \quad R_{\mu\nu}{}^{ab} \to -\frac{i}{2}R_{\mu\nu}{}^{ab}$

- When the commutative limit of the action is considered, it reduces to the Palatini action, which is equivalent to EG, with a cosmological constant term present.
- The tensor components transformations are given in: *Manousselis, Manolakos, Z, '18* (See App. I).

Unification of gravity theories with Internal Interactions

- So far in the gauge theoretic approach of gravity, general relativity is described by gauging the symmetry of the tangent manifold in four dimensions.
- The dimension of the tangent space is considered to be equal to the dimension of the curved manifold. However, the tangent group of a manifold of dimension d is not necessarily SO_d.

  Weinberg '84

- It has been suggested that by gauging an enlarged symmetry of the tangent space in four dimensions one could unify gravity with internal interactions.

  Chamseddine, Mukhanov '10

- We aim to unify gravities as a gauge theory with internal interactions under one unification gauge group.
- Further attempts of unification for the case of Einstein gravity: *Percacci, '91; Manolakos et al, '23; Konitopoulos, Roumelioti, Z, '23.*

Unification of Conformal and Fuzzy Gravities with Internal Interactions

Unification Group

- Conformal gravity is based on gauging the $SO(2,4)$, while Fuzzy gravity on $SO(2,4) \times U(1)$.
- Internal Interactions by $SO(10)$ (GUT).
- Spontaneous symmetry breakings are used in all cases.

Roumelioti, Stefas, Z, '24 (CG)
Roumelioti, Stefas, Z, '24 (FG)

In order to have a chiral theory we need an $SO(4n+2)$ group. The smallest unification group in which we can accommodate chiral fermions is $SO(2,16)$ from which:

$$SO(2,16) \xrightarrow{SSB} SO(2,4) \times SO(12)$$

and

$$SO(12) \xrightarrow{SSB} SO(10) \times [U(1)] \, .$$

Breakings and branching rules

We start from $SO(2,16) \sim SO(18)$

- For CG we gauge $SO(2,4) \sim SU(2,2) \sim SO(6) \sim SU(4)$
- For FG we gauge $SO(2,4) \times U(1) \sim SO(6) \times U(1) \sim U(4)$
- For internal interactions we require $SO(10)$ GUT.

Breakings and branching rules (Continued)

$$SO(18) \supset SU(4) \times SO(12)$$

$$\begin{aligned}
18 &= (6,1) + (1,12) && \text{vector} \\
153 &= (15,1) + (6,12) + (1,66) && \text{adjoint} \\
256 &= (4,\overline{32}) + (\overline{4},32) && \text{spinor} \\
170 &= (1,1) + (6,14) + (20',1) + (1,77) && \text{2nd rank symmetric}
\end{aligned}$$

VEV in the $\langle 1,1 \rangle$ component of a scalar in 170 leads to $SU(4) \times SO(12)$.

Breakings and branching rules (Continued)

We break $SO(12)$ down to $SO(10) \times U(1)$ or to $SO(10) \times [U(1)]_{global}$ given the branching rules

$$
\begin{aligned}
&SO(12) \supset SO(10) \times U(1) \\
&\qquad 66 = (1)(0) + (10)(2) + (10)(-2) + (45)(0) \\
&\qquad 77 = (1)(4) + (1)(0) + (1)(-4) + (10)(2) + (10)(-2) + (54)(0)
\end{aligned}
$$

by giving VEV to the $\langle(1,0)\rangle$ of the 66 rep or the $\langle(1,4)\rangle$ of 77 respectively.
Also,

$$
\begin{aligned}
SO(12) &\supset SO(10) \times [U(1)] \\
32 &= \overline{16}(1) + 16(-1).
\end{aligned}
$$

Breakings and branching rules (Continued)

We break $SU(4)$ in 2 steps:

- First step: Breaking $SU(4) \to Sp_4$:

$$\begin{aligned} SU(4) &\supset Sp_4 \\ 4 &= 4 \\ 6 &= 1 + 5 \end{aligned}$$

  giving VEV to a scalar in 6 rep in the $\langle 1 \rangle$ component, the $SU(4)$ breaks down to the Sp_4.

- Second step: Breaking $Sp_4 \to SU(2) \times SU(2)$

$$\begin{aligned} Sp_4 &\supset SU(2) \times SU(2) \\ 5 &= (1,1) + (2,2) \\ 4 &= (2,1) + (1,2). \end{aligned}$$

  giving VEV in $\langle 1, 1 \rangle$ of a scalar in the 5 rep we obtain eventually the Lorentz group $SU(2) \times SU(2) \sim SO(1,3)$.

Fermions

Weyl condition: $\Gamma^{D+1}\psi_\pm = \pm\psi_\pm, \quad D = even.$

Note that since $\Gamma^{D+1} = \gamma^5 \otimes \gamma^{d+1}$, the eigenvalues of γ^5 and γ^{d+1} are interrelated. However the choice of the eigenvalue of Γ^{D+1} does not impose the eigenvalue on γ^5!

Majorana condition: $\psi = C\overline{\psi}^T$

Weyl-Majorana spinors in $SO(t,s)$ exist only for $(s-t) = 0 \mod 8$.

Chapline & Slansky, 1982; Polchinski, 1998 (book); D'Auria et al., 2001; Figueroa-O'Farrill (notes); Ortin, 2004 (book)

Successful example: $SO(1,17)$ *(See below).*

Fermions (Continued)

According to the above, and choosing only the negative eigenvalue of γ^5 , after the full sequence of breakings, we obtain

$$4 \times 16_L(-1)$$

$\rightarrow$ The degeneracy is reduced by starting from $SO(1,17)$ and imposing Weyl - Majorana condition.

- For the gravity part we gauge $SO(1,5) \sim SO(6) \sim SU(4)$
- For internal interactions we require $SO(10)$ GUT.

$$SO(1,17) \xrightarrow{SSB} SO(1,5) \times SO(12) \ ,$$

$$SO(12) \xrightarrow{SSB} SO(10) \times U(1) \ ,$$

$$SO(1,5) \xrightarrow{SSB} SO(1,4) \xrightarrow{SSB} SO(1,3)$$

$\rightarrow$ That way we obtain

$$2 \times 16_L(-1)$$

The remaining degeneracy can be lifted by introducing various scalar reps in the SSB of $SO(10)$.

Fermions in Fuzzy Gravity and Unification with Internal Interactions

- Fermions should be chiral in the original theory to have a chance to survive in low energies
- they should appear in a matrix representation since FG is a matrix model

Fortunately the way out was suggested in unification schemes with extra fuzzy dimensions *Chatzistavrakidis, Steinacker, Z '10*

Instead of using fermions in fundamental, spinor or adjoint reps of an $SU(N)$, we can use bi-fundamental reps of cross product $SU(N)$ groups.

Interesting example $N=1$, $SU(N)^k$ models:

$$SU(N)_1 \times SU(N)_2 \times ... \times SU(N)^k$$

with matter content

$$(N, \overline{N}, 1, ..., 1) + (1, N, \overline{N}, ..., 1) + ... + (\overline{N}, 1, 1, ..., N)$$

Ma, Mondragon, Z, '04

with successful phenomenology, $N=1$, $SU(3)^3$.

Fermions in Fuzzy Gravity and Unification with Internal Interactions (Continued)

- In FG choosing to start with the $SO(6) \times SO(12)$ as the initial gauge theory with fermions in the $(4, \overline{32}) + (\overline{4}, 32)$ we satisfy the criteria to obtain chiral fermions in tensorial representation.
- The gauge $U(1)$ of FG due to the anticommutation relations, is identified with the one appearing in the $SO(12) \supset SO(10) \times U(1)$.

Further studies:

- on various breakings and their scales, *Patellis, Trakas, Z, '24*
- including possible gravitational signals from cosmic strings due to the $SO(10)$ breakings, *Patellis, Roumelioti, Stefas, Z, '25*
- in Cosmology treating the heavy spin-2 graviton as DM (work in progress).

Possible solution of the ghost problems using bimetric theory (work in progress). *Hassan, Rosen*

Thank you for your attention!

Appendix I: Fields and transformations

- The gauge connection, A_μ, as an element of the $SO(2,4)$ algebra, can be expanded in terms of the generators as

$$A_\mu = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a + b_\mu{}^a K_a + \tilde{a}_\mu D,$$

- It obeys the following infinitesimal transformation rule,

$$\delta A_\mu = D_\mu \epsilon = \partial_\mu \epsilon + [A_\mu, \epsilon],$$

where $\epsilon = \epsilon(x)$ a gauge algebra parameter which be expanded too as,

$$\epsilon = \xi^a P_a + \frac{1}{2}\lambda^{ab}M_{ab} + \kappa D + \rho^a K_a.$$

- From the above, the transf rule of the fields can be found:

$$\begin{aligned}
\delta e_\mu{}^a &= \partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b - b_\mu \xi^a - \lambda^a{}_b e_\mu{}^b + \kappa e_\mu{}^a, \\
\delta \omega_\mu{}^{ab} &= \partial_\mu \lambda^{ab} - 2\omega_\mu{}^{ac}\lambda^b{}_c - 4f_\mu{}^{[a}\xi^{b]} - 4e_\mu{}^{[a}\rho^{b]}, \\
\delta \tilde{a}_\mu &= \partial_\mu \kappa - 2\xi^a f_{\mu a} + 2\rho^a e_{\mu a}, \\
\delta b_\mu{}^a &= \partial_\mu \rho^a + \omega_\mu{}^{ab}\rho_b + b_\mu \rho^a - \lambda^{ab} f_{\mu b} - \kappa f_\mu{}^a.
\end{aligned}$$

Appendix I: Fields and transformations (2)

The transformations of the fields:

$$\delta\omega_m^{\ ab} = -i[X_m, \lambda^{ab}] - i[a_m, \lambda^{ab}] + i[\epsilon_0, \omega_m^{\ ab}] - 2\{\xi^a, b_m^{\ b}\} - \frac{1}{2}\{\lambda^a_{\ c}, \omega_m^{\ bc}\}$$

$$- \frac{1}{2}\{\tilde{\xi}^a, e_m^{\ b}\} + i[\xi^c, e_m^{\ d}]\epsilon_{abcd} + \frac{i}{2}[\tilde{\epsilon}_0, \omega_m^{\ cd}]\epsilon_{abcd} + \frac{i}{2}[\lambda^{cd}, \tilde{a}_m]\epsilon_{abcd} - i[\tilde{\xi}^c, b_m^{\ d}]\epsilon_{abcd}$$

$$\delta e_m^{\ a} = -i[X_m, \tilde{\xi}^a] - i[a_m, \tilde{\xi}^a] + i[\epsilon_0, e_m^{\ a}] - \{\xi^a, \tilde{a}_m\} + \{\tilde{\epsilon}_0, b_m^{\ a}\} + \frac{1}{4}\{\lambda^a_{\ b}, e_m^{\ b}\}$$

$$- \frac{1}{4}\{\tilde{\xi}_b, \omega_m^{\ ab}\} + i[\xi^c, \omega_m^{\ bd}]\epsilon_{abcd} - i[\lambda^{cd}, b_m^{\ b}]\epsilon_{abcd}$$

$$\delta b_m^{\ a} = -i[X_m, \xi^a] - i[a_m, \xi^a] + i[\epsilon_0, b_m^{\ a}] - \{\xi_b, \omega_m^{\ ab}\} - 2\{\tilde{\epsilon}_0, e_m^{\ a}\} + \frac{1}{2}\{\lambda^a_{\ b}, b_m^{\ b}\}$$

$$+ \{\tilde{\xi}^a, \tilde{a}_m\} + i[\lambda^{bc}, e_m^{\ d}]\epsilon_{abcd} + i[\tilde{\xi}^b, \omega_m^{\ cd}]\epsilon_{abcd}$$

$$\delta a_m = -i[X_m, \epsilon_0] - i[a_m, \epsilon_0] + i[\xi^a, b_m^{\ a}] + i[\tilde{\epsilon}_0, \tilde{a}_m] + \frac{i}{2}[\lambda_{ab}, \omega_m^{\ ab}] + \frac{i}{2}[\tilde{\xi}_a, e_m^{\ a}]$$

$$\delta \tilde{a}_m = -i[X_m, \tilde{\epsilon}_0] - i[a_m, \tilde{\epsilon}_0] + i[\epsilon_0, \tilde{a}_m] + \{\xi_a, e_m^{\ a}\} - \{\tilde{\xi}_a, b_m^{\ a}\} + \frac{i}{2}[\lambda^{ad}, \omega_m^{\ bc}]\epsilon_{abcd}$$

(Transformations of the component of $\mathcal{B}_{mn}$ are calculated as well)

Appendix II: SSB of Weyl Gravity to EG

- The result of the breaking can be seen by considering the decomp. of **15** of $SU(4)$ under $SU(2) \times SU(2) \times U(1)$

$$SU(4) \xrightarrow{<\mathbf{15}>} SU(2) \times SU(2) \times U(1)$$
$$\mathbf{15} = [(\mathbf{3},\mathbf{1})_0 + (\mathbf{1},\mathbf{3})_0] + (\mathbf{1},\mathbf{1})_0 + (\mathbf{2},\mathbf{2})_2 + (\mathbf{2},\mathbf{2})_2\,,$$

$\rightarrow [(\mathbf{3},\mathbf{1})_0 + (\mathbf{1},\mathbf{3})_0]$ describes the generators of the Lorentz gauge group, M_{ab}
$\rightarrow (\mathbf{1},\mathbf{1})_0$ the generator of dilatations, D
$\rightarrow (\mathbf{2},\mathbf{2})_2$ the generators of the translations, P_a
$\rightarrow (\mathbf{2},\mathbf{2})_2$ the generators of conformal transformations, K_a

- The generators P_a and K_a are broken due to the SSB of the scalar **15**-plet

Appendix II: SSB of Weyl Gravity to EG (2)

- Similarly, the decomposition of the 15 generators of $SU(4)$ under the $SO(5)$ to which it breaks after the SSB of the scalar **6**-plet is,

$$SU(4) \xrightarrow{<\mathbf{6}>} SO(5)$$
$$\mathbf{15} = \mathbf{10} + \mathbf{5}\,,$$

$\rightarrow$ **10** the generators of the unbroken gauge group, $SO(5)$, and
$\rightarrow$ **5** the broken generators

- To identify the unbroken and the broken generators above we consider the decomp of reps **10** and **5** of $SO(5)$ under the $SU(2) \times SU(2)$ describing the Lorentz gauge group,

$$SO(5) \supset SU(2) \times SU(2)$$
$$\mathbf{10} = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})\,,$$
$$\mathbf{5} = (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})\,.$$

$\rightarrow$ Ten unbroken gen. from the SSB of the scalar **6**-plet correspond to M_{ab} and the P_a (which were broken by the $<\mathbf{15}>$)
$\rightarrow$ Five broken gen. are the $(\mathbf{1},\mathbf{1})$ of D and the $(\mathbf{2},\mathbf{2})$ of K_a.

Appendix II: SSB of Weyl Gravity to EG (3)

- In summary, $<\mathbf{15}>$ breaks the generators of P_a and K_a, leaving unbroken the Lorentz rotation generators, M_{ab} and the dilaton generator, D, while $<\mathbf{6}>$ breaks the dilaton generator, D and gives an additional contribution to the breaking of the generators K_a (and to the masses of the corresponding gauge bosons).

Appendix III: Equivalence of Gauge and Diffeo transf

- We calculate the difference between a diffeomorphism and a gauge transformation of the fields:

$$\tilde{\delta} e^a_\mu - \delta e^a_\mu = \left(v^\nu \partial_\nu e^a_\mu + \partial_\mu (v^\nu e^a_\nu) - v^\nu \partial_\mu e^a_\nu\right) - \left(\partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b - b_\mu \xi^a - \lambda^{ab} e^b_\mu + \kappa e^a_\mu\right)$$

- Setting $\xi^a = v^\mu e_\mu{}^a$, $\lambda^{ab} = v^\mu \omega_\mu{}^{ab}$, $\kappa = v^\mu b_\mu$, and $\rho^a = v^\mu f_\mu{}^a$:

$$\begin{aligned}\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a =& v^\nu \left(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\nu{}^b + \omega_\nu{}^a{}_b e_\mu{}^b + b_\mu e_\nu{}^a - b_\nu e_\mu{}^a\right) \\ =& - v^\nu \tilde{R}_{\mu\nu}{}^a .\end{aligned}$$

$\rightarrow$ the constraint needed for getting rid of the translational part, with a coordinate transformation making up for them, is the vanishing of the torsion,

$$\tilde{R}_{\mu\nu}{}^a = 0 .$$

Appendix III: Equiv. of Gauge and Diffeo transf (2)

- Similarly, the difference between a diffeomorphism and the gauge transformation $\tilde{\delta} b_\mu{}^a - \delta b_\mu{}^a$ leads to

$$R_{\mu\nu}{}^a = 0\,,$$

while the corresponding difference $\tilde{\delta}\omega_\mu{}^{ab} - \delta\omega_\mu{}^{ab}$ results to

$$R_{\mu\nu}{}^{ab} = 0\,.$$

- As already mentioned the generators P_a and K_a are broken due to the SSB of the scalar 15-plet, i.e. the two torsionless conditions are resulting from the SSB of the scalar 15-plet.
- The two torsionless conditions and the vanishing of the curvature tensor (which is satisfied on-shell) guarantee the equivalence of the diffeomorphisms and gauge transformations.
  $\rightarrow$ The gauge theory based on the $SO(2,4)$ group describes the 4-d conformal gravity.

Appendix IV: Unification based on $SO(1,17)$

We start from $SO(1,17) \sim SO(18)$

- For the gravity part we gauge $SO(1,5) \sim SO(6) \sim SU(4)$
- For internal interactions we require $SO(10)$ GUT.

$$SO(1,17) \xrightarrow{SSB} SO(1,5) \times SO(12)$$

and

$$SO(12) \xrightarrow{SSB} SO(10) \times U(1) \ ,$$

$$SO(1,5) \xrightarrow{SSB} SO(1,4) \xrightarrow{SSB} SO(1,3)$$

Appendix IV: Unification based on $SO(1,17)$ (II)

The new features in the present case are:

- The E-H action we end up with describes a dS spacetime. To change to AdS (which would be balanced to the $SO(10)$ breaking vevs) one could use the three possible SSB terms *(Wilczek '98)*

$$S_1 = \int d^4x \varepsilon^{\mu\nu\rho\sigma} \, \mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}\phi) \ ,$$

$$S_2 = \int d^4x \varepsilon^{\mu\nu\rho\sigma} \, \mathrm{Tr}(F_{\mu\nu}D_\rho\phi D_\sigma\phi\phi) \ ,$$

$$S_2 = \int d^4x \varepsilon^{\mu\nu\rho\sigma} \, \mathrm{Tr}(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi) \ ,$$

- Due to the signature of the initial group now we can apply both Majorana and Weyl conditions ending up with 2 families of fermions.