A CFT CONSTRUCTION OF QUANTUM GATES BY BRAIDING FIBONACCI ANYONS

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Ivan Todorov memorial conference
Sofia, 26-30 May 2026

QFT seminar in r. 300 of INRNE, almost half a century ago

MSc Diploma thesis 1973

First meeting with Prof. Ivan Todorov - end of June 1971

MSc thesis 1973

Lorentz-invariant expansion of the scattering amplitude for particles of any spin (Communications JINR E2-7461, with B. Aneva and S. Mavrodiev)

...

PhD thesis - Exact solutions of some quantum and classical field theories

D.T. Stoyanov, L.K. Hadjiivanov,
On the quantum stress-energy tensor in the Thirring model, Communication JINR P2-84-466 (1984)

2D physics - various aspects

I. Exactly solvable "toy models"

2D QFT

(Massless) Thirring model (1958) - solutions K. Johnson (1961), C.R. Hagen (1967), B. Klaiber (1968)
- bosonization (exponents of 2D free massless scalar field)

A.S. Wightman Cargèse Lectures in Theoretical Physics 1964, 1966
- foundations of Constructive QFT

Sugawara construction: the stress-energy tensot in terms of currents (H. Sugawara 1968)

Solvable statistical models

Ising (1D E. Ising 1925, 2D L. Onsager 1944)

Heisenberg (anti)ferromagnetic spin chain (H. Bethe 1931)

E.H. Lieb, B. Surtherland (1967); R. Baxter (1971)

2D - physics and mathematics

II. Strings (1984) M.B. Green, J.H. Schwarz ; E. Witten (1987);

2D critical phenomena - BPZ, minimal models;
WZW model, KZ equation (1984)

Math
Affine Lie ("current") algebras - V. Kac, R. Moody (1967-1968)
Virasoro algebra - I.M. Gelfand, D. Fuchs (1969), M.A. Virasoro (1970)
M. Lüscher, G. Mack (Hamburg 1976, unpublished manuscript); I. Todorov Phys. Lett. B 153 (1985) 77

V. Jones - subfactors of II_1 von Neumann factors (1983)
- knots and braids (Jones polynomial) (1985)

V.G. Drinfeld - Quantum Groups (1986)
Faddeev-Reshetikhin-Takhtajan construction (1989)
$R\, T_1\, T_2 = T_2\, T_1\, R$, YBE for R

2D physics

III. Real 2D physical phenomena

- Integer Quantum Hall Effect,
  Klaus von Klitzing (1980), Nobel prize 1985

- Fractional Quantum Hall Effect,

D.C. Tsui, H.L. Störmer, A.C. Gossard (1982)

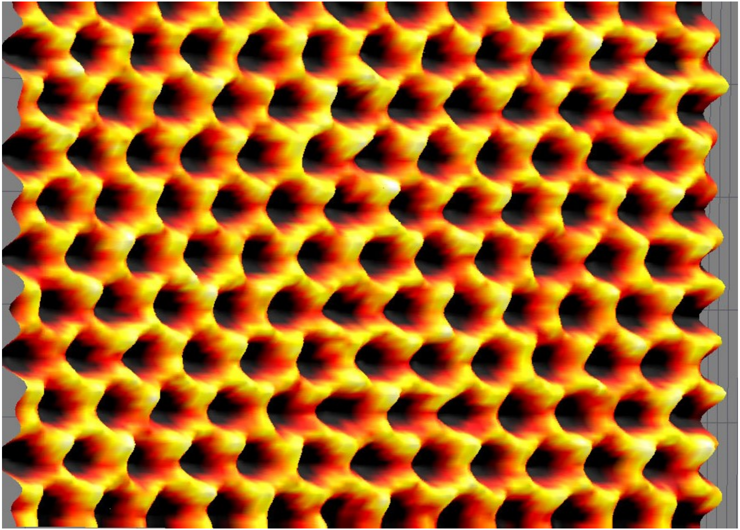
F. Wilczek (1982) - anyons,
hypothetical quasiparticles existing exclusively in 2D space,
exhibiting both fractional statistical phases (braiding)
and fractional electrical charges $\nu\, e$ (ν - filling factor)
R.B. Laughlin (1983) wave function for $\nu = 1/3$

Robert Laughlin, Horst Störmer, Daniel Tsui Nobel prize 1998, see H. Störmer's Nobel lecture, 8 Dec 1998

Topological quantum computing (A.Yu. Kitaev 1997)

- Graphene A. Geim, K. Novoselov 2004, Nobel prize 2010

Graphene under scanning tunneling microscope



The 2D CFT, QGs and Braid Group Statistics project

Roman Paunov

Yassen Stanev

Paolo Furlan (DFT, Trieste Univ. & INFN)

Alexey P. Isaev (BLTP, JINR Dubna)

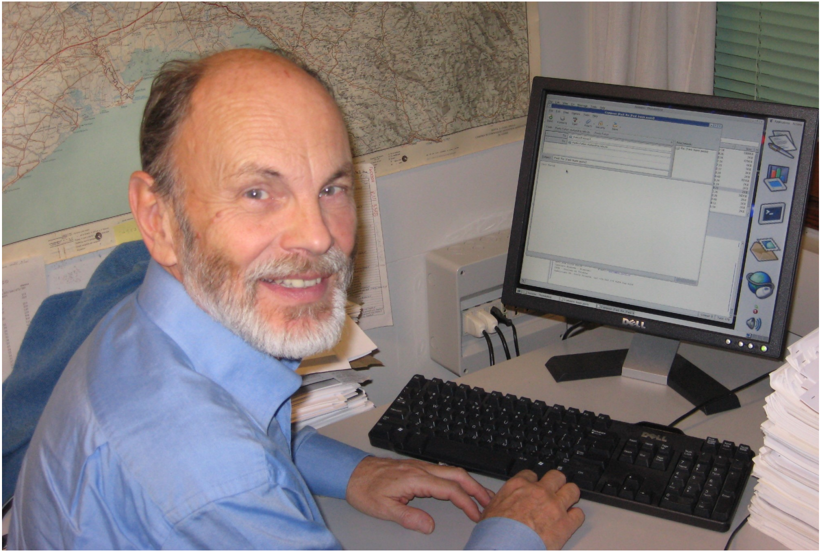
Pavel N. Pyatov (BLTP, JINR Dubna)

Oleg V. Ogievetsky (CPT Luminy, Marseille)

M. Dubois-Violette (LPT Paris XI, Orsay)

Molo Audace, Trieste 2003

Paolo Furlan



Prof. Paolo Furlan in his office at ICTP, 2007

2D CFT, quantum groups and braid statistics

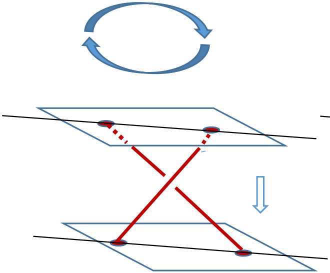
I. Todorov, L. Hadjiivanov, Quantum Groups and Braid Group Statistics in Conformal Current Algebra Models, EdUFES, Vitoria, Brazil (2010), 163 p.

L. Hadjiivanov, P. Furlan, Quantum groups as generalized gauge symmetries in WZNW models.

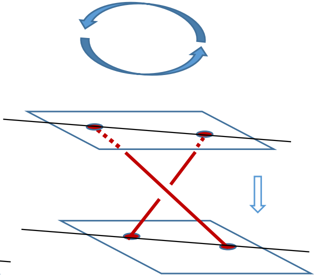
Part I. The classical model, Phys. Part. Nucl. 48:4 (2017) 509-563;

Part II. The quantized model, Phys. Part. Nucl. 48:4 (2017) 564-621.

Exchanging points as braiding strands



b



b^{-1}

Artin's braid group

The braid group B_n has been defined by Emil Artin in 1925 as the group with $n-1$ generators $b_1\,, \dots\,, b_{n-1}$ (b_i "braiding" the strands i and $i+1$) satisfying the following two sets of relations:

$$b_i\, b_j = b_j\, b_i \ , \quad |i-j| \geq 2 \ ,$$
$$b_i\, b_{i+1}\, b_i = b_{i+1}\, b_i\ b_{i+1} \ , \quad i = 1, \,, \dots, \, n-2 \ .$$

Let $\sigma\ :\ B_n \to S_n$ be the group homomorphism defined by

$$\sigma(b_i) = \sigma_i \ , \quad \sigma_i^2 = 1\ i = 1\,, \dots, n-1\,,$$

where σ_i are the transpositions exchanging points i and $i+1$ that generate S_n. The kernel of this homomorphism M_n is called the monodromy, or pure braid, group:

$$1 \to M_n \to B_n \overset{\sigma}{\to} S_n \to 1 \ .$$

Two dimensional configuration spaces

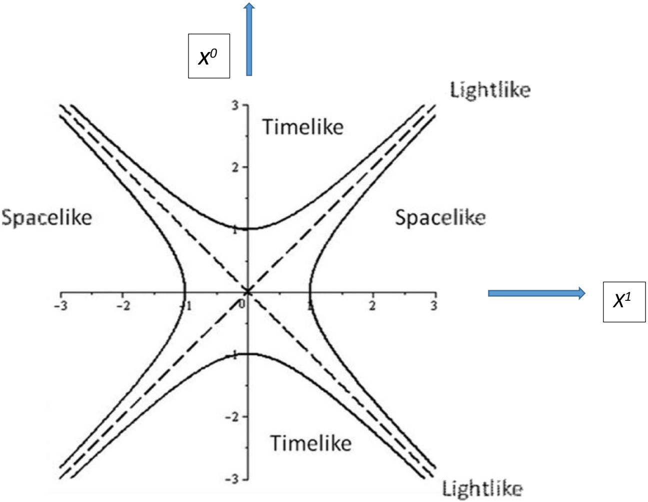
Using complex coordinates $z \in \mathbb{C}$ for the points in the plane $\mathbb{R}^2$, the n-point configuration space would be

$$Y_n = \mathbb{C}^n \backslash Diag = \{(z_1, \ldots, z_n) \,|\, i \neq j \Rightarrow z_i \neq z_j\}$$

The symmetric group S_n acts on the space Y_n by permutation of coordinates. The factor space $X_n = Y_n / S_n$ can be considered as the configuration space of n ”identical particles”. Theorem (V.I. Arnold 1968, 1970)

$$\pi_1(X_n) \simeq B_n \ , \qquad \pi_1(Y_n) \simeq M_n$$

D=1+1: light "cone"and Lorentz transformations



Exchange relations

$$\phi_\alpha(y)\,\phi_\beta(x) = \phi_\gamma(x)\,\phi_\rho(y)\,\hat{R}^{\gamma\rho}_{\alpha\beta}(x,y)\,, \;\; x \sim y$$

for $x = (x^0, x^1)\,, \;\; y = (y^0, y^1)$ **spacelike:**

$$x \sim y \quad \Leftrightarrow \quad (x^1 - y^1)^2 > (x^0 - y^0)^2$$

$\epsilon := sign\,(x^1 - y^1)$ **another Lorentz invariant for a spacelike interval in D** $= 1 + 1$

vanishing (anti)commutator reproduced for

$$\hat{R} = \pm \hat{P}\;, \quad \hat{P}^{\gamma\rho}_{\alpha\beta} = \delta^\gamma_\beta\,\delta^\rho_\alpha\;, \quad \hat{R}^2 = 1\,,$$

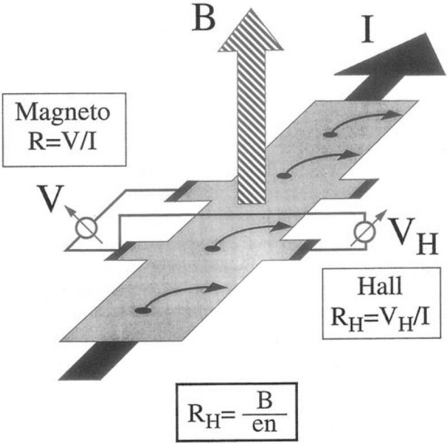
but in general, $\hat{R}\,(x,y) = \hat{R}\,(sign\,(x^1 - y^1))$

$$\hat{R}\,(-)^{-1} = \hat{R}\,(+) \equiv \hat{R} \quad (\hat{R}^\epsilon_{i\,i+1} =: b^\epsilon_i)$$

is required, as well as *(N.B.* $\epsilon_1, \epsilon_2, \epsilon_3$ *conspire!)*

$$\hat{R}_{12}(\epsilon_1)\,\hat{R}_{23}(\epsilon_2)\,\hat{R}_{12}(\epsilon_3) = \hat{R}_{23}(\epsilon_3)\,\hat{R}_{12}(\epsilon_2)\,\hat{R}_{23}(\epsilon_1)$$

Classical Hall effect - Edwin Hall 1879 (from H. Störmer's Nobel lecture 1998)



Classical Hall effect

E. Hall 1979: strip of thin gold leaf placed in a strong perpendicular magnetic field

Lorentz force $\vec{F} = e\,(\vec{E} + \vec{v} \times \vec{B})$

Equilibrium: $\vec{F} = 0$

Hall resistance: $\boxed{R_H := \frac{V_H}{I} = \frac{B}{e\,n}}$

V_H - the Hall voltage

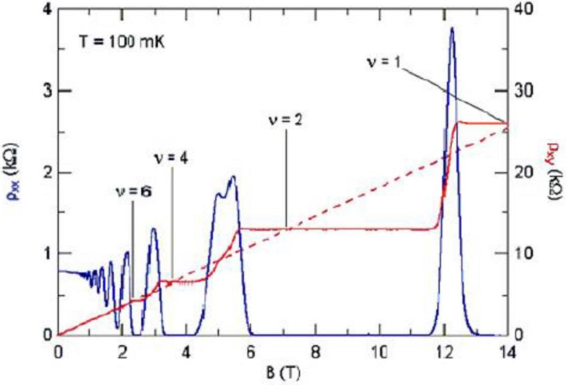
I - the (longitudinal) current

B - magnetic field strength

e - electron charge

n - electron density (per unit area!)

Integer Quantum Hall Effect (K. von Klitzing 1980)



Integral Quantum Hall Effect (K. von Klitzing 1980)

So classically, R_H is proportional to the magnetic field, $R_H \sim B$

Instead, von Klitzing et al.,

K. v. Klitzing, G. Dorda, M. Pepper, New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance, Phys. Rev. Lett. 45 (1980) 494–497

observe, in 2D MOSFET (Metal-Oxide-Semiconductor Field-Effect Transistor) structures, for $B \sim 10\, T$ and liquid helium temperatures $\sim 4\, K$), a step-wise behavior of R_H ,

$$R_H = \frac{1}{m}\frac{h}{e^2} \ , \qquad m = 1, 2, \dots$$

More surprisingly, the quantization of R_H at the plateaux is strictly equal, up to a few ppb (part per billion !!) to $(1/m)$-th of what became from 1990 the new resistance standard

$$\boxed{R_{K-90} := h/e^2 = 25812.807557(18)\ \Omega}$$

Integral Quantum Hall Effect

In the same time, the longitudal resistance $R = V/I$ drops to zero, i.e. in this direction the material becomes a superconductor.

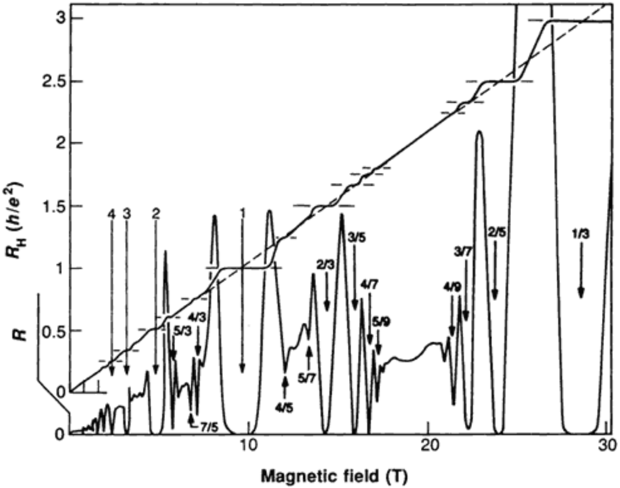
There is a simple quantum mechanical explanation (or hint) for this behavior - the motion of a particle of mass m and charge e, in magnetic field $\vec{B}$ quantized on (degenerate) cyclotron orbits (Landau levels)

$$E_n = \hbar\,\omega_c\,(n - \frac{1}{2})\ , \quad n = 1, 2, \dots\ , \quad \omega_c = \frac{e\,B}{m\,c}$$

with large gaps between them. Here 2D is crucial as motion in the third direction, along the magnetic field, effectively fills the gaps between the levels.

The full explanation including the existence of broad plateaux instead of points includes the effect of the so called "electron localization".

Fractional Quantum Hall Effect



Fractional Quantum Hall Effect

Fractional Quantum Hall Effect (1982)
H.L. Störmer, D.C. Tsui, A.C. Gossard
(unlike the integral one, unexpected)
in GaAs/AlGaAs heterostructures
at $B \sim 20\,T$ ($10^6\times$ Earth's magnetic field)
plateaux in $R_H = \frac{1}{\nu}\frac{h}{e^2}$, e.g. with $\nu = 1/3$
fractional filling factor ν instead of n integer

Phenomenological explanation: strongly correlated motion of electrons influenced by the strong magnetic field and the Coulomb repulsion between them forming an incompressible quantum fluid

Fractional Quantum Hall Effect - theory

Early theoretical explanation: R.B. Laughlin (1983), 1/3 Nobel prize 1998 proposes a wave function for states from the series $\nu = \frac{1}{m}\,,\ m$ odd (confirmed numerically by D. Haldane) - predicts an energy gap corresponding to quasiparticles with fractional charge, $\pm\nu\, e = \pm\frac{e}{m}$, forming a complex with m magnetic flux quanta

First experimental confirmations: existence
- of energy gap - 1993
- of fractional charge $\frac{1}{3}$ $(m = 3)$ - in shot noise experiments, quantum point contacts

Exotic statistics - "anyons"

Existence of anyons - recent studies?

Abelian anyons:

Anyon collider (ENS & C2N Paris), advanced Electronic Fabry-Perot interferometer (Purdue Univ., Indiana), Weizmann Institute, Stony Brook,... quasiparticles at $\nu = \frac{1}{3}\,, \nu = \frac{2}{5}$ - measuring charges, braiding phases

Non-abelian anyons:

- most probably, at $\nu = \frac{5}{2}$
- promising: the $\nu = \frac{12}{5}$ Read-Rezayi state

Classical and quantum computation

- Classical *registers* - info stored as a collection of bits
- Classical logical *gates* performing Boolean operations
- Initialization and reading (measurement)

- Qubits - quantum states in 2-dim Hilbert space
- Quantum registers - tensor product of qubit spaces
- Quantum gates - unitary operators acting on the collection of qubits
- Initialization and measurement - e.g. with Electronic Fabry-Perot interferometers

The qubit as the Bloch sphere

Density matrix ρ - mixed state, for any observable A, $\langle A \rangle_\rho = tr(\rho\, A)$; for a pure state $\rho_{|x\rangle} = P_{|x\rangle} = |x\rangle\langle x|$.

$$\rho^* = \rho \ , \quad \mathrm{tr}\, \rho = 1 \quad \Rightarrow \quad \rho = \frac{1}{2}\, (\mathbb{1} + \vec{a}\cdot\vec{\sigma}) \ , \quad \vec{a} = (a_1, a_2, a_3)$$

$$\rho \geq 0 \quad \Rightarrow \quad \det \rho = \frac{1}{4}\,(1 - \vec{a}^{\,2}) \geq 0 \ , \quad \text{i.e.} \ \ \vec{a}^{\,2} := a_1^2 + a_2^2 + a_3^2 \leq 0 \, .$$

Pure states:

$$\rho^2 = \rho \quad \Leftrightarrow \quad \vec{a}^{\,2} = 1 \quad (\textbf{the "Bloch sphere"})$$

The pair of vectors of the orthonormal basis

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

forming a bit correspond to the poles of the Bloch sphere:

$$\rho_{|0\rangle} = |0\rangle\langle 0| \ , \quad \vec{a}_{|0\rangle} = (0, 0, 1) \ , \quad \rho_{|1\rangle} = |1\rangle\langle 1| \ , \quad \vec{a}_{|1\rangle} = (0, 0, -1) \, .$$

Topological quantum computers (A. Kitaev 1997)

Topological Quantum Computers - operating with quantum gates, based on braiding of nonabelian anyons $\Rightarrow$ providing not just braiding phases but multidimensional braiding matrices

Idea (Kitaev): to address the main challenge to Quantum Computing - decoherence (instability, fragility), with topological protection

The crucial advantage comes from the stability of braiding against continuous deformations (homotopy classes) caused e.g. by temperature or EM fluctuations. Therefore Topological Quantum Computing is often characterized also as ”fault-tolerant”.

Fibonacci anyons

What is especially attractive theoretically in the $\nu = \frac{12}{5}$ RR state is that it is believed to contain quasiparticle excitations realizing the so called Fibonacci anyons.

Fibonacci anyons are known to be suitable for performing universal quantum computation. This means that their braiding gives rise to universal quantum gate sets (usually comprising a few single-qubit gates combined with a two-qubit entangling gate) that can execute any valid quantum algorithm.

From an abstract point of view, Fibonacci anyons are particles of two types, $[0]$ and $[1]$, with fusion rules

$$[0] \times [0] = [0] \,, \quad [0] \times [1] = [1] \,, \quad [1] \times [1] = [0] + [1] \,.$$

Fibonacci anyons - John Preskill

The simple form of the (non-abelian) Fibonacci anyon fusion rules allows one to find the basic *braiding* (R) and *fusion* (F) matrices by solving the corresponding pentagon and hexagon algebraic equations (consistency conditions), see e.g. John Preskill, Lecture Notes Ph219: Quantum Computation, Part III. Topological quantum computation, Caltech (2004) J. Preskill obtained

$$R = \begin{pmatrix} e^{\frac{4\pi i}{5}} & 0 \\ 0 & -e^{\frac{2\pi i}{5}} \end{pmatrix} , \quad F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

(up to complex conjugation), where

$$\tau = \frac{\sqrt{5}-1}{2} = \frac{1}{2\cos\frac{\pi}{5}} , \quad \tau^2 + \tau = 1 ,$$

is the inverse of the golden ratio.

Fibonacci anyons - E. Ardonne and K. Schoutens

$2D$ Conformal field theories (CFT) have been applied successfully to describe properties of FQH states since the 1990s.

N. Read, E. Rezayi, Beyond paired quantum Hall states: parafermions and incompressible states in the first excited Landau level, PR B59 (1999) 8084, [RR99] provided a plausible $\mathbb{Z}_k$-parafermion description of a Hall state as an incompressible fluid of k-electron clusters possessing excitations with non-abelian braid statistics

E.Ardonne, K.Schoutens, Ann.Phys. 322 (2007) 201, Wavefunctions for topological quantum registers

[AS07] proposed an Ansatz for the wavefunction of quasi-hole excitations (Fibonacci anyons) over the $k = 3$ Read–Rezayi state describing electrons at $\nu = \frac{12}{5}$

Fibonacci anyons - L.H., L.S. Georgiev

[LG24] L. Hadjiivanov, L.S. Georgiev,
Braiding Fibonacci anyons, JHEP 08 (2024) 084

On the general ground of the RR $\mathbb{Z}_3$-parafermion (chiral) CFT formulation for n quasi-holes (Fibonacci fields) ε and $N = 3r$ (presumably, a big number of) electrons ψ_1, [AS07] focus on the case $n = 4$.

In [LG24] these results are generalized to
- braiding of Fibonacci anyons for arbitrary n and r generating a "monodromy representation" of B_n
- the matrices of B_n braid generators are found explicitly (recall universal quantum computation)
- Preskill's matrices R and F are derived and the Artin braid relations verified for general n
- $\mathcal{N}$ qubit quantum registers are identified as computational subspaces in the space of $n = 2\mathcal{N} + 2$ anyon correlators

Conformal blocks of four Fibonacci anyons

Following [AS07], consider first the correlation function in the $n=4$ case:

$$\Phi^{(p)}(\{w\},\{\underline{z}\}) = \langle\, \varepsilon(w_1)\ldots\varepsilon(w_4) \prod_{j=1}^{3r} \psi_1(z_j)\,\rangle^{(p)}\,, \;\; p=0,1$$

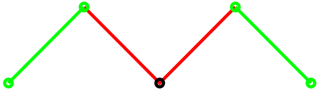
of $n=4$ Fibonacci anyons and $3\,r$ electrons.

The chiral fields ε and ψ_1 are $\mathbb{Z}_k$ parafermionic fields (A.B. Zamolodchikov, V.A. Fateev 1985) for $k=3$ with conformal dimensions

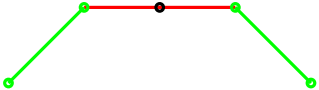
$$\Delta_{\psi_1} = \frac{2}{3}\,, \qquad \Delta_{\varepsilon} = \frac{2}{5}$$

4 anyons - Bratteli diagrams

$\Phi^{(0)}(\{w\},\{\underline{z}\}) =$



$\Phi^{(1)}(\{w\},\{\underline{z}\}) =$



Bratteli diagrams, $n = 4$

Explicit expressions for $n = 4\,, \ \ r = 0$

Following the prescriptions of [AS07], we start with the calculation for $r = 0$ (no electrons), only using a different harmonic ratio w.r.t. to their x, borrowed from KZ

$$\eta = \frac{w_{12}w_{34}}{w_{13}w_{24}} = \frac{x}{x-1}\ , \qquad x = \frac{w_{12}w_{34}}{w_{14}w_{32}}$$

to obtain (with $C = \frac{1}{2}\sqrt{\frac{\Gamma(\frac{1}{5})\,\Gamma^3(\frac{3}{5})}{\Gamma(\frac{4}{5})\,\Gamma^3(\frac{2}{5})}}$)

$$\Phi^{(0)}(\{w\}) = \frac{1}{2}\,(w_{12}w_{34})^{-\frac{4}{5}}\,(1-\eta)^{-\frac{2}{5}}\ \times$$
$$\times[\,(2-\eta)F(\frac{1}{5},\frac{4}{5},\frac{3}{5};\eta) - \frac{1}{3}\,\eta\,(1-2\eta)\,F(\frac{6}{5},\frac{4}{5},\frac{8}{5};\eta)\,]\ ,$$
$$\Phi^{(1)}(\{w\}) = \frac{C}{2}\ (w_{12}w_{34})^{-\frac{4}{5}}\,\eta^{\frac{2}{5}}(1-\eta)^{-\frac{4}{5}}\ \times$$
$$\times\,[\,(2-\eta)\,F(\frac{1}{5},\frac{4}{5},\frac{7}{5};\eta) - 2\,(1-2\eta)\,F(\frac{1}{5},-\frac{1}{5},\frac{2}{5};\eta)\,]\ .$$

Short distance behavior $\eta \to 0$

The short distance behavior of the two functions $\Phi(p)(\{w\})$, $p = 0, 1$, for $w_{12} \to 0$ or $w_{34} \to 0$ (both implying $\eta \to 0$) can be found from the hypergeometric series expansion

$$F(a, b, c; \eta) = 1 + \frac{ab}{c}\,\eta + \mathcal{O}(\eta^2)\ .$$

It gives

$$(w_{12}w_{34})^{\frac{4}{5}}\,\Phi^{(0)}(\{w\}) \underset{w_{12}\to 0}{\sim} 1\ ,$$

$$(w_{12}w_{34})^{\frac{4}{5}}\,\Phi^{(1)}(\{w\}) \underset{w_{12}\to 0}{\sim} \frac{C}{2}\,\eta^{\frac{2}{5}}\,(\frac{24}{7}\,\eta + \mathcal{O}\,(\eta^2)) =$$

$$= \frac{12}{7}\,C\,\eta^{\frac{7}{5}} + \mathcal{O}\,(\eta^{\frac{12}{5}})$$

As noted in [AS07], this indicates that the fusion of two ε fields is actually not $\varepsilon\,\varepsilon \sim \mathbf{1} + \varepsilon$ but $\varepsilon\,\varepsilon \sim \mathbf{1} + \varepsilon'$, $\Delta_{\varepsilon'} = \frac{7}{5}$.

Short distance behavior $\eta \to 0$

This follows from the fact that the fusion of $\varepsilon(w_1)$ and $\varepsilon(w_2)$ should reproduce the two- and the three-point function, respectively, which are fixed by conformal invariance:

$$\begin{aligned}
&\Phi^{(0)}(\{w\}) = \underset{w_{12}\to 0}{\sim} \; w_{12}^{-\frac{4}{5}} \, \langle \varepsilon(w_3)\, \varepsilon(w_4) \rangle = (w_{12} w_{34})^{-\frac{4}{5}} \;, \\
&\Phi^{(1)}(\{w\}) = \underset{w_{12}\to 0}{\sim} \; C_{\varepsilon'\varepsilon\varepsilon}\, w_{12}^{\frac{3}{5}} \, \langle \varepsilon'(w_2)\, \varepsilon(w_3)\, \varepsilon(w_4) \rangle = \\
&= (C_{\varepsilon'\varepsilon\varepsilon})^2\, (w_{12} w_{34})^{\frac{3}{5}}\, (w_{23} w_{24})^{-\frac{7}{5}} \; \underset{w_{12}\to 0}{\sim} \; (C_{\varepsilon'\varepsilon\varepsilon})^2\, (w_{12} w_{34})^{-\frac{4}{5}}\, \eta^{\frac{7}{5}} \;.
\end{aligned}$$

This complies with the operator product expansion (OPE)

$$\varepsilon(w_1)\, \varepsilon(w_2) \; \underset{w_{12}\to 0}{\sim} \; w_{12}^{-\frac{4}{5}}\, \mathbb{1} + C_{\varepsilon'\varepsilon\varepsilon}\, w_{12}^{\frac{3}{5}}\, \varepsilon'(w_2) \;, \qquad \Delta_{\varepsilon'} = \frac{7}{5} \;.$$

Explanation: the 3-state Potts model

See Vl.S. Dotsenko, Nucl. Phys. B 235:1 [FS11] (1984) 54 or Table 7.5 in [DiFMS97] where ε' is denoted by X, and ψ, by Z)

$\mathbb{Z}_k$ ZF parafermion model for $k = 3$ with central charge $c_k = \frac{2(k-1)}{k+2}$ coincides with the (unitary minimal) 3-state Potts model, with $c_m = 1 - \frac{6}{m(m+1)}$ for $m = 5$. Here is the complete list of non-trivial fusion relations in this sector of $\mathbb{Z}_3$ Potts Virasoro fields ($\Delta_Y = 3$):

$$\varepsilon\,\varepsilon \sim \mathbb{1} + \varepsilon' \,, \quad \varepsilon'\,\varepsilon \sim Y + \varepsilon \,, \quad \varepsilon\,Y \sim \varepsilon' \,,$$
$$\varepsilon'\,Y \sim \varepsilon \,, \quad \varepsilon'\varepsilon' \sim \mathbb{1} + \varepsilon' \,, \quad Y\,Y \sim \mathbb{1} \,.$$

In effect, in the $\mathbb{Z}_3$-parafermionic CFT realization of the Fibonacci model the $[0]$ sector is generated by $\{\mathbb{1}, Y\}$ and $[1],$, by $\{\varepsilon\,, \varepsilon'\}$. This complication does not affect braiding.

Kac table for the 3-state Potts model

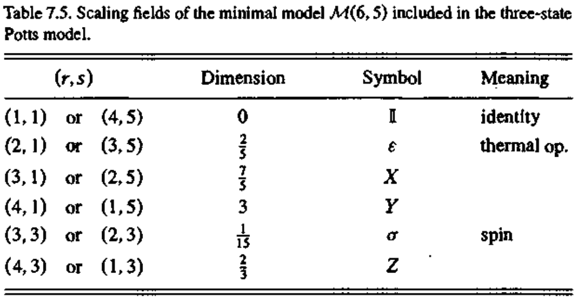


Table 7.5. Scaling fields of the minimal model $\mathcal{M}(6,5)$ included in the three-state Potts model.

(r,s)	Dimension	Symbol	Meaning
$(1,1)$ or $(4,5)$	0	$\mathbb{I}$	identity
$(2,1)$ or $(3,5)$	$\frac{2}{5}$	ε	thermal op.
$(3,1)$ or $(2,5)$	$\frac{7}{5}$	X	
$(4,1)$ or $(1,5)$	3	Y	
$(3,3)$ or $(2,3)$	$\frac{1}{15}$	σ	spin
$(4,3)$ or $(1,3)$	$\frac{2}{3}$	Z	

Braiding

The following two braidings (homotopy classes of analytic continuation) are diagonal in the Φ basis:

$$\begin{aligned}
&b_1 \equiv b_{12}\, : \;\; w_{12} \xrightarrow{\curvearrowleft} w_{21} := e^{i\pi} w_{12}\ , \\
&b_3 \equiv b_{34}\, : \;\; w_{34} \xrightarrow{\curvearrowleft} w_{43} := e^{i\pi} w_{34} \\
&(b_i - q^{-4})\, \Phi^{(0)}(\{w\}) = 0\ , \\
&(b_i - q^{3})\, \Phi^{(1)}(\{w\}) = 0\ ,\ \ i = 1, 3\ ,
\end{aligned}$$

where $q = e^{i\frac{\pi}{5}}$, **or**

$$b_i\, \Phi(\{w\}) = R\, \Phi(\{w\})\ , \qquad i = 1, 3\ ,$$

$$\Phi(\{w\}) := \begin{pmatrix} \Phi^{(0)}(\{w\}) \\ \Phi^{(1)}(\{w\}) \end{pmatrix}\ , \qquad R = \begin{pmatrix} q^{-4} & 0 \\ 0 & q^{3} \end{pmatrix}\ .$$

recovering the Preskill's braid R matrix.

Fusion matrix

The Φ basis of four Fibonacci field conformal blocks is well adapted to study the $\eta \sim 0$ behaviour (which means small w_{12} or w_{34}).

The braiding of the two middle fields is however related to small w_{23} or $\eta \sim 1$. This requires the introduction of a "dual"basis Θ such that

- The vectors $\Theta^{(p)}(\{w\})\,,\ p = 0, 1$ are linear combinations of $\Phi^{(s)}(\{w\})\,,\ s = 0, 1$.
- The short distance asymptotics of the Θ basis vectors for $w_{23} \to 0$ reproduces the relevant two- and three-point functions
- The braiding b_2 acts on $\Theta(\{w\})$ as

$$b_2\, \Theta(\{w\}) = R\, \Theta(\{w\})\,, \qquad R = \begin{pmatrix} q^{-4} & 0 \\ 0 & q^3 \end{pmatrix} .$$

The fusion matrix F

Technically, we just need to recast the expressions for $\Phi^{(s)}(\{w\})\,,\ s=0,1$ by replacing $w_{12}w_{34}$ with $w_{23}w_{14}$,

$$w_{12}w_{34} = w_{23}w_{14}\,\frac{\eta}{1-\eta}$$

and express the corresponding hypergeometric series with argument η as linear combinations of those with argument $1-\eta$. The result which can be written compactly as

$$\begin{pmatrix}\Phi^{(0)}(\{w\})\\ \Phi^{(1)}(\{w\})\end{pmatrix} = \begin{pmatrix}\tau & \sqrt{\tau}\\ \sqrt{\tau} & -\tau\end{pmatrix}\begin{pmatrix}\Theta^{(0)}(\{w\})\\ \Theta^{(1)}(\{w\})\end{pmatrix}\ , \quad \text{or}$$

$$\Phi(\{w\}) = F\,\Theta(\{w\})\ , \qquad F = \begin{pmatrix}\tau & \sqrt{\tau}\\ \sqrt{\tau} & -\tau\end{pmatrix}$$

recovering the Preskill's fusion F matrix.

The monodromy representation of B_4

The three conditions we imposed on the dual basis are easily verified.

Expressing the action of the b_2 braiding in the Φ basis (the matrix F is involutive) and adding the diagonal action derived for b_1 and b_3, we obtain

$$\pi^{(4)}(b_2) = B \ , \qquad B := F\,R\,F = \begin{pmatrix} q^4\tau & q^{-3}\sqrt{\tau} \\ q^{-3}\sqrt{\tau} & -\tau \end{pmatrix} \ .$$

$$\pi^{(4)}(b_1) = \pi^{(4)}(b_3) = R = \begin{pmatrix} q^{-4} & 0 \\ 0 & q^3 \end{pmatrix} \ ,$$

To verify Artin's relations, we only need to check the matrix equality $R\,B\,R = B\,R\,B$ $(\text{for } B = F\,R\,F)$. by using that $\tau = \frac{1}{q+q^{-1}}$ $(= q^2 + q^{-2})\,.$

Braiding of arbitrary number of anyons n

The case $n = 4$ and r arbitrary (i.e., considering the general situation of four anyons, already with electron fields) turns out to be doable, albeit with another technique, reproducing the same result for the braiding.

To treat the most general case (arbitrary number of anyons n), we need to introduce a basis in the space of conformal blocks (corresponding to all admissible Bratteli diagrams). To this end, introduce

$$\Phi^{0\,1\,\alpha_2\,\alpha_3\,\alpha_4\,\dots\,\alpha_{n-2}\,1\,0} = \Phi^{0\,1\,\alpha_2\,\alpha_3\,\alpha_4\,\dots\,\alpha_{n-2}\,1\,0}(\{w\},\{z\}) :=$$

$$\langle 0|\, \varepsilon(w_1)\Pi_1\varepsilon(w_2)\Pi_{\alpha_2}\dots\ \varepsilon(w_{n-1})\Pi_1\,\varepsilon(w_n)\prod_{i=1}^{3r}\psi_1(z_i)\,|0\rangle$$

(with Π_α , $\alpha = 0, 1$ orthogonal projectors on the corresponding sector $[\alpha]$).

The general case

For any n, the ordered set of indices α_i (0 or 1) starts with $0\,1$, ends with $1\,0$ and zeroes are not allowed to appear as neighbors (the latter follows from the fusion rules). Denote by V_{d_n} the (d_n-dimensional) vector spaces spanned on these objects (counting only the anyons, keeping the number of electrons $3r$ fixed).

As $n = 2$ and $n = 3$ correspond to the two- and three-point functions, we have

$$d_2 = 1 = d_3 \, .$$

The corresponding braid group follow immediately:

$$\pi^{(2)}(b_1) = q^{-4} \, , \qquad \pi^{(3)}(b_1) = q^3 = \pi^{(3)}(b_2) \, .$$

On the other hand, it is easy to realize that

$$d_n = d_{n-2} + d_{n-1} \, .$$

The braid group B_n

The latter construction clarifies the name "Fibonacci". As $V_{d_n} \simeq V_{d_{n-2}} \oplus V_{d_{n-1}}$, it is clear that keeping the same ordering of the basis vectors at every step, one can easily build recursively a block matrix structure of braid group representations corresponding to inclusions

$$\mathcal{B}_2 \subset \mathcal{B}_3 \subset \cdots \subset \mathcal{B}_{n-1} \subset \mathcal{B}_n \ ,$$

each of the subgroups $\mathcal{B}_i$ of $\mathcal{B}_n\,,\ 2 \le i \le n$ being generated by the first $i-1$ generators $b_j\,,\ \ j \le i-1\,.$

We end up by illustrating the natural idea of building computational spaces (quantum registers) based on the conformal blocks of Fibonacci anyons.

Block matrix structure of B_n representation

For $n \geq 5$, one derives the block matrix structure of the B_n representation in the standard basis:

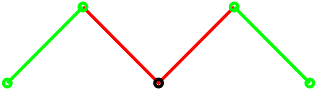
$$\pi^{(n)}(b_i) \quad = \quad \begin{pmatrix} \pi^{(n-2)}(b_i) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \pi^{(n-3)}(b_i) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \pi^{(n-2)}(b_i) \end{pmatrix} \; ,$$

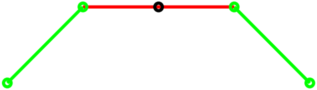
$$i = 1, \quad \ldots \; , \quad n-3 \; ,$$

$$\pi^{(n)}(b_{n-2}) \quad = \quad \begin{pmatrix} {B^0}_0 \; \mathbb{1}_{d_{n-2}} & \mathbf{0} & {B^0}_1 \; \mathbb{1}_{d_{n-2}} \\ \mathbf{0} & q^3 \; \mathbb{1}_{d_{n-3}} & \mathbf{0} \\ {B^1}_0 \; \mathbb{1}_{d_{n-2}} & \mathbf{0} & {B^1}_1 \; \mathbb{1}_{d_{n-2}} \end{pmatrix} \; ,$$

$$\pi^{(n)}(b_{n-1}) \quad = \quad \begin{pmatrix} q^{-4} \; \mathbb{1}_{d_{n-2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & q^3 \; \mathbb{1}_{d_{n-3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & q^3 \; \mathbb{1}_{d_{n-2}} \end{pmatrix} \; .$$

Four anyons, single qubit

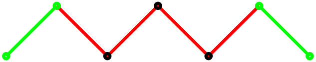
$\Phi^{01010} =$  $= |0\rangle$

$\Phi^{01110} =$  $= |1\rangle$,

$\mathcal{N} = 1\,,\ 4$ **anyons,** $\dim\ V_{d_4} = 2$

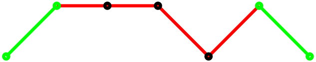
Six anyons, two qubits

$\Phi^{0101010} = \langle\varepsilon|\,(\varepsilon\;0\;\varepsilon)\,(\varepsilon\;0\;\varepsilon)\,|\varepsilon\rangle =$



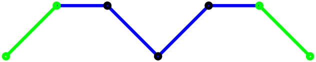
$= |00\rangle$

$\Phi^{0111010} = \langle\varepsilon|\,(\varepsilon\;1\;\varepsilon)\,(\varepsilon\;0\;\varepsilon)\,|\varepsilon\rangle =$



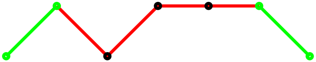
$= |10\rangle$

$\Phi^{0110110} = \langle\varepsilon|\,\varepsilon\,(\varepsilon\;0\;\varepsilon)\,\varepsilon\,|\varepsilon\rangle =$



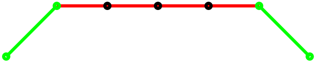
$= |NC\rangle$

$\Phi^{0101110} = \langle\varepsilon|\,(\varepsilon\;0\;\varepsilon)\,(\varepsilon\;1\;\varepsilon)\,|\varepsilon\rangle =$



$= |01\rangle$

$\Phi^{0111110} = \langle\varepsilon|\,(\varepsilon\;1\;\varepsilon)\,(\varepsilon\;1\;\varepsilon)\,|\varepsilon\rangle =$



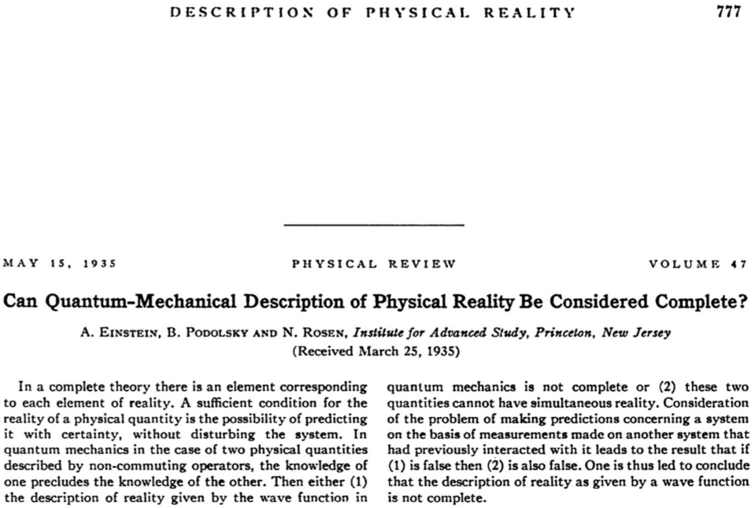
$= |11\rangle$

$\mathcal{N} = 2\,,\;\; 6$ **anyons,** dim $V_{d_6} = 5$

2-**qubit computational vectors in red, NC in blue**

THANK YOU!

EPR 1935



DESCRIPTION OF PHYSICAL REALITY 777

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

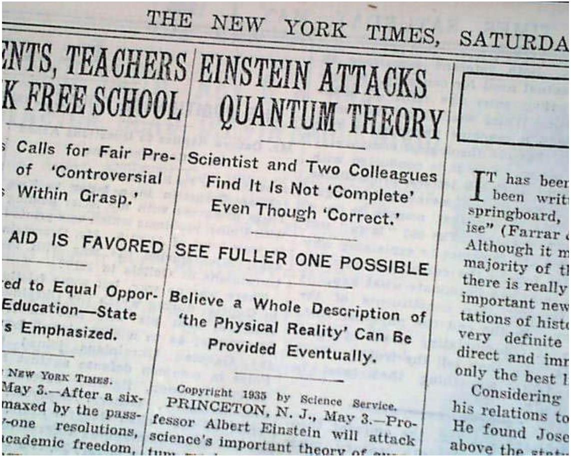
Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

EPR announced in The NY Times - Saturday, May 4, 1935



THE NEW YORK TIMES, SATURDA

NTS, TEACHERS
K FREE SCHOOL

Calls for Fair Pre-
of 'Controversial
Within Grasp.'

AID IS FAVORED

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's Emphasized.

New York Times.
May 3.—After a six-
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cademic freedom,

EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

Copyright 1935 by Science Service.

PRINCETON, N. J., May 3.—Professor Albert Einstein will attack science's important theory of

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EPR, Bell inequality violation

L. Hadjiivanov, I. Todorov, Quantum entanglement, Bulg. J. Phys. 42 (2015) 128-142, `arXiv:1506.04262[physics.hist-ph]`

After the Nobel prize 2022 awarded to John Klauser, Alain Aspect and Anton Zeilinger

” ... FOR EXPERIMENTS WITH ENTANGLED PHOTONS ESTABLISHING THE VIOLATION OF THE BELL INEQUALITIES AND PIONEERING QUANTUM INFORMATION SCIENCE ... ”

- two separate popular articles (in Bulgarian) in The World of Physics v.1 (2023)

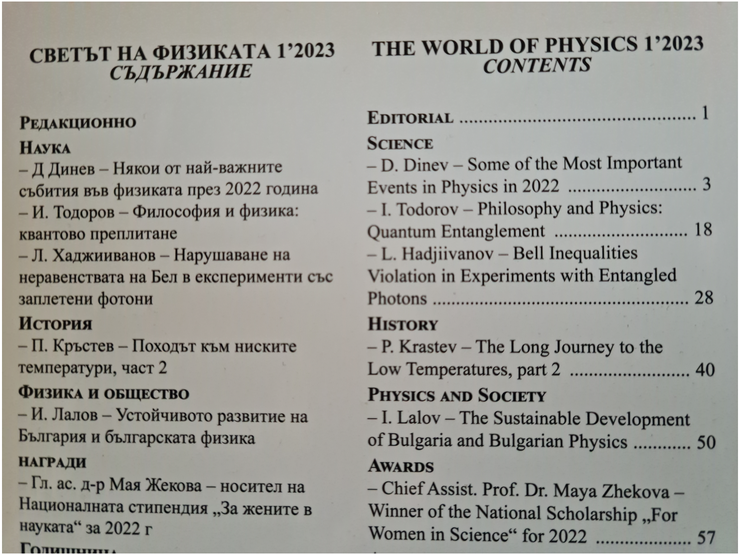
Classical vs quantum computation

During half a century scientists were convinced that any computer is a realization of the universal machine described in 1936 by Alan Turing.

In his keynote talk at the 1st conference on Physics and Computation, MIT 1981 Richard Feynman noted that the behaviour of entangled photons cannot be imitated by such a classical machine and should be used to construct a quantum computer. Feynman thinks of simulating quantum phenomena which do not admit a classical realization in order to better understand quantum theory.

Mathematicians (Yu.I. Manin, 1980) were interested in new possibilities for calculations using the “greater capacity of quantum states”.

The World of Physics (2023) v.1 (in Bulgarian)



СВЕТЪТ НА ФИЗИКАТА 1’2023
СЪДЪРЖАНИЕ

THE WORLD OF PHYSICS 1’2023
CONTENTS

Ivan Todorov - the beauty of formulae

Often used "увлекателно" (for a calculation)

- meaning captivating your attention, absorbing, gripping, ...

Loved beautiful formulae

Wightman 2pt function of free massive scalar field $\varphi(x)$ in 4D

$$\mathcal{L}(x) = -(\partial\varphi^*\partial\varphi + m^2\varphi^*\varphi) \qquad \big((g_{\mu\nu} = diag\,(1,1,1,-1)\ \big)\ ,$$

$$w(x_{12}) = \langle 0|\,\varphi(x_1)\,\varphi^*(x_2)|0\rangle = \int e^{ip\,x_{12}}(dp)_m\ , \qquad x_{12} := x_1 - x_2$$

$$(2\pi)^3(dp)_m := [\int_0^\infty \delta(p^2+m^2)\,dp^0\,]\,d^3p = \frac{d^3p}{2\omega_{\mathbf{p}}}$$

$$\boxed{w(t,\mathbf{x}) = \frac{m}{4\pi^2\sqrt{\mathbf{x}^2-(t-i0)^2}}\,K_1(m\sqrt{\mathbf{x}^2-(t-i0)^2})}$$

$K_1(z)$ **the modified Bessel function of the second kind**

T.D. Lee's two laws of physicists

Mathematics - Mathematical Physics -
Phenomenology - Experiment -
(Engineering - Applications)

T.D. Lee's (Nobel prize in Physics in 1957 with C.N. Yang) two laws of physicists:

”Without experimentalists, theorists tend to drift. Without theorists, experimentalists tend to falter.”

"drift (v)" - to move without a specific destination or control;

"falter" - to move unsteadily, to stumble

(T.D. Lee, History of the weak interactions, Talk at the "Jackfest"marking the 65th birthday of Jack Steinberger, see e.g. CERN Courier, January-February 1987.)