

A short trip to cumuland

An hommage to the memory of Ivan Todorov

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Sofia, 30 May 2026

Recollections of Ivan T. Todorov

Starting in the mid 80's

A frequent visitor of our lab
in Saclay, an old friend of
Claude Itzykson (1938–1995).

Because of causality, the limit in (A17) is the light-cone limit $x^2 = 4x_+^2 - x_-^2 \rightarrow 0$. It indeed follows from (A14) and (A17) that

$$\pi \tilde{W}(x, \lambda) \sim \delta(x) \frac{1}{2} \epsilon(\lambda) f(\lambda) \quad (\text{A23})$$

for $x \sim 0$. We therefore have determined the leading singularity of \tilde{W} on the light cone and, by (A20), the large- λ behavior of its coefficient. The behavior of W near the light cone can also be determined from the configuration-space form

$$\pi \tilde{W}(x^2, z_0) = -\frac{1}{2} \int d a d b \sigma(a, b) \exp(-i b z_0) \Delta(x; a + b^2) \quad (\text{A24})$$

of (A1) and the behavior (3.58) of Δ . These give

$$\pi \tilde{W}(x, \lambda) \sim \delta(x) L(\lambda) \quad (\text{A25})$$

for $x \sim 0$, where

$$L(\lambda) = -[i \epsilon(\lambda) / 4\pi] \int d a d b \sigma(a, b) \exp(-i b \lambda). \quad (\text{A26})$$

Equations (A6) and (A9) give

$$L(\lambda) \xrightarrow{\lambda \rightarrow \infty} -2^{-\alpha} i \text{sgn}(\lambda) |\lambda|^\alpha. \quad (\text{A27})$$

In view of (A22), Eqs. (A25) and (A27) are in perfect agreement with (A23) and (A20).

Three-Dimensional Formulation of the Relativistic Two-Body Problem and Infinite-Component Wave Equations

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A relativistic quasipotential equation is derived from the conventional Hamiltonian formalism and old-fashioned "noncovariant" off-energy-shell perturbation theory in a similar way to that by which the four-dimensional Bethe-Salpeter equation is obtained from the off-mass-shell Feynman rules. The three-dimensional equation for the (off-energy-shell) scattering amplitude appears as a straightforward generalization of the nonrelativistic Lippmann-Schwinger equation. The corresponding homogeneous equation for the bound-state wave function and the normalization condition for its solutions are derived from the equation for the complete four-point Green's function. In order to obtain a solvable model, we consider a simplified version of the quasipotential equation which still reproduces correctly the on-shell scattering amplitude and is consistent with the elastic unitarity condition. It involves a "local" approximation to the potential $V(p-q)$ which defines the kernel of our integral equation (the integration being carried over a two-sheeted hyperboloid in the energy-momentum space). It is shown that for the scalar Coulomb potential $V(p-q) = \alpha / (p-q)^2$, our model equation is equivalent to a simple infinite-component wave equation of the type considered by Nambu, Barut, and Frossdal. The energy eigenvalues for the bound-state problem are calculated explicitly in this case and are found to be $O(4)$ degenerate (just as in the nonrelativistic Coulomb problem and in Wick and Cutkosky's treatment of the Bethe-Salpeter equation in the same approximation).

I. INTRODUCTION

THE purpose of this paper is to show the relationship between a modification of the quasipotential approach to the relativistic two-body problem developed in Refs. 1-3 and the infinite-component wave equations

for the "relativistic hydrogen atom" of the type considered in Refs. 4-6.

A three-dimensional relativistic quasipotential equation for the two-particle scattering amplitude and for the bound-state wave function was first proposed by Logunov and Tavkhelidze.⁷ It was derived in the framework of the Bethe-Salpeter equation using the non-uniqueness of the off-shell extrapolation of the scattering amplitude.

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¹ V. G. Kadyshevsky, Nucl. Phys. B6, 125 (1968); V. G. Kadyshevsky and N. D. Mattev, Nuovo Cimento 55A, 275 (1968).

² V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, Nuovo Cimento 55A, 233 (1968).

³ M. D. Matveev, R. M. Mir-Kasimov, M. Fretzman, Joint Institute for Nuclear Research, Dubna, USSR, Report No. P2-4107, 1968 (unpublished).

⁴ Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppl. 37-38, 368 (1966); Phys. Rev. 160, 1171 (1967).

⁵ C. Frossdal, Phys. Rev. 156, 1665 (1967).

⁶ A. O. Barut and H. Kleinert, Phys. Rev. 157, 1180 (1967); 160, 1149 (1967); H. Kleinert, Fortsch. Physik 16, 1 (1968); A. O. Barut and A. Bajuni, Phys. Rev. 184, 1347 (1969).

⁷ A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento 29, 380 (1963); A. A. Logunov et al., Nuovo Cimento 30, 134 (1963).

High reputation of
(field and group) theorist
and conformal field theorist..

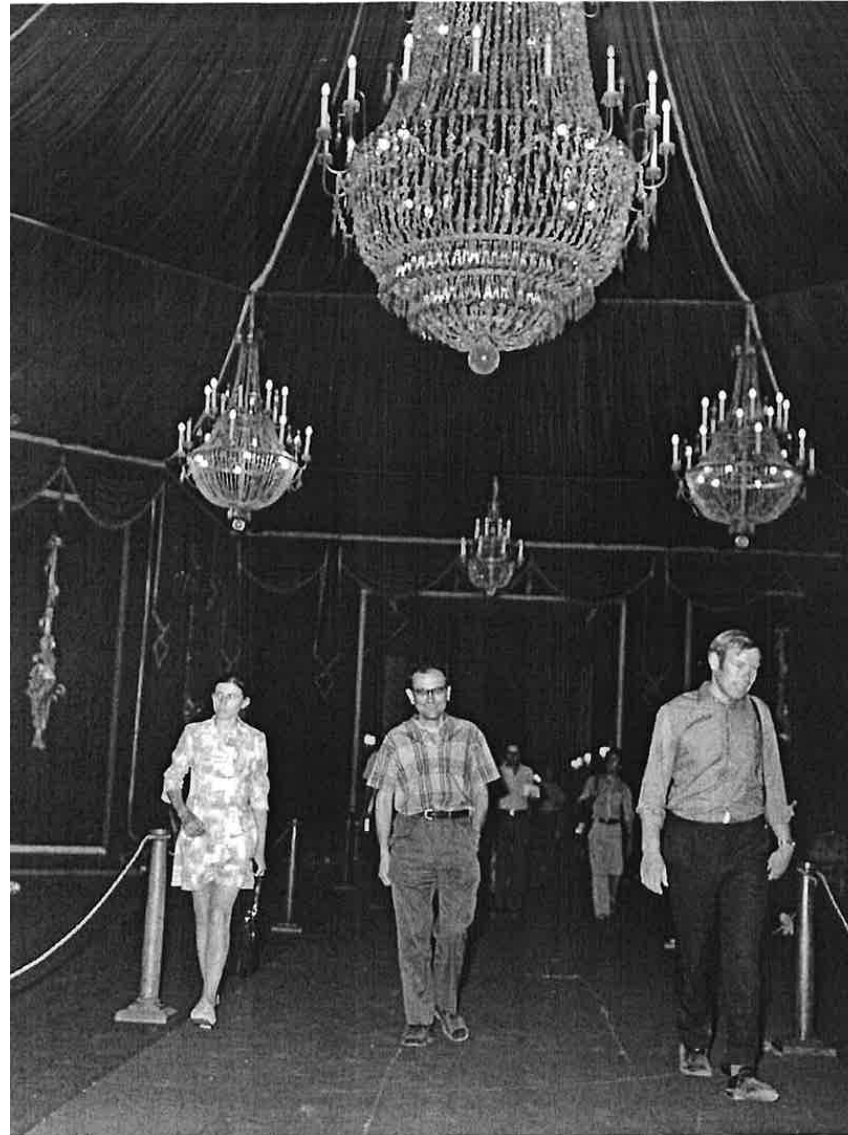
SCUOLA NORMALE SUPERIORE
PISA
CLASSE DI SCIENZE

I.T. TODOROV
M.C. MINTCHEV - V.B. PETKOVA

**CONFORMAL INVARIANCE
IN
QUANTUM FIELD THEORY**

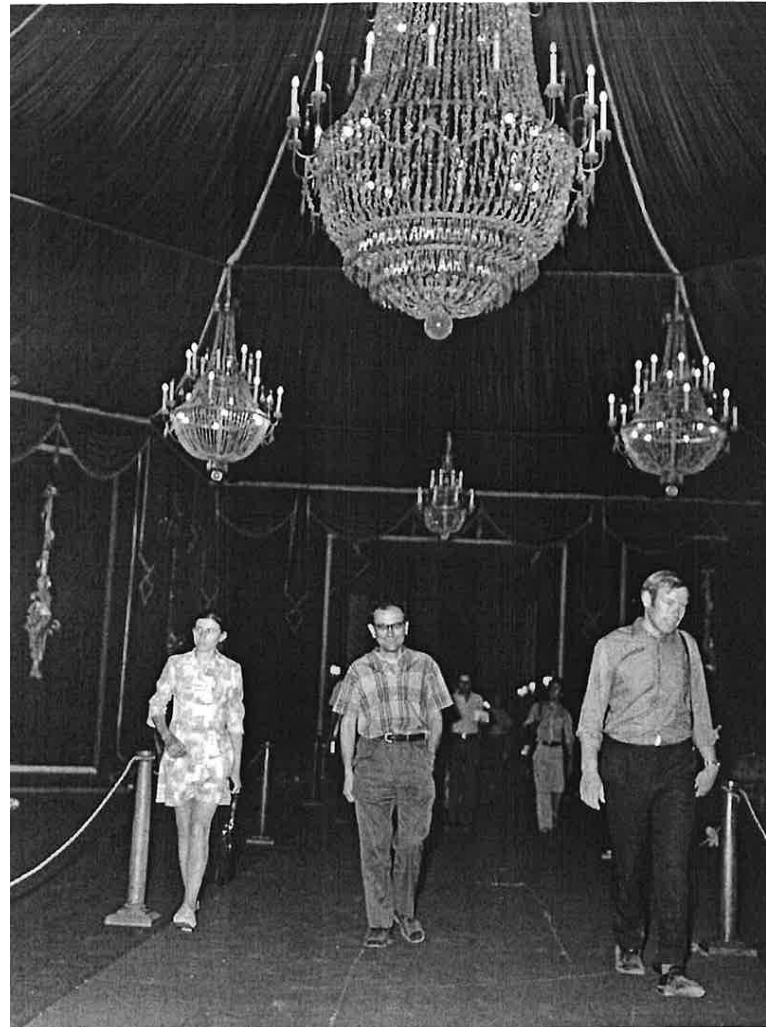
PISA · 1978

High reputation of
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Who ? Where ?

High reputation of
(field and group) theorist
and conformal field theorist..



Tehran 1972 or 1973 ?

with Valentina Petkova and Werner Rühl

Many encounters in various
places,
Varna (1987),
Sofia (1989),
Durham (1991),
Vienna (1999),
CERN (2000), Trieste (200?),

Saclay, Orsay,
Bures-sur-Yvette,



Sofia 1989



Durham 1991

A man of indefatigable curiosity . . .

A man of culture . . .

The father of a school of mathematical physics here in Bulgaria. . .

**Respect and admiration . . .
and gratitude . . .**

A short trip to cumuland



with Sylvain Lacroix

math-ph/2508.21483

I. Classical cumulants

In probability, statistics, statistical physics...

X random variable, $m_n(X) = \langle X^n \rangle$ its moments;

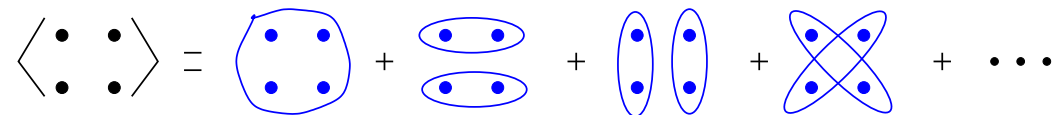
if several such r.v. X_i , mixed moments $m_n(X_{i_1}, \dots, X_{i_n})$.

Decompose moments following *partitions* of the set $\{1, \dots, n\}$

$$m_n(X_{i_1}, \dots, X_{i_n}) = \sum_{\pi \in P[n]} \prod_{B \in \pi} c_{|B|}(X_{i_a}, a \in B)$$

Thus

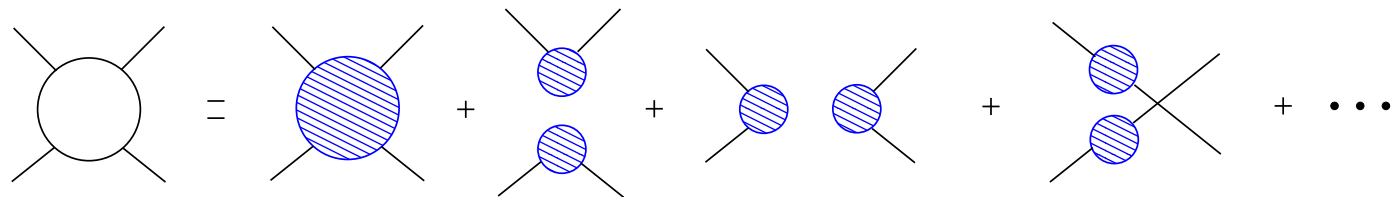
$$\begin{aligned}
 m_1(X) &= c_1(X) \\
 m_2(X) &= c_2(X) + c_1^2(X) & m_2(X, Y) &= c_2(X, Y) + c_1(X)c_1(Y) \\
 &\vdots \\
 m_4(X) &= c_4(X) + 3c_2^2(X) + \underbrace{\dots\dots\dots}_{0 \text{ if } c_1(X)=0}
 \end{aligned}$$



This defines recursively the cumulants c_n as (signed) homogeneous polynomials of the moments m_k 's.

Thus $c_2(X, Y) = \langle XY \rangle - \langle X \rangle \langle Y \rangle$ is the (co-)variance, etc.

In QFT or particle physics, write the 4-point function (or the scattering amplitude) as



“Exponential” generating functions

$$Z_X(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} m_n(X) = \langle e^{Xt} \rangle$$

$$W_X(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} c_n(X)$$

$$Z_X(t) = e^{W_X(t)}$$

Fundamental property, in connection with independence.
 X, Y independent variables: for all (polynomials) f and g ,

$$\langle f(X)g(Y) \rangle = \langle f(X) \rangle \langle g(Y) \rangle \quad (1)$$

X, Y independent $\implies c_n(X + Y) = c_n(X) + c_n(Y)$

$$e^{W_{X+Y}(t)} = \langle e^{t(X+Y)} \rangle = \langle e^{tX} \rangle \langle e^{tY} \rangle = e^{W_X(t)} e^{W_Y(t)} = e^{W_X(t) + W_Y(t)}$$

X, Y independent \iff all mixed cumulants vanish [Rota]

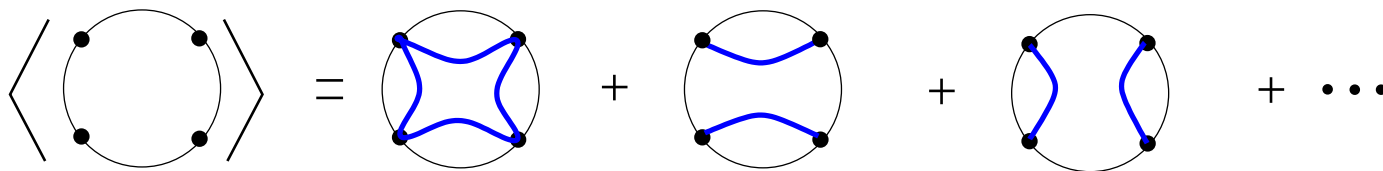
II. Free cumulants

Consider now *non commuting* random variables X_1, X_2, \dots
 again $m_n(X_1, \dots, X_n) = \langle X_1 \cdots X_n \rangle$

Change the rule: replace the set $P[n]$ of all partitions by the set $NC[n]$ of *non-crossing* partitions [Kreweras '72]

$$m_n(X_{i_1}, \dots, X_{i_n}) = \sum_{\pi \in NC[n]} \prod_{B \in \pi} \kappa_{|B|}(X_{i_a}, a \in B)$$

Thus now $m_4(X) = \kappa_4(X) + 2\kappa_2^2(X) + \dots$



This defines recursively the free cumulants κ_n as (signed) homogeneous polynomials of the moments.

Ordinary probability

$$m_n = \sum_{\pi \in P[n]} \prod_{B \in \pi} c_{|B|}$$

Exp. g.f. $Z_X(t) = e^{W_X(t)}$

X, Y independent

$$\implies c_n(X + Y) = c_n(X) + c_n(Y)$$

\iff all mixed cumulants vanish

Free probability

$$m_n = \sum_{\pi \in NC[n]} \prod_{B \in \pi} \kappa_{|B|}$$

Ordinary g.f. $M_X(t) = 1 + K_X(tM_X(t))$

X, Y free

$$\implies \kappa_n(X + Y) = \kappa_n(X) + \kappa_n(Y)$$

\iff all mixed free cumulants vanish

III. Freeness and Random Matrices

Large Random Matrices : natural setting of Free Probability

[Voiculescu '91]

Sequences of $N \times N$ matrices A_N , assume convergence $m_n(A_N) = \frac{1}{N} \text{Tr} A_N^n \rightarrow m_n(A)$ (“limiting distribution”).

In particular, take two such sequences of Hermitian matrices A_N and B_N with (at least) B_N with a $U(N)$ -invariant distribution.

Theorem Then A_N and B_N are *asymptotically free* (as $N \rightarrow \infty$).

[Voiculescu '91]

In particular, A_N, B_N deterministic (and with limiting distribution), A_N and $U_N B_N U_N^\dagger$, with U_N Haar-uniform in $U(N)$, are free.

Thus $\lim_{N \rightarrow \infty} \left[\int_{U(N)} DU \kappa_n(A_N + U_N B_N U_N^\dagger) - \kappa_n(A_N) - \kappa_n(B_N) \right] = 0$.

Thus $\lim_{N \rightarrow \infty} \left[\int_{\mathcal{U}(N)} DU \kappa_n(A_N + U_N B_N U_N^\dagger) - \kappa_n(A_N) - \kappa_n(B_N) \right] = 0$

Question Can one find *at finite* N , invariant functions $K_n(A)$ such that
$$\int_{\mathcal{U}(N)} DU K_n(A_N + U_N B_N U_N^\dagger) - K_n(A_N) - K_n(B_N) = 0 \quad (*)$$
 holds true, and $K_n(A_N) - \kappa(A_N) = O(N^{-2})$?

The answer is **yes** and the solution is surprisingly simple. For $n \leq N$ $K_n(A) =$ coefficient of $\frac{Nt^n}{n} \text{Tr} C^n$ in $Z(A, C; t) := \int_{\mathcal{U}(N)} DU e^{Nt \text{Tr} AUCU^\dagger}$

More generally, for a partition $\alpha = (\alpha_1, \dots, \alpha_\ell) = [n^{\hat{\alpha}_n} \dots 1^{\hat{\alpha}_1}]$ of n ($n \leq N$), $n = \alpha_1 + \dots + \alpha_\ell$, define

$$K_\alpha^{(N)}(A) := \frac{\prod_{k=1}^n k^{\hat{\alpha}_k} \hat{\alpha}_k!}{N^\ell} [t^n \text{Tr}(C^{\alpha_1}) \dots \text{Tr}(C^{\alpha_\ell})] Z(A, C; t).$$

Proof of (*)

$$\int DV Z(A + VBV^\dagger, C; t) = \int DUDV e^{Nt \text{Tr}(AUCU^\dagger) + BV^\dagger UCU^\dagger V} = \int DUDV e^{Nt \text{Tr}(AUCU^\dagger + BV^\dagger CV)} = Z(A, C; t) Z(B, C; t)$$

and identify the “simple trace terms” $\text{Tr} C^n$ in the two sides:

$$1 + \sum_n \frac{Nt^n}{n} \text{Tr} C^n \int DV K_n(A + VBV) + \dots = (1 + \sum_n \frac{Nt^n}{n} \text{Tr} C^n K_n(A) + \dots) (1 + \sum_n \frac{Nt^n}{n} \text{Tr} C^n K_n(B) + \dots).$$

□

The functions $K_n(A)$ and $K_\alpha(A)$ (symmetric functions of the eigenvalues) enjoy all kinds of nice properties, anticipating those of free cumulants:

$\lim_{N \rightarrow \infty} K_n = \kappa_n$ (whence the name “precursors of free cumulants”)

That the “simple trace terms”, or the “low rank” case, in the large N limit of $Z(A, C; t)$, are given by free cumulants has been known for long [Itzykson–Z 1980, Marinari–Parisi–Rittort 1994, P. Zinn-Justin 1999, 2002, Guionnet–Maida '05, Collins–Śniady '02,'07, Tanaka '08]

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For example

$$K_1 = m_1 = \kappa_1$$

$$K_2 = \frac{N^2}{N^2 - 1} (m_2 - m_1^2) = \frac{N^2}{N^2 - 1} \kappa_2$$

$$K_3 = \frac{N^4}{(N^2 - 1)(N^2 - 4)} (m_3 - 3m_2m_1 + 2m_1^3) = \frac{N^4}{(N^2 - 1)(N^2 - 4)} \kappa_3,$$

$$\begin{aligned} K_4 &= \frac{N^4}{(N^2 - 1)(N^2 - 4)(N^2 - 9)} \left((N^2 + 1)(m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4) + 5(m_2 - m_1^2)^2 \right) \\ &= \frac{N^4}{(N^2 - 1)(N^2 - 4)(N^2 - 9)} \left((N^2 + 1)\kappa_4 + 5\kappa_2^2 \right) \end{aligned}$$

etc.

The functions $K_n(A)$ and $K_\alpha(A)$ (symmetric functions of the eigenvalues) enjoy all kinds of nice properties, anticipating those of free cumulants:

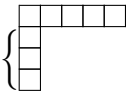
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$K_n(A_N) - \kappa(A_N) = O(N^{-2})$, a computable $\frac{1}{N^2}$ -expansion, in terms of “monotone Hurwitz numbers” [Goulden–Gay–Paquet–Novak, 2011-17]

K_n explicitly known in terms of Schur functions

$$K_n(A) = \frac{1}{N} \sum_{t=0}^{n-1} \frac{(-1)^t}{C_{\lambda_{n,t}}(N)} s_{\lambda_{n,t}}(A), \quad \underbrace{C_{\lambda_{n,t}}(N)}_{\text{“content”}} = N^{-n} \frac{(N+n-t-1)!}{(N-t-1)!}.$$

$\lambda_{n,t}$: “hook diagrams” with n boxes t 

G.f. of the K_n : $\int DU e^{N\text{Tr}AU^\dagger} \frac{1}{N} \text{Tr} \frac{1}{1-tU} = 1 + \sum_{n=1} t^n K_n(A)$

(which implies a novel form of the g.f. of free cumulants

$$\lim_{N \rightarrow \infty} \int DU e^{N\text{Tr}AU^\dagger} \frac{1}{N} \text{Tr} \frac{1}{1-tU} = 1 + \sum_{n=1} t^n \kappa_n(A))$$

Define $A^U := UAU^\dagger$. Then $K_n(A) = N^{n-1} \int DU A_{i_1 i_2}^U A_{i_2 i_3}^U \cdots A_{i_n i_1}^U$.

Known at large N : [Collins et al '06, Maillard et al '19, Bernard–Hruza '24]

and more generally, for $\sigma \in S_n$, $[\sigma]$ its cycle decomposition

$$K_{[\sigma]}(A) = N^{n-\ell([\sigma])} \int_{U(N)} DU A_{1\sigma(1)}^U A_{2\sigma(2)}^U \cdots A_{n\sigma(n)}^U.$$

“Wick property”. Take A a random Gaussian Hermitian matrix (“GUE(N)”). Then $\mathbb{E}_{A \sim \text{GUE}(N, \sigma)} [K_\alpha(A)] = \delta_{\alpha, [2n]} \sigma^{2n}$.

Approach to additivity.

Let $\delta K_n(A, B; U) := K_n(A + UBU^\dagger) - K_n(A) - K_n(B)$, where U is taken randomly and uniformly following the Haar measure of $U(N)$.

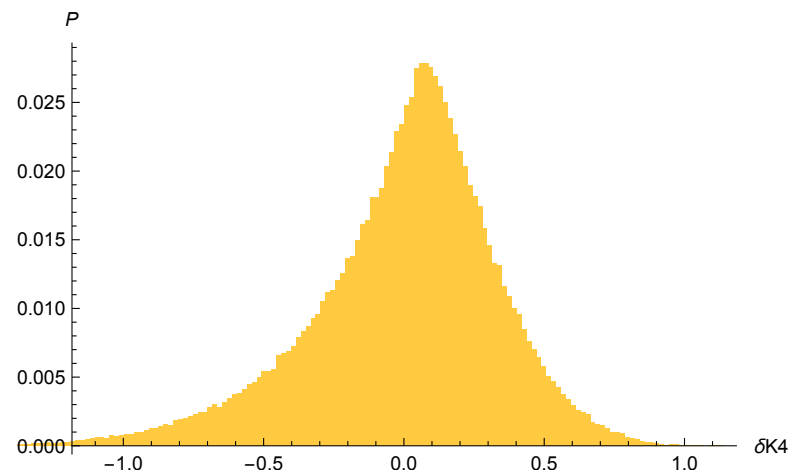
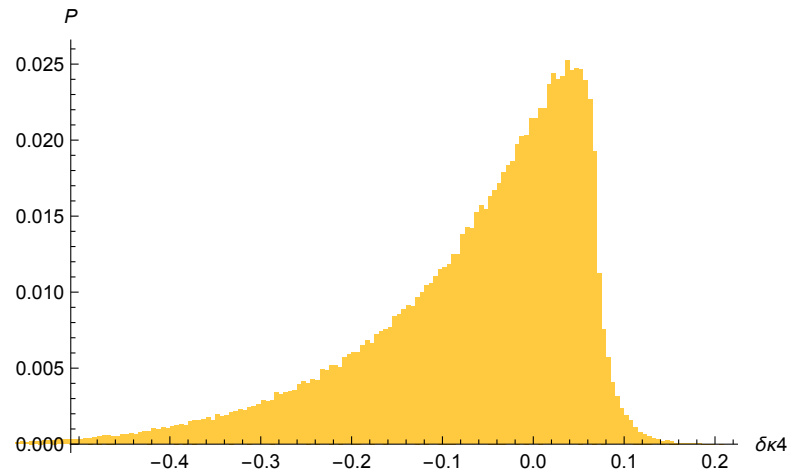
$\mathbb{E}_U(\delta K_n(A, B; U)) = 0$. Higher (classical) cumulants ?

For large N , $\text{var}(\delta K_n) = O(N^{-2})$ and the higher (classical) cumulants of δK_n are more and more suppressed: $c_{2k}(\delta K_n) = O(N^{-2k})$, $c_{2k+1}(\delta K_n) = O(N^{-2k-2})$ (Narrow Gaussian).

Application: (probabilistic) Horn problem: distribution of eigenvalues of the sum of two (Hermitian) matrices A and B of given spectrum, *i.e.*, sum of *orbits* of A and B . At finite N , non trivial distribution with a convex support.

For large N , spectrum of $A + B$ must localise on the “free convolution” of the spectra of A and B .

At finite N , $\delta K_n(A, B)$ closer to Gaussian variable than $\delta \kappa_n(A, B)$.



A numerical experiment with 4×4 matrices A and B of regularly spaced spectrum on $[-1, 1]$ and a sample of 200,000 Haar-distributed matrices U , showing the histograms of $\delta\kappa_4(U)$ (left), $\delta K_4(U)$ (right). The improvement of the latter two with respect to the former is manifest: smaller skewness, distribution closer to Gaussian, etc.

Are there other applications of these *finite N precursors* of free cumulants ?

Recently many appearances of free cumulants in various **physical** contexts

– large time behaviour of diffusion with a random diffusion coefficient

[Guéneau–Majumdar–Schehr, '25]

– “quantum exclusion processes” [D. Bernard–Hruza, '24-25]

– “ETH” (Eigenstate Thermalization Hypothesis) [Pappalardi–Foini–Kurchan, '22]

⋮

Algebraic aspects

A (new) coproduct on the algebra $\mathcal{A}^{(N)}$ of symmetric functions

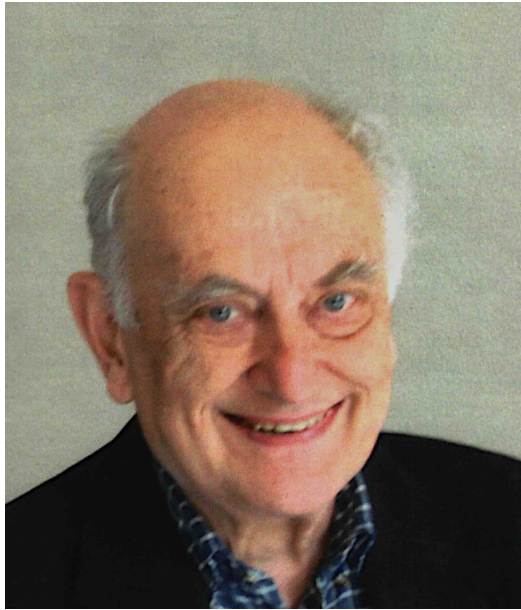
$$\mathcal{A}^{(N)} \rightarrow \mathcal{A}^{(N)} \otimes \mathcal{A}^{(N)} \quad : \quad \Delta f(A, B) = \int_{\mathbf{U}(N)} DU f(A + UBU^\dagger),$$

for which the K_n are the primitive elements

$$\Delta K_n = K_n \otimes 1 + 1 \otimes K_n$$

⋮

more in [math-ph/2508.21483](#)



Thank you !