# Jordan Algebraic Formulation of the Non-commutative Landau Problem with a Harmonic Potential

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**Abstract:** We give a Jordan algebraic,non-associative formulation of the exotic (non-commutative) Landau problem coupled with a harmonic potential. The non-commutative parameter can be expressed in terms of associators in the Jordan algebraic setting. We briefly comment on the construction of a Hilbert space of states for this problem and discuss pure and mixed states expressed in terms of density matrices. Split operators for our Hamiltonian are given explicitly and the Jordan-Schrödinger time-evolution equation satisfied by density matrices is written in terms of associators.

This is based on joint work with E.S.Yörük.

# Bloch waves and non-commutative tori of magnetic translations

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ABSTRACT

We review the Landau problem of an electron in a constant uniform magnetic field. The magnetic translations are the invariant transfor- mations of the free Hamiltonian. A Kähler polarization of the plane has been used for the geometric quantization. Under the assumption of quasi-periodicity of the wavefunction, the Zak's magnetic translations in the Bravais lattice generate a non-commutative quantum torus. We concentrate on the case when the magnetic flux density is a rational number. The Bloch wavefunctions form a finite-dimensional module of the noncommutative torus of magnetic translations as well as of its commutant, which is the non-commutative torus of magnetic translations in the dual Bravais lattice. The bi-module structure of the Bloch waves is shown to be the connecting link between two Morita equivalent non-commutative tori. The main focus of our review is the Kähler structure on the Hilbert space of Bloch waves and its inherent quantum toric geometry. We reveal that the metaplectic group $Mp(2,\mathrm{R})$ of the automorphisms of magnetic translation algebras is represented by the quantum optics squeezing operators.

# A remarkable dynamical symmetry of the Landau problem

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Abstract. We show that the dynamical group of an electron in a constant magnetic field is the group of symplectomorphisms Sp(4,R). It is generated by the spinorial realization of the conformal algebra so(2,3) considered in Dirac's seminal paper "A Remarkable Representation of the 3 + 2 de Sitter Group". The symplectic group Sp(4,R) is the double covering of the conformal group SO(2,3) of 2+1 dimensional Minkowski spacetime which is in turn the dynamical group of a hydrogen atom in 2 space dimensions. The Newton-Hooke duality between the 2D hydrogen atom and the Landau problem is explained via the Tits-Kantor- Koecher construction of the conformal symmetries of the Jordan algebra of real symmetric 2 × 2 matrices. The connection between the Landau problem and the 3D hydrogen atom is elucidated by the reduction of a Dirac spinor to a Majorana one in the Kustaanheimo-Stiefel spinorial regularization.

## Introduction

A non-associative formulation of quantum mechanics dates back to 1934 [1]. It was assumed that physical observables form an algebra under the symmetric Jordan product of self-adjoint operators

$$A \bullet B = \frac{1}{2}(AB + BA). \tag{1}$$

Clearly the Jordan product is commutative but only power associative:

$$(A \bullet B) \bullet A^2 = A \bullet (B \bullet A^2). \tag{2}$$

The interest in non-associative structures with physical applications had seen a revival during the 1980's [2]. See also [3],[4]. In particular non-associative algebras are used for a quantum description of point magnetic monopole dynamics [5]. On the other hand, many more recent work on non-associative algebras relate with string models and double field theories [6],[7]. We should rather be dealing with Jordan algebras equipped with a Banach space structure as operator algebras arising in quantum mechanics are often infinite dimensional. We refer to previous work for mathematical details [8],[9],[10]. Here we concentrate on a concrete physical application dealing with the non-commutative Landau problem with a harmonic potential [11],[12].

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## Non-commutative Landau Problem

Let $\mathcal{B}$ be the $C^*$-algebra of operators on $\mathcal{H}$ with self-adjoint part $\mathcal{A}$, which is considered as the Jordan algebra of observables. Now we consider a point electric charge $(\mu, e)$ moving in the $xy$-plane under the influence of a constant, uniform magnetic field $B$ pointing along the positive $z$ direction that is described in the symmetric gauge by the magnetic vector potential $\mathcal{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right) = (A_x, A_y)$. Furthermore, for generality, we will assume the presence of a harmonic potential. Then the dynamics will be determined by the following Hamiltonian

$$H_\theta = \frac{1}{2\mu}\left(p_x - \frac{eB}{2c}y\right)^2 + \frac{1}{2\mu}\left(p_y + \frac{eB}{2c}x\right)^2 + \frac{1}{2}\mu\omega^2(x^2 + y^2), \quad (3)$$

where , $\omega_L = \frac{eB}{\mu c}$ is the Larmor frequency and $\omega$ is an independent parameter. The position and momentum operators satisfy the following nonzero commutation relations of the non-commutative Heisenberg algebra for the problem described above:

$$[x, y] = i\theta, \quad [x, p_x] = [y, p_y] = i\hbar.$$

Next, we define new operators in $\mathcal{B}$, corresponding to complex variables, related to chiral decomposition of the described Landau problem. It is worth remarking that our approach here is reminiscent of the introduction of complex coordinates in the Landau problem pertaining to the Bloch waves quasi-periodicity on a torus. Precisely, we have

$$z = x + iy, \quad \overline{z} = x - iy. \tag{4}$$

and the corresponding momentum operators

$$p_z = \frac{1}{2}(p_x - ip_y), \quad p_{\overline{z}} = \frac{1}{2}(p_x + ip_y). \tag{5}$$

These operators can be shown to satisfy the following non-vanishing commutation relations in $\mathcal{B}$

$$[z, p_z] = [\overline{z}, p_{\overline{z}}] = i\hbar \quad , \quad [z, \overline{z}] = 2\theta. \tag{6}$$

We next introduce two pairs of creation and annihilation operators in $\mathcal{B}$:

$$\begin{aligned} A_+ &= \frac{\xi}{2}\overline{z} + \frac{i}{\xi\hbar}p_z, \qquad A_+^\dagger = \xi\frac{z}{2} - \frac{i}{\xi\hbar}p_{\overline{z}}, \\ A_- &= \frac{\xi}{2}z + \frac{i}{\xi\hbar}p_{\overline{z}}, \qquad A_-^\dagger = \frac{\xi}{2}\overline{z} - \frac{i}{\xi\hbar}p_z, \end{aligned} \tag{7}$$

where

$$\xi = \left[\frac{\mu^2\omega^2/\hbar^2 + \mu^2{\omega_L}^2/4\hbar^2}{1 - (\mu\omega_L\theta/2) + (\mu^2\omega^2\theta^2/16) + (\mu^2{\omega_L}^2\theta^2/64)}\right]^{1/4}.$$

Conversely we have

$$
\begin{aligned}
x &= \frac{1}{\sqrt{2}\xi}(A_- + A_+ + A_+^\dagger + A_-^\dagger), \\
y &= \frac{i}{\sqrt{2}\xi}(-A_- + A_+ - A_+^\dagger + A_-^\dagger), \\
p_x &= \frac{i\xi\hbar}{2\sqrt{2}}(-A_- - A_+ + A_+^\dagger + A_-^\dagger), \\
p_y &= \frac{\xi\hbar}{2\sqrt{2}}(-A_- + A_+ + A_+^\dagger - A_-^\dagger).
\end{aligned}
\tag{8}
$$

Then the non-vanishing commutation relations between these operators are

$$
[A_+, A_+^\dagger] = 1 \quad , \qquad [A_-, A_-^\dagger] = 1. \tag{9}
$$

Finally, defining $\Omega_\pm := \Omega \pm \frac{\tilde{\omega}_L}{2}$, where

$$\Omega = \left[\omega^2 + \frac{\omega_L^2}{4} - \frac{\mu\omega_L\omega^2\theta}{2} - \frac{\mu\omega_L^3\theta}{8} + \left(\omega^2 + \frac{\omega_L^2}{4}\right)\left(\left(\frac{\mu\omega\theta}{4}\right)^2 + \left(\frac{\mu\omega_L\theta}{8}\right)^2\right)\right]^{1/2},$$

$$\tilde{\omega}_L = \omega_L\left(1 - \left(\frac{\omega_L}{4} + \frac{\omega^2}{\omega_L}\right)\mu\theta\right),$$

one can decouple the Hamiltonian as

$$H_\theta = \hbar\Omega_+\left(N_+ + \frac{1}{2}\right) + \hbar\Omega_-\left(N_- + \frac{1}{2}\right), \qquad N_\pm = A_\pm^\dagger A_\pm. \qquad (10)$$

We set

$$H_\pm = \hbar\Omega_\pm\left(N_\pm + \frac{1}{2}\right).$$

Then the operator $H_+ \otimes \mathbb{I} + \mathbb{I} \otimes H_-$ will be the infinitesimal generator of the strongly continuous one-parameter unitary group $e^{itH_+} \otimes e^{itH_-}$. By Stone's theorem, $H_+ \otimes \mathbb{I} + \mathbb{I} \otimes H_-$ is self-adjoint. Using this result, the Hamiltonian (1) can be written as

$$H_\theta = H_+ \otimes \mathbb{I}_{\widehat{\mathcal{H}}_{J,-}} + \mathbb{I}_{\widehat{\mathcal{H}}_{J,+}} \otimes H_-.$$

Here $\widehat{\mathcal{H}}_{J,+} \otimes \widehat{\mathcal{H}}_{J,-}$ is the real Jordan algebraic Hilbert space for the problem which is the domain of $H_\theta$, and $\mathbb{I}_{\widehat{\mathcal{H}}_{J,\pm}}$ are the corresponding identity operators.

**Proposition:** The canonical commutation relations in $\mathcal{B}$, given by (2) are equivalent to the following associator relations in $\mathcal{A}$:

$$[y_L, x, y_R] = \theta, \quad [p_{xL}, x, p_{xR}] = \hbar, \quad [p_{yL}, y, p_{yR}] = \hbar, \tag{11}$$

where $y_L = p_{xL} = p_{yL} = 4H_\theta$, and

$$y_R = \left(\frac{1}{2\hbar\Omega_-} - \frac{1}{2\hbar\Omega_+}\right) x - \left(\frac{1}{\xi^2\hbar^2\Omega_+} + \frac{1}{\xi^2\hbar^2\Omega_-}\right) p_y,$$

$$p_{xR} = \left(\frac{\xi^2}{4\Omega_+} + \frac{\xi^2}{4\Omega_-}\right) x + \left(\frac{1}{2\hbar\Omega_+} - \frac{1}{2\hbar^2\Omega_-}\right) p_y,$$

$$p_{yR} = \left(\frac{\xi^2\hbar}{4\Omega_+} + \frac{\xi^2\hbar}{4\Omega_-}\right) y + \left(\frac{1}{2\Omega_-} - \frac{1}{2\Omega_+}\right) p_x.$$

*Proof:* Express the associator as a double commutator $[A, B, C] = \frac{1}{4}[B, [A, C]]$ and the rest is algebra.

## Density Matrices in the JBW-algebra Setting

It turns out that one needs a notion of density matrix in the JBW-algebra $\mathcal{A}$ in order to describe the state of a quantum system.

**Theorem:** Let $\mathcal{A}$ be a commutative, order unit algebra that is the dual of a base norm space $V$ such that the multiplication in $\mathcal{A}$ is separately $w^*$- continuous. Then for each $a \in \mathcal{A}$ and each $\epsilon > 0$ there are orthogonal projections $p_1, p_2, \ldots, p_n$ in the $w^*$-closed subalgebra generated by $a$ and 1, and scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\|a - \sum_{i=1}^{n} \lambda_i p_i\| < \epsilon.$$

The set of finite linear combinations of orthogonal projections is norm dense in $\mathcal{A}^{**}$, here $\mathcal{A}^{**}$ denotes the bidual of $\mathcal{A}$, which is a JBW-algebra.

We will assume in the above theorem that any element can be written as a finite linear combination of orthonormal projection operators. In particular, for any $B \in \widehat{\mathcal{H}}_J$ we have

$$B = \sum_{i=1}^{n} \lambda_i P_i, \tag{12}$$

where $\lambda_i$'s are real scalars, and $P_i$'s denoting the orthonormal operators. And for a general element, we use the fact that it can be approximated in norm by linear combinations of orthonormal projections. So, the general case follows from the norm continuity of multiplication.

**Definition:** A self-adjoint operator $\rho \in \mathcal{A}$ is a density matrix if $\rho$ is non-negative and $tr(\rho) = 1$.

**Proposition** Assume that $\rho$ is a density matrix on $\widehat{\mathcal{H}}_J$, then the map $\Phi_\rho : \mathcal{A} \to \mathbb{R}$ given by $\Phi_\rho(A) = tr(\rho \bullet A)$ is a family of expectation values.

**Definition:** A density matrix is called a pure state if it can be written as a projection operator in $\mathcal{A}$, otherwise it is called a mixed state.

**Proposition:** Given the time evolution equation $\dot{\rho} = \frac{1}{i\hbar}[H, \rho]$ and a Hamiltonian of the form $H = i[R, S]$ where $R$ and $S$ are self-adjoint operators, we have

$$\dot{\rho} = -\frac{4}{\hbar}[R, \rho, S]\,. \tag{13}$$

Now, we can revisit the non-commutative Landau problem, where the Hilbert space for the composite system is the Hilbert tensor product $\widehat{\mathcal{H}}_{J,+} \otimes \widehat{\mathcal{H}}_{J,-}$ of the Hilbert spaces $\widehat{\mathcal{H}}_{J,+}$ and $\widehat{\mathcal{H}}_{J,-}$ describing the subsystems. Assuming $\rho$ is a density matrix on $\widehat{\mathcal{H}}_{J,+} \otimes \widehat{\mathcal{H}}_{J,-}$, there exists a unique density matrix $\rho_+$ on $\widehat{\mathcal{H}}_{J,+}$ with the property that $tr(\rho_+ \bullet A) = tr(\rho \bullet (A \otimes \mathbb{I}_{\widehat{\mathcal{H}}_{J,-}}))$ for all $A \in \mathcal{B}(\widehat{\mathcal{H}}_{J,+})$. Similarly, there exists a unique density matrix $\rho_-$ on $\widehat{\mathcal{H}}_{J,-}$ with the property that $tr(\rho_- \bullet B) = tr(\rho \bullet (\mathbb{I}_{\widehat{\mathcal{H}}_{J,+}} \otimes B))$ for all $B \in \mathcal{B}(\widehat{\mathcal{H}}_{J,-})$. Since, the state of the first system is independent of the state of the second system, the density matrix $\rho$ is of the form $\rho = \rho_+ \otimes \rho_-$ for this problem.

**Proposition:** A pure state is being represented in $\widehat{\mathcal{H}}_{J,+} \otimes \widehat{\mathcal{H}}_{J,-}$ by $\pm\mathcal{P}_{0,+} \otimes \pm\mathcal{P}_{0,-}$, where $\mathcal{P}_{0,+}$ and $\mathcal{P}_{0,-}$ are idempotent elements on $\widehat{\mathcal{H}}_{J,+}$ and $\widehat{\mathcal{H}}_{J,-}$, respectively.

Now, we define new observables before stating our next result giving the time-evolution equation for the state vectors of the non-commutative Landau problem. Let

$$X_+ = \sqrt{\frac{\hbar}{2\mu\Omega_+}}(A_+^\dagger + A_+), \qquad X_- = \sqrt{\frac{\hbar}{2\mu\Omega_-}}(A_-^\dagger + A_-),$$

be the position operators corresponding to subsystems. And

$$P_+ = i\sqrt{\frac{\hbar\mu\Omega_+}{2}}(A_+^\dagger - A_+), \qquad P_- = i\sqrt{\frac{\hbar\mu\Omega_-}{2}}(A_-^\dagger - A_-),$$

be the corresponding momentum operators. We have

$$H_+ = i[R_{1_+}, S_{1_+}] + i[R_{2_+}, S_{2_+}] \quad , \quad H_- = i[R_{1_-}, S_{2_-}] + i[R_{2_-}, S_{2_-}],$$

where

$$R_{1_+} = -\frac{\Omega_+}{4}(X_+A_+ + A_+^\dagger X_+), \quad S_{1_+} = P_+A_+ + A_+^\dagger P_+,$$

$$R_{2_+} = -\frac{\Omega_+}{4}(P_+A_+ + A_+^\dagger P_+), \quad S_{2_+} = 2\sqrt{\frac{\hbar}{2\mu\Omega_+}}A_+^\dagger A_+,$$

and

$$R_{1_-} = -\frac{\Omega_-}{4}(X_-A_- + A_-^\dagger X_-), \quad S_{1_-} = P_-A_- + A_-^\dagger P_-,$$

$$R_{2_-} = -\frac{\Omega_-}{4}(P_-A_- + A_-^\dagger P_-), \quad S_{2_-} = 2\sqrt{\frac{\hbar}{2\mu\Omega_-}}A_-^\dagger A_-.$$

Therefore, the time-evolution equation satisfied by a state vector $v_+ \otimes v_- \in \widehat{\mathcal{H}}_{J,+} \otimes \widehat{\mathcal{H}}_{J,-}$ can be written as follows:

$$\begin{aligned} \frac{d}{dt}(v_+ \otimes v_-) &= -\frac{4}{\hbar}\{(\left[R_{1_+}, v_+, S_{1_+}\right] + \left[R_{2_+}, v_+, S_{2_+}\right]) \otimes v_- \\ &+ v_+ \otimes (\left[R_{1_-}, v_-, S_{1_-}\right] + \left[R_{2_-}, v_-, S_{2_-}\right])\}. \end{aligned}$$

## Conclusion

A key algebraic fact underlying the above result is that every self-adjoint operator $H$ in a Jordan $C^*$-algebra can be written as a finite sum of commutators, $H = i\sum_j [H_{L_j}, H_{R_j}]$, with self-adjoint summands. While we guarantee this decomposition in general, our main contribution here is to construct the split operators $R_{1\pm}, S_{1\pm}, R_{2\pm}, S_{2\pm}$ explicitly for the non-commutative Landau Hamiltonian. This turns an abstract existence result into a concrete decomposition that drives both the associator representation of the non-commutative parameter and the Jordan–Schrödinger time-evolution equation.