I will begin this presentation with an overview of the research program that we initiated together with Ivan Todorov twenty-six years ago. One may refer to it is as Rational Quantum Field Theory, although it has been developed primarily within the framework of Conformal Field Theory.

Rational Quantum Field Theory

Nikolay M. Nikolov

Institute for Nuclear Research and Nuclear Energy
of the Bulgarian Academy of Sciences

Conference in memory of Ivan Todorov

Sofia, May 26, 2026

Rational QFT

It is devoted to a class of quantum field models in which the singularities of correlation functions and products of fields have a rational structure.

More specifically, the vacuum expectation values of products of fields, namely the Wightman functions, are realized as boundary values of rational functions of the following type. Their denominators are given by powers of the light quadric, that is, the Minkowski-space interval.

$$\langle \Omega | \phi_1(x_1) \cdots \phi_n(x_n)\, \Omega \rangle = \frac{\mathrm{Polynomial}\,(x_1, \ldots, x_n)}{\prod\limits_{j \neq k} \mathrm{dist}\,(x_j, x_k)^{2N}}$$

Rational QFT

$$\phi_1(x_1)\cdots\phi_n(x_n)\,\Omega = \frac{\text{Analytic}\, F(x_1,\dots x_n)}{\prod\limits_{j\neq k} \mathrm{dist}\,(x_j,x_k)^{2N}}$$

The field products acting on the vacuum possess a similar analytic structure, in which the singularities at coincident points are governed by the same rational denominators.

We often express the light quadric as the scalar square x^2 associated with the Minkowski inner product $\mathrm{x} \cdot \mathrm{y}$. As already mentioned, the denominators are integer powers of such polynomial factors.

Rational QFT

$$\langle \Omega | \phi_1(x_1) \cdots \phi_n(x_n)\, \Omega \rangle = \frac{\mathrm{Polynomial}\,(x_1, \ldots x_n)}{\prod\limits_{j \neq k} \left((x_j - x_k)^2 \right)^N}$$

Rational QFT

$$\phi_1(x_1)\cdots\phi_n(x_n)\,\Omega = \frac{\text{Analytic } F(x_1,\ldots x_n)}{\prod\limits_{j\neq k}\left((x_j-x_k)^2\right)^N}$$

$$\Longleftrightarrow \quad \left((x_j-x_k)^2\right)^N \left[\phi_j(x_j)\,,\phi_k(x_k)\right] = 0$$

$$N \gg 0$$

This occurs when the Huygens locality principle is satisfied: quantum fields commute at non-lightlike separations. Physically, this expresses the condition that any causal correlation may propagate only at the speed of light — neither faster nor slower.

The free electromagnetic field — pure light — as well as other free massless fields, namely the scalar and Dirac fields, possess this property.

Rational QFT

$$\phi_1(x_1)\cdots\phi_n(x_n)\,\Omega = \frac{\text{Analytic}\,F(x_1,\ldots x_n)}{\prod\limits_{j\neq k}\left((x_j-x_k)^2\right)^N}$$

$$\Longleftrightarrow \qquad \left((x_j-x_k)^2\right)^N\left[\phi_j(x_j)\,,\,\phi_k(x_k)\right] = 0$$

$$N \gg 0$$

From a mathematical perspective, rationality may be regarded as a starting model framework, potentially broader than the framework of free fields. In a sense, this is analogous to mathematical analysis, where one begins with rational numbers and rational functions and subsequently enlarges them — algebraically and transcendently. Algebraic extensions still retain a constructive character. Transcendental extensions, in turn, may also remain constructive, as in the case of differential-algebraic extensions leading to the so-called periods, or, period-numbers and special functions. One may also encounter completely non-constructive transcendental extensions, such as topological completions. Thus, one may expect a similar hierarchy in the construction of models in Quantum Field Theory.

Rational QFT

Let . . .

start with rational singularities

Together with Ivan Todorov, we initiated this program devoted to the search for rational quantum field models in 2001, and later developed it further together with Bojko Bakalov, Karl-Henning Rehren, and Yassen Stanev.

Rational QFT

Let . . .

start with rational singularities

One of the directions within this program was the development of a theory of vertex algebras in higher dimensions. Thanks to the rational structure, one obtains a notion of vertex algebra which, in many aspects, is parallel to the one-dimensional chiral case. There is, however, one important missing ingredient: the connection with Lie algebras — in particular, the Virasoro algebra, Kac–Moody algebras, current algebras, and so on. Remarkably, in quantum field theory, because of locality, quantum noncommutativity is encoded entirely in the singular numerical denominator. However, in higher dimensions, the structure of singularities is governed by a type of structure more complex than Lie algebras. My main goal here is to provide an intuitive picture of what such a structure might be.

Rational OPE and Vertex Algebras

$$\phi_1(x_1)\cdots\phi_n(x_n)\,\Omega = \frac{\text{Analytic } F(x_1,\dots x_n)}{\prod\limits_{j\neq k}\left((x_j-x_k)^2\right)^N}$$

$$\Longleftrightarrow \qquad \left((x_j-x_k)^2\right)^N \left[\phi_j(x_j)\,,\phi_k(x_k)\right] = 0$$

$$N \gg 0$$

We shall see that an operadic approach, based on the geometry of configuration spaces of mutually non-isotropic sets of points, is especially convenient for this purpose.

Rational OPE and Vertex Algebras

$$\phi_1(x_1)\cdots\phi_n(x_n)\,\Omega = \frac{\text{Analytic } F(x_1,\dots x_n)}{\prod\limits_{j\neq k}\left((x_j-x_k)^2\right)^N}$$

$$\Longleftrightarrow \qquad \left((x_j-x_k)^2\right)^N\left[\phi_j(x_j),\phi_k(x_k)\right]=0$$

$$N \gg 0$$

Rational OPE and Vertex Algebras

The operadic approach that I propose is based on a generalization of the field product-operations in a vertex algebra to what I shall call OPE operations.

$$\langle \Omega | \phi_1(x_1) \cdots \phi_k(x_k) \underbrace{\phi_{k+1}(x_{k+1}) \cdots \phi_{k+\ell}(x_{k+\ell})} \cdots \phi_n(x_n) \Omega \rangle$$

$$\tilde{\Gamma} \Big\downarrow \text{ an OPE operation}$$

$$\langle \Omega | \phi_1(x_1) \cdots \phi_k(x_k) \, \psi(y) \, \phi_{k+\ell+1}(x_{k+\ell+1}) \cdots \phi_n(x_n) \Omega \rangle$$

Rational OPE and Vertex Algebras

These operations extract from a correlation function involving a given collection of fields a new correlation function in which this collection is replaced by the field corresponding to the OPE operation performed on that collection.

$$\langle \Omega \,|\, \phi_1(x_1) \cdots \phi_k(x_k)\, \underbrace{\phi_{k+1}(x_{k+1}) \cdots \phi_{k+\ell}(x_{k+\ell})} \cdots \phi_n(x_n)\, \Omega \rangle$$

$$\tilde{\Gamma} \Big\downarrow \ \text{an OPE operation}$$

$$\langle \Omega \,|\, \phi_1(x_1) \cdots \phi_k(x_k)\, \psi(y)\, \phi_{k+\ell+1}(x_{k+\ell+1}) \cdots \phi_n(x_n)\, \Omega \rangle$$

Rational OPE and Vertex Algebras

This construction is performed simultaneously for all correlation functions involving the given collection of fields, and in particular for their own correlation function. In fact, all the remaining ones are naturally induced by this correspondence.

$$\langle \Omega | \underbrace{\phi_{k+1}(x_{k+1}) \cdots \phi_{k+\ell}(x_{k+\ell})} \Omega \rangle$$

$$\tilde{\Gamma} \downarrow \text{an OPE operation}$$

$$\langle \Omega | \psi(y) \Omega \rangle$$

Here I would like to emphasize an important technical point. I shall work not only with translation-invariant local quantum fields, but with general local quantum fields occurring in the theory. Usually, translation invariance simplifies the theoretical constructions. In this particular case, however, the notion of OPE operations becomes conceptually simpler when formulated for general local fields.

Rational OPE and Vertex Algebras

$$\langle \Omega | \, \phi_{k+1}(x_{k+1}) \cdots \phi_{k+\ell}(x_{k+\ell}) \, \Omega \rangle$$

$$\Gamma \; \Big\downarrow \; \text{an OPE operation}$$

$$\langle \Omega | \, \psi(y) \, \Omega \rangle$$

Rational OPE and Vertex Algebras

Thus, we would like to think of OPE operations on quantum fields as an operadic representation of abstract operadic operations that are maps between rational functions.

$$\begin{tikzcd} \phi_{k+1}(x_{k+1})\cdots\phi_{k+\ell}(x_{k+\ell}) \arrow[d, mapsto, "op_{\Gamma}"', "\text{an OPE operation}"] \\ \psi(y) \end{tikzcd}$$

Rational OPE and Vertex Algebras

Notice that rational functions with singularities at coincident arguments are mapped to polynomial, or more generally, regular functions. The latter occurs because one obtains the correlation function of a single field. It would reduce to a constant if we were working only with translation-invariant fields.

$$
\begin{array}{c}
\langle \Omega | \, \phi_{k+1}(x_{k+1}) \cdots \phi_{k+\ell}(x_{k+\ell}) \, \Omega \rangle \\
\Gamma \; \Big\downarrow \; \text{an OPE operation} \\
\langle \Omega | \, \psi(y) \, \Omega \rangle
\end{array}
$$

Rational OPE and Vertex Algebras

Thus, we would like to think of OPE operations on quantum fields as an operadic representation of abstract operadic elements that are maps between rational functions.

$$\begin{array}{c} G(x_1,\dots,x_\ell) \\ \Gamma \;\Big\downarrow\; \text{an OPE operation} \\ F(y) \end{array}$$

The linear maps inducing the OPE operations are not arbitrary. They are, what I call, semi-differential operators.

Rational OPE and Vertex Algebras

$$\begin{tikzcd} \mathbb{K}[x_1,\dots,x_\ell]\Big[\prod\limits_{j\neq k}\big((x_j-x_k)^2\big)^{-1}\Big] \arrow[d, "\text{Semi-differential operator } \Gamma"', "\Rightarrow \text{ OPE operation}"] \\ \mathbb{K}[y] \end{tikzcd}$$

I have not written down the formal definition here, since it is quite simple and I shall explain it.

If we were dealing with classical field theory, where singularities at coincident arguments are absent, this would simply reduce to an ordinary differential operator acting on G along the diagonal.

Rational OPE and Vertex Algebras

$$G(x_1,\dots,x_\ell)$$

Semi-differential operator $\Gamma \ \Big\downarrow$ $\Rightarrow$ OPE operation

$$F(y)$$

In the presence of singularities, however, we replace this condition by a weaker one: for every fixed rational function G, if we multiply it by a polynomial p (i.e., by a regular function), then the resulting expression should define a differential operator acting on p.

This is the definition of a semi-differential operator.

Rational OPE and Vertex Algebras

$$p(x_1,\dots,x_\ell)\, G(x_1,\dots,x_\ell)$$

Semi-differential operator Γ $\bigg\downarrow$ $\Rightarrow$ OPE operation

$$F(y)$$

Here is the explicit representation of the induced differential operator on the multiplier p in an integral form.

Rational OPE and Vertex Algebras

$$p(x_1,\dots,x_\ell)\, G(x_1,\dots,x_\ell)$$

$$\Big\downarrow$$

Semi-differential operator Γ $\Rightarrow$ OPE operation

$$\int d^Dx_1 \cdots d^Dx_\ell \sum_{s,r_1,\dots,r_\ell} \mathcal{E}_{\{G\};s,r_1,\dots,r_\ell}(y) \prod_k \delta^{(r_k)}(y-x_k)$$

$$\times\, p(x_1,\dots,x_\ell)$$

Rational OPE and Vertex Algebras

The corresponding integral kernel $\gamma_{\{G\}}$ depends on G and

$$p(x_1,\dots,x_\ell)\,G(x_1,\dots,x_\ell)$$

$$\big\downarrow$$

Semi-differential operator Γ $\Rightarrow$ OPE operation

$$\int d^D x_1 \cdots d^D x_\ell\ \gamma_{\{G\}}(y;x_1,\dots,x_\ell)\,p(x_1,\dots,x_\ell)$$

Rational OPE and Vertex Algebras

$$p(x_1,\dots,x_\ell)\, G(x_1,\dots,x_\ell)$$

$$\longmapsto \quad \text{Semi-differential operator } \Gamma \Rightarrow \text{OPE operation}$$

$$\int d^D x_1 \cdots d^D x_\ell \, \underbrace{\delta_{\{G\}}(y; x_1,\dots,x_\ell)}_{\text{supported at } \{y = x_1 = \cdots = x_\ell\}} \, p(x_1,\dots,x_\ell)$$

is an arbitrary distribution supported on the diagonal, with polynomial coefficients.

Thus, we arrive at an equivalent definition of a semi-differential operator: it is a linear map from rational functions to distributions supported on the diagonal,

Rational OPE and Vertex Algebras

$$G(x_1,\dots,x_\ell)$$

$$\text{Semi-differential operator } \Gamma \;\Big\downarrow\; \Rightarrow \text{ OPE operation}$$

$$\underbrace{\delta_{\{G\}}(y;x_1,\dots,x_\ell)}_{\text{supported at } \{y=x_1=\dots=x_\ell\}}$$

such that it is also a module map with respect to multiplication by polynomials.

Rational OPE and Vertex Algebras

$$\mathbb{K}[x_1,\dots,x_\ell]\Big[\prod_{j\neq k}\big((x_j-x_k)^2\big)^{-1}\Big]$$

Semi-differential operator Γ $\Big\downarrow$ $\Rightarrow$ OPE operation

$$\mathcal{D}'\{y=x_1=\cdots=x_\ell\}$$

Renormalization and Vertex Algebras

$$\mathbb{K}\left[x_1,\dots,x_\ell\right]\left[\prod_{j\neq k}\left((x_j-x_k)^2\right)^{-1}\right]$$

$$\Big\downarrow \quad \text{Semi-differential operator } \Gamma \;\Rightarrow\; \text{renormalization change}$$

$$\mathcal{D}'\left\{y=x_1=\dots=x_\ell\right\}$$

We now arrive at an interesting link with another area of Quantum Field Theory, namely the theory of renormalization. Maps of this type appear there as changes of renormalization procedure.

In this setting, the rational functions at the input are the integral kernels of Feynman amplitudes. Renormalization amounts to extending each such rational function to a distribution.

Renormalization and Vertex Algebras

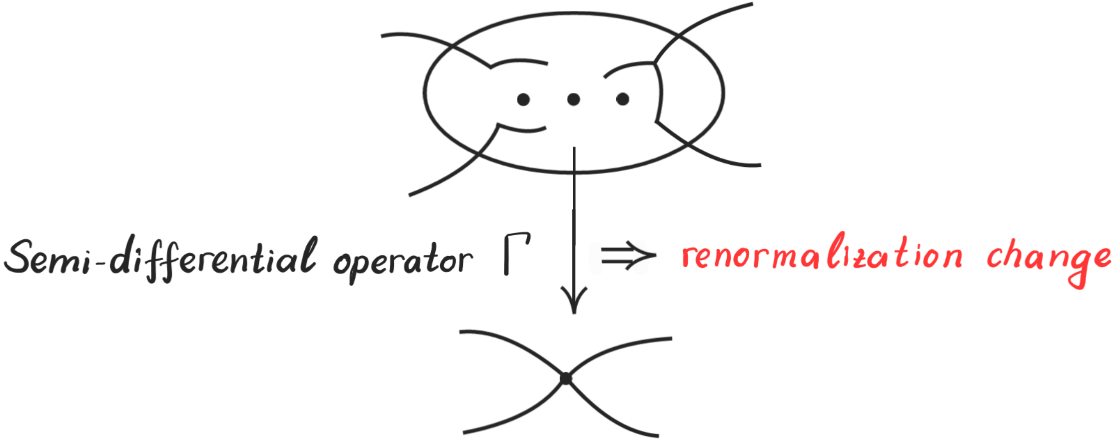
$$\mathbb{K}[x_1,\dots,x_\ell]\Big[\prod_{j\neq k}\big((x_j-x_k)^2\big)^{-1}\Big]$$

Semi-differential operator Γ $\big\downarrow$ $\Rightarrow$ renormalization change

$$\mathcal{D}'\{y=x_1=\dots=x_\ell\}$$

This procedure is recursive and contains a built-in ambiguity: the extension can be changed by adding a distribution supported on the diagonal.

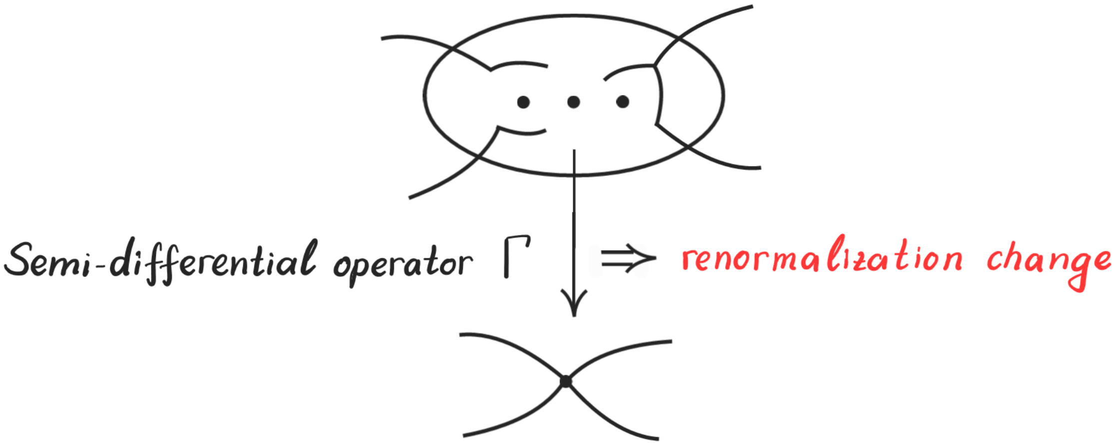
Renormalization and Vertex Algebras



Within this framework, the general renormalization procedure is formulated in terms of a system of so-called renormalization maps. The corresponding theory was developed through a series of works, culminating in a joint paper with Raymond Stora and Ivan Todorov.

It was a great honor for me to work on this project together with another outstanding theoretical physicist, Raymond Stora. Unfortunately, the difference in our generations did not allow this deeply valuable scientific collaboration to continue for longer.

Renormalization and Vertex Algebras

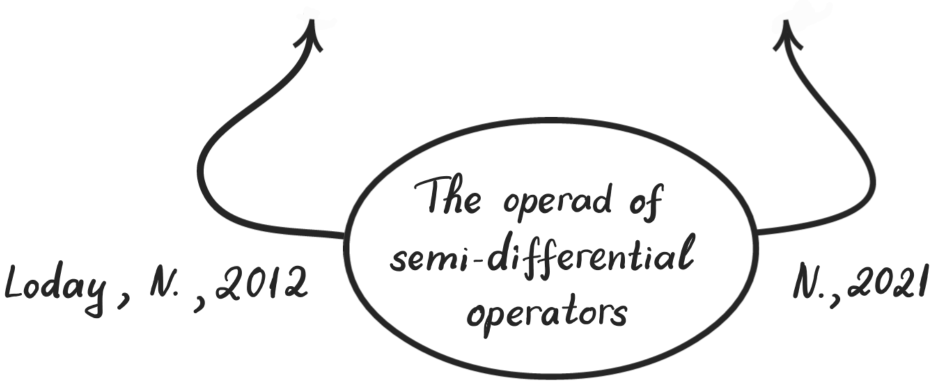


[N., 2007, 2009; N., Stora, Todorov, 2014]

Thus, the spaces of semi-differential operators naturally form an operad. This operad plays an important role both in the theory of vertex algebras and in the theory of the renormalization group.

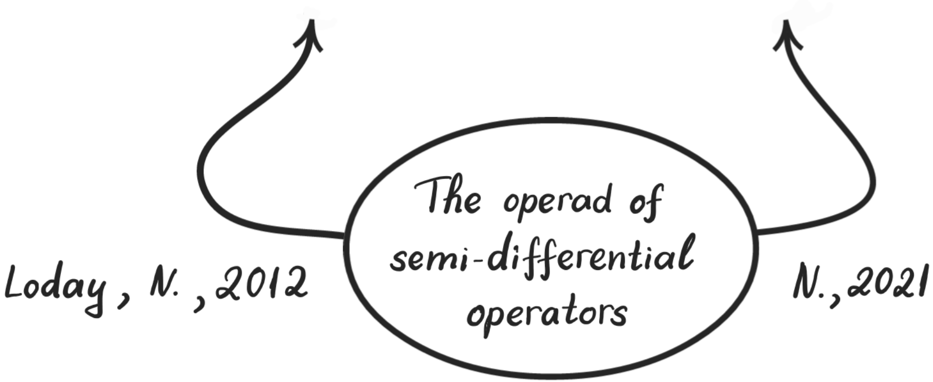
On the one hand, vertex algebras become algebras over this operad. On the other hand, the operad itself gives rise to the renormalization group.

Renorm. Group and Vertex Algebras



Concerning the latter point, let me mention my joint work with Jean-Louis Loday, in which we constructed a general functor producing groups from operads. Applied to the present operad, it yields precisely the renormalization group in the sense of Bogoliubov and Stueckelberg. Unfortunately, Loday passed away shortly thereafter, which was a tremendous loss.

Renorm. Group and Vertex Algebras



Returning to the operad that we constructed, its operadic spaces turn out to be extremely large: they are uncountably infinite-dimensional vector spaces. This is a drawback which, however, can be overcome in several ways.

Moreover, the explicit description of semi-differential operators in this generality becomes exceedingly complicated and is still unknown beyond the three-point case.

Fortunately, there is a way out here: for the theory of vertex algebras, restricting oneself to the third operadic space is entirely sufficient.

Indeed, these belong to the class of so-called quadratic algebras, which are defined by systems of binary operations, while the relations between them are encoded in the third operadic space.

Renormalization and Vertex Algebras

$$\mathbb{K}\left[x_1,\dots,x_\ell\right]\Big[\prod_{j\neq k}\big((x_j-x_k)^2\big)^{-1}\Big]$$

Semi-differential operator Γ $\Big\downarrow$ $\Rightarrow$ renormalization change

$$\mathcal{D}'\{y=x_1=\dots=x_\ell\}$$

The Role of Cohomologies of Configuration Spaces

$$F_n = \left\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \right\} \Big/ \mathbb{K}^D \text{ (translations)}$$

This brings us naturally to the concluding part of my talk, where the new contributions to this line of research are located.

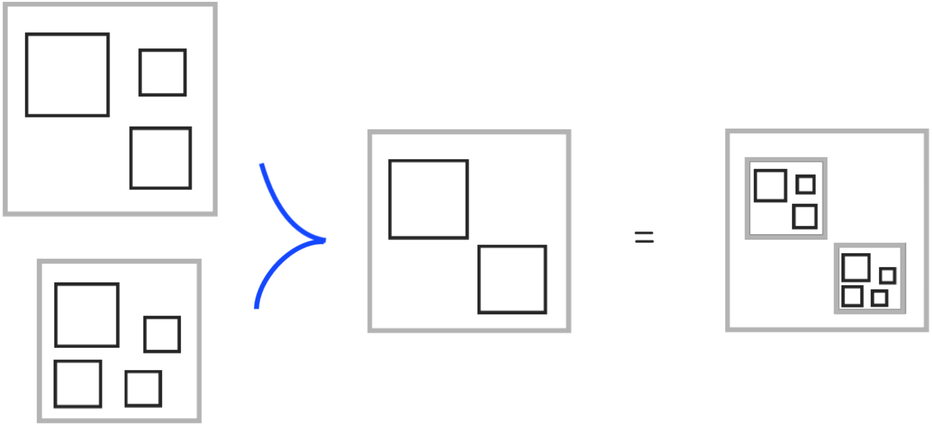
For unraveling the general structure of semi-differential operators, as well as for their applications, the problem of determining the (co)homologies of the associated configuration spaces turns out to be crucial.

The rational functions that we consider are functions on spaces of mutually non-isotropic configurations of points. These are precisely the configuration spaces mentioned earlier, additionally factorized here by translations.

It is well known that such systems of spaces provide a rich source of structures with numerous applications. One classical example is the operad of little cubes, whose construction I have sketched here.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j\neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \text{ (translations)}$$



The Role of Cohomologies of Configuration Spaces

$$F_n = \left\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \right\} \Big/ \mathbb{K}^D \text{ (translations)}$$

$$\Rightarrow \quad \big(H_\bullet (F_n) \big)_{n=1}^{\infty} \text{ is an operad}$$

Thus, the homologies of the latter set-theoretic operad again form an operad — this time an algebraic operad with finite-dimensional operadic spaces.

In an appropriate sense, this operad is homologically equivalent to the operad of semi-differential operators.

Here I shall present a new result concerning the complete determination of this operad up to level 3. The computation is carried out for an even number of dimensions, which is both the richer and the physically more interesting case.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \text{ (translations)}$$

$$\Rightarrow \big(H_\bullet(F_n) \big)_{n=1}^{\infty} \text{ is an operad}$$

Let us "truncate" it up to $n \leq 3$.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \ (\text{translations})$$

$$\Rightarrow \left(H_\bullet (F_n) \right)_{n=1}^{\infty} \ \text{is an operad. For } \textcolor{red}{\text{even } D}$$

r	0	1	2	3		D-1	D	D+1	D+2		2D-2	2D-1	2D
$H_r(F_2)$	1	1			...0...	1	1			...0...			
$H_r(F_3)$	1	3	3	1	...0...	3	8	6	1	...0...	2	5	3

First, you see the corresponding Betti numbers. An interesting feature appears in the lowest dimensions, namely 2 and 4: in the presented table, certain columns overlap. Whenever this occurs, the corresponding Betti numbers simply add up.

However, this overlap has an important consequence for the operadic structure. In dimension 2, one obtains an additional operadic identity leading to a factorization of the operad, completely analogous to the chiral splitting in two-dimensional theories.

The operad becomes a Hadamard product of two Lie operads — or, more precisely, of two Gerstenhaber operads, since a grading shift is involved.

However, such an operadic splitting is no longer valid for $D > 2$.

The second operadic space is the homology space of the second configuration space. Its elements are the binary operations of the new "H"-operad.

These are cycles in the configuration space, and they may also be used as integration cycles for correlation functions, leading to an integral realization of semi-differential operators.

For example, the top cycle corresponds to the so-called higher-dimensional residue, introduced jointly with Bojko Bakalov. It is obtained by integration over the compactified real Minkowski space embedded into complex Minkowski space.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \text{ (translations)}$$

$$H_\bullet(F_2) = \mathbb{K}\check{\varepsilon} \oplus \mathbb{K}\check{\lambda} \oplus \mathbb{K}\check{\sigma} \oplus \mathbb{K}\check{\rho}$$

$\check{\rho} \rightarrow$ residue

even D

r	0	1	2	3		$D-1$	D	$D+1$	$D+2$		$2D-2$	$2D-1$	$2D$
$H_r(F_2)$	1	1			⋯0⋯	1	1			⋯0⋯			
$H_r(F_3)$	1	3	3	1	⋯0⋯	3	8	6	1	⋯0⋯	2	5	3

In four dimensions, however, there are no operadic relations for this residue, as follows from the top homologies of the third operadic space.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \text{ (translations)}$$

$$H_\bullet(F_2) = \mathbb{K}\check{\varepsilon} \oplus \mathbb{K}\check{\lambda} \oplus \mathbb{K}\check{\sigma} \oplus \mathbb{K}\check{\rho}$$

$\check{\rho}$ → residue ⤷ no relations

even D

r	0	1	2	3		$D-1$	D	$D+1$	$D+2$		$2D-2$	$2D-1$	$2D$
$H_r(F_2)$	1	1			…0…	1	1			…0…			
$H_r(F_3)$	1	3	3	1	…0…	3	8	6	1	…0…	2	5	3

In general, one may define a "residue" of a rational function of the type considered here as a linear functional that vanishes on every partial derivative. In other words, it is a translation-invariant linear functional. It turns out that such linear functionals are precisely semidifferential operators, and they span a finite-dimensional space naturally isomorphic to the top homology space of the corresponding configuration space.

Thus, integration over cycles representing classes in the top homology spaces provides a way to describe the above-mentioned OPE operations, much as in chiral CFT.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \ (\text{translations})$$

$$H_\bullet(F_2) = \mathbb{K}\check{\varepsilon} \oplus \mathbb{K}\check{\lambda} \oplus \mathbb{K}\check{\sigma} \oplus \mathbb{K}\check{\rho}$$

even D

residue

no relations

r	0	1	2	3		$D-1$	D	$D+1$	$D+2$		$2D-2$	$2D-1$	$2D$
$H_r(F_2)$	1	1			…0…	1	1			…0…			
$H_r(F_3)$	1	3	3	1	…0…	3	8	6	1	…0…	2	5	3

Next, there is a one-dimensional cycle dual to the dlog of the Minkowski interval. For this cycle as well, no operadic relations arises.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \text{ (translations)}$$

$$H_\bullet(F_2) = \mathbb{K}\check{\varepsilon} \oplus \mathbb{K}\check{\lambda} \oplus \mathbb{K}\check{\sigma} \oplus \mathbb{K}\check{\rho}$$

no relations

even D

r	0	1	2	3		$D-1$	D	$D+1$	$D+2$		$2D-2$	$2D-1$	$2D$
$H_r(F_2)$	1	1		…0…		1	1		…0…				
$H_r(F_3)$	1	3	3	1	…0…	3	8	6	1	…0…	2	5	3

On the other hand, for the codimension-one cycle one does obtain an operadic relation — namely, a Lie operation.

This operation is, in fact, precisely the operation describing current algebras in higher dimensions.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \ (\text{translations})$$

$$H_\bullet(F_2) = \mathbb{K}\check{\varepsilon} \oplus \mathbb{K}\check{\lambda} \oplus \mathbb{K}\check{\sigma} \oplus \mathbb{K}\check{\rho}$$

even D

Lie operation

r	0	1	2	3		$D-1$	D	$D+1$	$D+2$		$2D-2$	$2D-1$	$2D$
$H_r(F_2)$	1	1			…0…	1	1			…0…			
$H_r(F_3)$	1	3	3	1	…0…	3	8	6	1	…0…	2	5	3

Now, if we take the binary operations corresponding to the basis cycles of dimension one and codimension one, respectively, we obtain the following operadic relation between them. There is an unspecified sign factor here, depending on a possible choice of grading shift. This operadic identity splits into two identities in the degenerate case D=2, and this is precisely what leads to the factorization of the homological operad in that case. For D>2, the mathematical and physical interpretation of this operadic relation remains unclear at present.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \ (\text{translations})$$

$$H_\bullet(F_2) = \mathbb{K}\check{\varepsilon} \oplus \underbrace{\mathbb{K}\check{\lambda}}_{\blacksquare} \oplus \underbrace{\mathbb{K}\check{\sigma}}_{\text{Lie operation } [\bullet,\bullet]} \oplus \mathbb{K}\check{\rho} \qquad \text{even } D$$

$$\sum_{cycl(1,2,3)} (-1)^{??} \Big([\phi_1, \phi_2 \blacksquare \phi_3] - \phi_1 \blacksquare [\phi_2, \phi_3] \Big) = 0$$

This is also the current research frontier concerning the structural theory of vertex algebras from an operadic perspective. A possible continuation of this line of investigation is to consider semi-differential operators not only on rational functions, but also on the corresponding de Rham complexes. Homologically, this would lead precisely to the operad that we have just described geometrically in terms of configuration spaces. One may then seek a construction inducing the large operad from its homological counterpart. This would be the direct analogue of the construction of vertex algebras induced from vertex Lie algebras in the chiral case D=1.

The Role of Cohomologies of Configuration Spaces

$$F_n = \Big\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \Big\} \Big/ \mathbb{K}^D \text{ (translations)}$$

$$\mathcal{H}_\bullet(F_2) = \mathbb{K}\check{\varepsilon} \oplus \underbrace{\mathbb{K}\check{\lambda}}_{\blacksquare} \oplus \underbrace{\mathbb{K}\check{\sigma}}_{\text{Lie operation } [\bullet,\bullet]} \oplus \mathbb{K}\check{\rho} \qquad \text{even } D$$

$$\sum_{cycl(1,2,3)} (-1)^{??} \Big([\phi_1, \phi_2 \blacksquare \phi_3] - \phi_1 \blacksquare [\phi_2, \phi_3] \Big) = 0$$

Finally, I would also like to briefly mention the significance of the above homological operad for renormalization theory. As was shown in my works from 2007 and 2009, these (co)homologies are directly responsible for the differential anomalies arising in ultraviolet renormalization. The latter manifest themselves through the failure of differentiation to commute with the renormalization maps and, in particular, with the Euler operator, that is, with dilations. However, this remains a work in progress.

The Role of Cohomologies of Configuration Spaces

$$F_n = \left\{ \prod_{j \neq k} (x_j - x_k)^2 \neq 0 \right\} \Big/ \mathbb{K}^D \text{ (translations)}$$

From the point of view of renormalization theory:

these (co)homologies are responsible for the "differential" anomalies in perturbative QFT

Here are my conclusions.

Rational QFT : Conclusion

- Rationality has both - physical and mathematical foundations.
  - Huygens principle ,
  - structural hierarchy .
- Rationality makes it possible to work in an algebraic and constructive manner.
- Rationality is rich in geometric and algebraic structures

However, the challenge of constructing non free models still remains