A Z_3-GRADED APPROACH TO QUARK COLOR DYNAMICS

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"Mathematical Quantum Field Theory" in memory of Ivan Todorov.
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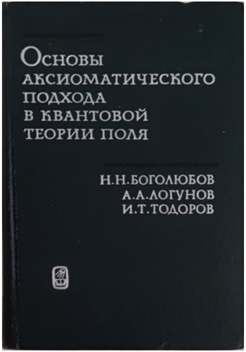
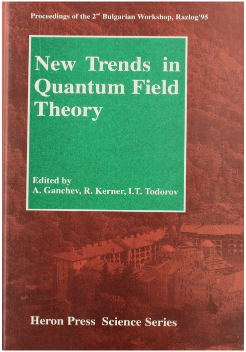
Professor Ivan T. Todorov 1933 – 2025

The undisputed father

of Bulgarian Theoretical and Mathematical Physics





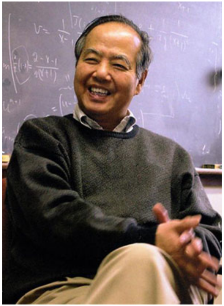
Left: **N.N. Bogolyubov, A.A. Logunov, I.T. Todorov**
Foundations of the axiomatic approach to QFT, "Nauka", 1965

Right: **A. Ganchev, R. Kerner, I. T. Todorov**
"New Trends in QFT", Heron Press, 1996

“Malkite Pet Kyosheta”, Ivan’s home,

where I had the privilege to visit Ivan and Boryana more than once.

We propose a description of colour triplets of quarks by entangled Z_3-graded Lee-Wick type fields, one with real mass and the two remaining ones with mutually conjugate complex masses. This is obtained by attributing colour degrees of freedom to six Pauli spinors, three endowed with colours and three with anti-colours, which are united into one 12-component generalized "coloured Dirac spinor".



Tsung-Dao Lee (1926 – 2024) **Gian Carlo Wick (1909 – 1992)**

T.D. Lee and G.C. Wick, *Negative metric and the unitarity of the S-matrix*, Nucl.Phys. B 9 (1969) 209.

T.D. Lee and G.C. Wick, *Finite theory of quantum electrodynamics*, Phys. Rev. D 2 (1970) 1033.

The Lee-Wick models are special higher-derivative theories (here up to the sixth order), and are claimed to lead to a perturbatively unitary S matrix

The higher-derivative kinetic Lagrangian terms are supposed to improve the ultraviolet behaviors of the Feynman diagrams and may turn nonrenormalizable theories into renormalizable ones.

The Lee-Wick models have been also considered by many authors with the hope of possible physical applications, including Quantum Electrodynamics, the Standard Model, Grand Unified Theories and Quantum Gravity .

- **The Lee-Wick model is based on the following scalar field lagrangian:**

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \left(1 + \frac{\Box^2}{M^2}\right) \partial^\mu \varphi - \frac{m^2}{2}\varphi \left(1 + \frac{\Box^2}{M^2}\right) \varphi - \frac{\lambda}{4}\varphi^4. \quad (1)$$

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- The lagrangian (1) is relativistic-invariant, and leads to the sixth-order equations of motion. The corresponding propagator - or rather, its Fourier transform -displays six simple poles in the complex p_0 plane:

$$\hat{D}(p^2, l^2, M) = \frac{M^4}{(p_0^2 - \mid \mathbf{p} \mid^2 - m^2)\left((p_0^2 - \mid \mathbf{p} \mid^2)^2 + M^4\right)} \quad (2)$$

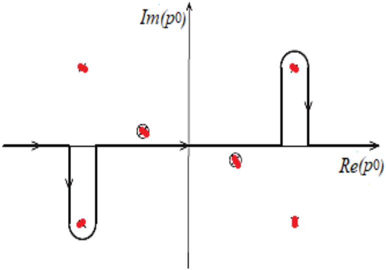
Propagators

A typical propagator of momentum **p** is equal to the standard propagator times a real function of p^2 that has no poles on the real axis. For concreteness, we take

$$D(p^2, m^2, \epsilon) = \frac{M^4}{(p^2 - m^2 + \epsilon)((p^2)^2 + M^4)} \tag{3}$$

More general propagators can be considered. In particular renormalization may lead to Green functions such as

$$D(p^2, m^2, \epsilon) = \frac{M^4}{(p^2 - m^2 + \epsilon)((p^2 - \mu^2)^2 + M^4)} \tag{4}$$



The poles of the propagator **(3)** are

$$p^0 = \pm\omega_m(\mathbf{p}) \mp i\epsilon, \quad p^0 = \pm\Omega_M(\mathbf{p}), \quad p^0 = \pm\bar{\Omega}_M(\mathbf{p})$$

where $\omega_m = \sqrt{p^2 + m^2}$ and $M(p) = \sqrt{p^2 + iM^2}$. Their locations are shown in Figure above.

In its original version the Lee-Wick model was conceived to describe only one scalar field, to be eventually extended to the electromagnetic field.

Until recently, it was unclear whether it can lead to a unitary S-matrix.

Recent papers by Anselmi and Piva claim that the unitarity can be achieved due to special integration paths in complex plane for the Fourier transform of propagators.

Our aim is to generalize the Lee-Wick model by extending it to fermions corresponding to coloured quark fields.

- **In currently widely accepted Quantum Chromo-Dynamics (QCD) the supposed fundamental elementary carriers of strong nuclear force named quarks are considered as Dirac fermions. The extra color variable and the new symmetry it represents are taken into account by introducing three Dirac spinors, $\psi^A, \quad A = 1, 2, 3$, and the free Lagrangian is invariant under the action of the fundamental representation of the $SU(3)$ group:**

$$\psi^{B'} = U^{B'}_{A} \, \psi^A.$$

- In currently widely accepted **Quantum Chromo-Dynamics** (QCD) the supposed fundamental elementary carriers of **strong nuclear force** named **quarks** are considered as **Dirac fermions**. The extra color variable and the new symmetry it represents are taken into account by introducing **three Dirac spinors**, $\psi^A, \quad A = 1, 2, 3$, and the free Lagrangian is invariant under the action of the fundamental representation of the $SU(3)$ **group**:

$$\psi^{B'} = U^{B'}_{A} \, \psi^A.$$

- The action of the Lorentz group (identically on each of the Dirac spinors forming the color triplet) commutes with the action of the $SU(3)$ group.

Explicitly, the fundamental representation of the $SU(3)$ group acts on the following triplet of Dirac spinors:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{5}$$

The Lorentz group acts simultaneously on each of the "coloured" Dirac spinors via its standard 4-D spinorial representation

The standard Dirac equation for the electron (or any spin $\frac{1}{2}$ particle with non-zero mass m) may be interpreted as a pair of coupled equations involving two Pauli spinors,

$$\psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix} \quad \text{and} \quad \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix},$$

$$E\psi_+ = mc^2\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\;\psi_-,$$

$$E\psi_- = -mc^2\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\;\psi_+,$$

where as usual

$$E = -i\hbar\,\partial_t, \quad \mathbf{p} = -i\hbar\,\mathbf{grad}$$

- **The relativistic invariance is now achieved: due to the negative mass term in the second equation, the iteration leads to the separation of variables, and all the components satisfy the desired relation**

$$[E^2 - c^2\,\mathbf{p}^2]\psi_+ = m^2 c^4\;\psi_+, \quad [E^2 - c^2\,\mathbf{p}^2]\,\psi_- = m^2 c^4\;\psi_-. \tag{6}$$

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$$[E^2 - c^2\,\mathbf{p}^2]\psi_+ = m^2 c^4\;\psi_+, \quad [E^2 - c^2\,\mathbf{p}^2]\,\psi_- = m^2 c^4\;\psi_-. \tag{6}$$

- In a more appropriate basis the Dirac equation becomes manifestly relativistic: $[\gamma^\mu p_\mu - mc]\;\psi = 0$, with $p_0 = \frac{E}{c}$,

$$\gamma^0 = \sigma_3 \otimes \mathbb{1}_2 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = (i\sigma_2)\otimes\sigma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}.$$

It can be written in a compact way as follows:

$$\gamma^\mu \, p_\mu \, \psi = mc \, \psi \quad \text{with} \quad \psi = (\psi_+, \psi_-)^T \,, \tag{7}$$

where $p_\mu = -i\hbar\partial_\mu$, $\psi_\pm$ are two complex 2-component Pauli spinors, and as Dirac matrices γ^μ one can choose

$$\gamma^0 = \sigma_3 \otimes 1\!\!1_2, \qquad \gamma^k = (i\sigma_2) \otimes \sigma^k, \tag{8}$$

where $\sigma_0 = 1\!\!1_2$, and σ^k $(k=1,2,3)$ are Pauli matrices. The Dirac matrices realize the 4-dimensional Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu} 1\!\!1_4, \quad \eta^{\mu\nu} = \mathrm{diag}(+,-,-,-). \tag{9}$$

- The structure of Dirac's gamma-matrices displays a fundamental $Z_2 \times Z_2$ symmetry. The two discrete symmetries leaving invariant the Dirac equation under the action (separate or simultaneous) of two involutions: the half-integer spin and particle-antiparticle symmetry.

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- Spin symmetry:

$$\mathbf{p} \rightarrow -\mathbf{p}, \quad \boldsymbol{\sigma} \rightarrow -\boldsymbol{\sigma}$$

Particle-antiparticle symmetry, or charge conjugation:

$$\psi_+ \rightarrow \psi_-, \quad \psi_- \rightarrow \psi_+, \quad m \rightarrow -m.$$

It is also worth mentioning is that the spin reflection is a Z_2 representation via multiplication by a COMPLEX NUMBER $e^{\pi i} = -1$, and not by a reflection in imaginary axis.

The structure of the tensor products reflects the $Z_2 \times Z_2$ symmetry: the first one corresponding to the half-integer spin of the Dirac particle and the charge conjugation, corresponding to particle-antiparticle symmetry.

The determinant of both sides of the system **(7)** produces a **4-th order** characteristic equation

$$\det(\gamma^\mu p_\mu) = \left(\frac{E^2}{c^2} - \mid \mathbf{p} \mid^2\right)^2 = \det(mc\mathbb{1}_4) = m^4 c^4, \tag{10}$$

displaying double degeneracy, being the equality of full squares, so that the effective characteristic equation is $E^2 - \mid \mathbf{p} \mid^2 c^2 = m^2 c^4$.

- **The Z_3 symmetry can be combined with the Z_2 symmetry; 3 and 2 being prime numbers, the Cartesian product of the two is isomorphic with another cyclic group,**

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- **The generalized Dirac equation involving six Pauli spinors will be invariant under the discrete group $Z_3 \times Z_2 \times Z_2 \simeq Z_6 \times Z_2$ (which is not isomorphic with Z_{12} because 6, being divisible by 2 and by 3, is not a prime number).**

The cyclic group Z_6 is represented in the complex plane by the **generator** $q = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}}$, and its powers from 1 **to** 6. In terms of the Z_3 **group** generated by j and Z_2 **group** generated by -1, we have $q = -j^2, \; q^2 = j, \; q^3 = -1, \; q^4 = j^2, \; q^5 = -j, \; q^6 = 1$, as shown in the figure **(1)** below.

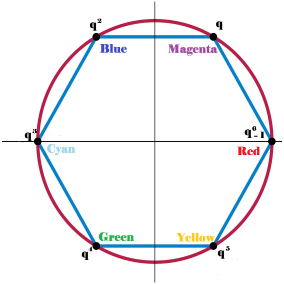

Figure: The six complex numbers q^k can be put into correspondence with three colours and three anti-colours.

The generalized Dirac equation incorporating colour degrees of freedom in a Z_3-symmetric way was proposed in our former publications; after introducing three pairs of independent Pauli spinors:

$$\varphi_+ = \begin{pmatrix} \varphi_+^1 \\ \varphi_+^2 \end{pmatrix}, \;\; \varphi_- = \begin{pmatrix} \varphi_-^1 \\ \varphi_-^2 \end{pmatrix}, \;\; \chi_+ = \begin{pmatrix} \chi_+^1 \\ \chi_+^2 \end{pmatrix},$$
$$\chi_- = \begin{pmatrix} \chi_-^1 \\ \chi_-^2 \end{pmatrix}, \;\; \psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix}, \;\; \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix}. \qquad (11)$$

with Pauli sigma-matrices acting on them in a natural way.

- The three Pauli spinors φ_+, χ_+ and ψ_+ are conventionally named "red", "blue" and "green", while their antiparticle counterparts φ_-, χ_- and ψ_- are called, respectively, "cyan", "yellow" and "magenta".

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- The cyclic group Z_3 is represented on the complex plane by multiplicative group of three complex numbers, generated by powers of $j = e^{\frac{2\pi i}{3}}$, namely:

$$j = e^{\frac{2\pi i}{3}}, \;\; j^2 = e^{\frac{4\pi i}{3}}, \;\; j^3 = 1, \;\; 1 + j + j^2 = 0. \tag{12}$$

The resulting system of equation is as follows:

$$\begin{aligned}
E\,\varphi_+ &= mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_-,\\
E\,\chi_- &= -j\;mc^2\,\chi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+,\\
E\,\psi_+ &= j^2\;mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_-,\\
E\,\varphi_- &= -mc^2\,\varphi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_+\\
E\,\chi_+ &= j\;mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,\\
E\,\psi_- &= -j^2\;mc^2\,\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,.\varphi_+
\end{aligned}\tag{13}$$

The color content is better seen in the following alternative basis:

$$
\begin{aligned}
E\,\varphi_+ &= mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_-,\\
E\,\varphi_- &= -mc^2\,\varphi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_+\\
E\,\chi_+ &= j\ mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,\\
E\,\chi_- &= -j\ mc^2\,\chi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+\\
E\,\psi_+ &= j^2\ mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_-,\\
E\,\psi_- &= -j^2\ mc^2\,\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,.\varphi_+
\end{aligned}
\tag{14}
$$

This system is invariant under the following discrete symmetries:

- \- the particle-antiparticle symmetry Z_2 **group** realized as $(1, \hat{\tau})$, where $\hat{\tau}$ denotes the involution $(\hat{\tau}^2 = 1)$ and is represented on the set **(13)** by

$$\hat{\tau}: \quad m \rightarrow -m, \quad (\varphi_\pm, \ \chi_\pm, \ \psi_\pm) \rightarrow (\varphi_\mp, \ \chi_\mp, \ \psi_\mp) \qquad (15)$$

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- \- the colour mixing symmetry Z_3 is realized by the following simultaneous maps:

$$\hat{a}: \quad m \to j\ m, \quad \varphi_\pm \to \chi_\pm \to \psi_\pm \to \varphi_\pm,$$

$$\hat{a}^2: \quad m \to j^2\ m, \quad \varphi_\pm \to \psi_\pm \to \chi_\pm \to \varphi_\pm, \tag{16}$$

- Besides, an extra Z_2 symmetry is inherited from the ordinary Dirac equation, related to the half-integer spin: $\boldsymbol{\sigma} \rightarrow -\boldsymbol{\sigma}, \;\; \mathbf{p} \rightarrow -\mathbf{p}.$

- Besides, an extra Z_2 symmetry is inherited from the ordinary **Dirac equation**, related to the half-integer spin: $\boldsymbol{\sigma} \rightarrow -\boldsymbol{\sigma}, \;\; \mathbf{p} \rightarrow -\mathbf{p}$.
- One can show that each of the 12 components of the "**colour Dirac spinor**", e.g. $\varphi^1_+, \; \varphi^2_+, \; \varphi^1_-, \; \varphi^2_-,$ etc., satisfies the following sixth-order equation: which plays the role of the **Klein-Gordon** equation for the standard **Dirac** equation:

$$\left(E^6 - c^6 \mathbf{p}^6\right) \; \Psi = m^6 c^{12} \; \Psi. \tag{17}$$

where $\Psi = (\varphi_+, \; \varphi_-, \; \chi_+, \; \chi_-, \; \psi_+, \; \psi_-)^T$

According to the correspondence principle, in Schrödinger's picture, the energy and momentum are represented by the following differential operators:

$$E \to -i\hbar \frac{\partial}{\partial t}, \quad \mathbf{p} \to -i\hbar \boldsymbol{\nabla}. \tag{18}$$

The corresponding differential equation **(17)** is an analogue of the Klein-Gordon equation (with $\hbar$ and c put equal to 1):

$$\left[\partial_t^2 - \Delta - m^2\right] \Psi_k = 0 \;\to\; \left[\partial_t^6 - \Delta^3 - m^6\right] \Psi_k = 0 \tag{19}$$

By analogy with the Klein-Gordon equation, the following colour generalization of the relativistic dispersion rule, relating **energy, momentum** and **mass**:

$$\omega = \sqrt{\mathbf{p}^2 + m^2 c^2}, \;\to\; \Omega = \sqrt[6]{\mid \mathbf{p} \mid^6 + m^6}, \tag{20}$$

The system (13) can be encoded into what we can call the "colour Dirac equation" using generalized Dirac matrices and the relativistic 4-vector notation. The standard form of the Dirac equation written by means of the first-order matrix-valued differential operator, acting on the four-component Dirac spinors:

$$\gamma^{\mu} p_{\mu} \, \psi = mc \mathbb{1}_4 \, \psi, \quad \mu, \nu, .. = 0, 1, 2, 3 \tag{21}$$

where $p_0 = E/c$ and the four Dirac matrices are defined by means of the 2×2 hermitian Pauli matrices as follows:

$$\gamma^0 = \sigma_3 \otimes \mathbb{1}_2, \quad \gamma^k = (i\sigma_2) \otimes \sigma^k.$$

The system **(13)** of twelve equations intertwining six Pauli spinors can be represented as a 12×12 **matrix valued operator**, linear in **energy, momentum** and **mass**. Instead of a tensor product the usual Dirac operator with the 3×3 unit matrix,$\mathbb{1}_3$, **the** $Z_3 \times Z_2 \times Z_2$ symmetry introducing the three colour degrees of freedom, we have here the following matrix structure:

$$E \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \, \Psi = Q_3 \otimes \sigma_1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p} \, \Psi + mc^2 \, B \otimes \sigma_3 \otimes \mathbb{1}_2 \, \Psi. \quad (22)$$

Here B and Q_3 are the following 3×3 matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad \text{and} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{23}$$

Their products and powers generate the $U(3)$ **Lie group algebra**, or the $SU(3)$ **algebra** if we remove the unit matrix; the full multiplication table, as well as the relation of this $SU(3)$ matrix basis with the currently used Gell-Mann matrices can be found in our previous papers.

To get as close as possible to the relativistic form of Dirac's equation, we should multiply all matrix operators by the inverse of the 12×12 matrix standing before the mass term mc^2. The inverse is given by the following tensor product of matrices:

$$(B \otimes \sigma_3 \otimes \mathbb{1}_2)^{-1} = B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2, \tag{24}$$

with as usual, $\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

Multiplying the system **(22)** by $B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$ from the left and dividing by c , we get the new form of our equation,

$$\frac{E}{c}\, B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes \boldsymbol{\sigma} \cdot \mathbf{p} = mc\mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \qquad (25)$$

where we used the fact that under matrix multiplication,
$\sigma_3\sigma^3 = \mathbb{1}_2$, $\quad B^\dagger B = \mathbb{1}_3$ and $B^\dagger Q_3 = Q_2$. with

$$Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}.$$

The system **(13)** can be now represented in a Dirac-like form as follows:

$$\Gamma^{\mu} p_{\mu} \Psi = mc \mathbb{1}_{12} \, \Psi, \tag{26}$$

where Ψ is the generalized 12-component spinor made of 6 Pauli spinors and the generalized 12×12 Dirac matrices Γ^{μ} are constructed as follows:

$$\Gamma^0 = B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \Gamma^i = Q_2 \otimes (i\sigma_2) \otimes \sigma^i, \tag{27}$$

Taking the determinant on both sides we get:

$$\det\left(\Gamma^{\mu} p_{\mu}\right) = (p_0^6 - \mid \mathbf{p} \mid^6)^2 = m^{12} c^{12}, \tag{28}$$

which is *the square* of the characteristic equation $E^6/c^6 - \mid \mathbf{p} \mid^6 = m^6 c^6$, or in terms of Fourier transforms, $k_0^6 - \mid \mathbf{k} \mid^6 = m^6$. The sixth-order expression on the left-hand side of **(28)** can be decomposed into a product of three second-order ones,

$$k_0^6 - \mid \mathbf{k} \mid^6 = (k_0^2 - \mid \mathbf{k} \mid^2)(k_0^2 - j \mid \mathbf{k} \mid^2)(k_0^2 - j^2 \mid \mathbf{k} \mid^2), \tag{29}$$

The first multiplier is manifestly Lorentz-invariant, while the second and third ones are invariant under complex representations of a Z_3-**covering** of the Lorentz group (see the detailed discussion in *R. Kerner and J. Lukierski, Nuclear Physics B, 2021*)

The lagrangian serving as the integrand for the variational principle in 4 dimensions leading to the generalized colour Dirac equation can be easily introduced following the same prescription that led to the usual Dirac's lagrangian:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + m\bar{\psi}\psi, \tag{30}$$

where $\bar{\psi}$ is the Dirac conjugate of ψ, $\bar{\psi} = \psi^{\dagger}\gamma^{o}$, and $\psi^{\dagger}$ is the hermitian conjugate of the Dirac spinor ψ.

In our case the Dirac conjugate of the generalized 12-component spinor should be defined as follows.

$$\bar{\Psi} = \Psi^{\dagger} B \otimes \sigma_3 \otimes \mathbb{1}_2. \tag{31}$$

Now the lagrangian cen be written in the following concise form:

$$\mathcal{L}_{color} = i \left(\bar{\psi} \Gamma^{\mu} \partial_{\mu} \Psi \right) + \bar{\Psi} \mu c \Psi. \tag{32}$$

Written explicitly, it shows how the original form of the system is recovered under the variational principle:

$$\Psi^{\dagger} B \otimes \sigma_3 \otimes \mathbb{1}_2 \left[B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes \boldsymbol{\sigma} \cdot \mathbf{p} \right] \Psi =$$

$$\Psi^{\dagger} p_0 \Psi - \Psi^{\dagger} Q_3 \otimes \sigma_1 \otimes \boldsymbol{\sigma} \cdot \mathbf{p} \Psi, \tag{33}$$

In this form, which ensures the energy positivity (p_0 is diagonal), the original system of equations (14) is clearly visible again.

Propagators

Let recall the concise form of the coloured generalization of the Dirac equation written in a manifestly 4-dimensional form with Minkowskian space-time indices $\mu, \nu = 0, 1, 2, 3$:

$$\Gamma^{\mu} p_{\mu} \, \Psi = mc \, \mathbb{1}_{12} \, \Psi, \quad \text{with} \;\; p^0 = \frac{E}{c}, \;\; p^k = [\, p^x, p^y, p^z \,]. \qquad (34)$$

with 12×12 **matrices** Γ^{μ} $(\mu = 0, 1, 2, 3)$ **defined as follows:**

$$\Gamma^0 = B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad \Gamma^k = Q_2 \otimes (i\sigma_2) \otimes \sigma^k \qquad (35)$$

To obtain a general solution of the colour Dirac equation one should use its Fourier transformed version, using the momentum representation:

$$\left(\Gamma^{\mu}\, p_{\mu} - m\mathbb{1}_{12}\right)\ \hat{\Psi}(p) = 0. \tag{36}$$

The sixth power of the matrix $\Gamma^{\mu} p_{\mu}$ is diagonal and proportional to m^6, so that we have

$$(\Gamma^{\mu} p_{\mu})^6 - m^6 \mathbb{1}_{12} = \left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right) \mathbb{1}_{12} = 0. \tag{37}$$

To find the inverse of the matrix $(\Gamma^\mu p_\mu - m\mathbb{1}_{12})$. let us note that the sixth-order expression on the left-hand side in (37) can be factorized as follows:

$$(\Gamma^\mu p_\mu)^6 - m^6 = \left((\Gamma^\mu p_\mu)^2 - m^2\right) \left((\Gamma^\mu p_\mu)^2 - j\ m^2\right) \left((\Gamma^\mu p_\mu)^2 - j^2\ m^2\right). \tag{38}$$

The first factor can be expressed as the product of two linear operators, one of which defines the colour Dirac equation (34), (36):

$$(\Gamma^\mu p_\mu)^2 - m^2 = (\Gamma^\mu p_\mu - m)\ (\Gamma^\mu p_\mu + m) \tag{39}$$

The inverse of the Fourier transform of the above linear operator **(36)** is given by the following matrix:

$$[\Gamma^\mu p_\mu - m]^{-1} = \frac{(\Gamma^\mu p_\mu + m)\ \left((\Gamma^\mu p_\mu)^2 - j\ m^2\right)\ \left((\Gamma^\mu p_\mu)^2 - j^2\ m^2\right)}{\left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right)}. \tag{40}$$

Similarly,

$$[\Gamma^\mu p_\mu - jm]^{-1} = \frac{(\Gamma^\mu p_\mu + jm)\ \left((\Gamma^\mu p_\mu)^2 - m^2\right)\ \left((\Gamma^\mu p_\mu)^2 - j\ m^2\right)}{\left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right)}. \tag{41}$$

It is important to stress that the overall power counting in **(41)** shows that the dominant behavior is like p^{-1}, just like in the case of the propagator of the classical Dirac's equation.

To find the propagators in the spacetime representation let us use the inverse Fourier transformation technique. The Fourier image of the usual Dirac field propagator can be written as

$$(\gamma^\mu p_\mu - m)^{-1} = \frac{(\gamma^\mu p_\mu + m)}{(p^\nu p_\nu - m^2)} = \frac{(\gamma^\mu p_\mu + m)}{(p_0^2 - \mid \mathbf{p} \mid^2 - m^2)}, \tag{42}$$

To obtain the Dirac propagator, we determine the spacetime propagator of the denominator, i.e. the inverse of a scalar Klein-Gordon field, e.g. the Feynman propagator $\Delta_F(x^\lambda)$, and then act on it with the differential operator $(\gamma^\mu p_\mu + m)$:

$$S_F(x) = (\gamma^\mu \partial_\mu + m)\, \Delta_F \mathbb{1}_4, \tag{43}$$

- The same strategy works in the case of the colour Dirac equation: obtain first the spacetime representation of the inverse of the sixth-order polynomial in the space of momenta, applying the inverse Fourier transformation, then act on it with the fifth-order differential operator serving as numerator in (40)

- The same strategy works in the case of the colour Dirac equation: obtain first the spacetime representation of the inverse of the sixth-order polynomial in the space of momenta, applying the inverse Fourier transformation, then act on it with the fifth-order differential operator serving as numerator in (40)
- The inverse of the sixth-order polynomial can be decomposed into a sum of three expressions with second-order denominators, multiplied by the common factor of order 4. Let us denote by Ω the sixth root of $(\mid \mathbf{p} \mid^6 + m^6)$,

$$\Omega = \sqrt[6]{\mid \mathbf{p} \mid^6 + m^6}, \tag{44}$$

along with five other root values obtained via multiplication by consecutive powers of the sixth root of unity, $q = e^{\frac{2\pi i}{6}}$.

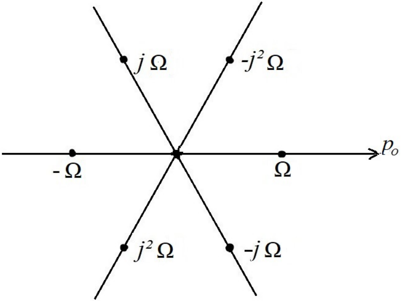


Figure: **The six poles, akin to the Lee-Wick poles, are displayed in the Figure below. The difference with the the Lee-Wick model is that they are disposed symmetrically, invariant under the action of the Z_3 group.**

- In order to introduce the propagators in the coordinate space, one has to perform the contour integrals in complex energy plane.The inverse Fourier transformation from the 4-momentum into the space-time dependent functions implies the extension of the p_0 **component** (the energy) into the complex domain.

- In order to introduce the propagators in the coordinate space, one has to perform the contour integrals in complex energy plane. The inverse Fourier transformation from the 4-momentum into the space-time dependent functions implies the extension of the p_0 component (the energy) into the complex domain.
- The first term in the decomposition of the coulour Dirac propagator presents two simple poles on the real line, while the second and the third terms display two simple poles each, located on complex straight lines $Imp_0 = jRep_0$ and $Imp_0 = j^2 Rep_0$.

In the massless case, the operator equation reduces to

$$\left[\frac{1}{c^6}\frac{\partial^6}{\partial t^6} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^3\right] G(t, \mathbf{r}) = \delta^4(x).$$

Using the Fourier transformation method, we can write:

$$\left[\frac{\omega^6}{c^6} - \mid \mathbf{p} \mid^6\right] \hat{G}(p_0, \mathbf{p}) = 1, \quad \text{where} \;\; p_0 = \frac{\omega}{c}, \tag{45}$$

from which we get

$$\hat{G}(p_0, \mathbf{p}) = \frac{1}{p_0^6 - \mid \mathbf{p} \mid^6} + \Phi(p_0, \mathbf{p}), \tag{46}$$

where $\Phi(p_0, \mathbf{p})$ is a solution of the homogeneous equation,

$$\left[p_0^6 - \mid \mathbf{p} \mid^6\right] \Phi(p_0, \mathbf{p}) = 0 \;\; \rightarrow \;\; \Phi(p_0, \mathbf{p}) = \delta(p_0^6 - \mid \mathbf{p} \mid^6). \tag{47}$$

Remembering that in the case of the Klein-Gordon equation, the homogeneous solution in the momentum space, $\delta(p_0^2 - \mid \mathbf{p} \mid^2)$, could be decomposed into a sum of two delta-functions corresponding to the factorization of the quadratic relativistic invariant into two linearly independent factors, which gives, neglecting numerical factors,

$$\delta(p_0^2 - \mid \mathbf{p} \mid^2) \simeq \delta(p_0 - \mid \mathbf{p} \mid) + \delta(p_0 + \mid \mathbf{p} \mid). \tag{48}$$

corresponding to retarded or advanced solutions.

The sixth-order polynomial $p_0^6 - \mid \mathbf{p} \mid^6$ can be split into the product of three second-order factors as follows:

$$p_0^6 - \mid \mathbf{p} \mid^6 = (p_0^2 - \mid \mathbf{p} \mid^2)\,(p_0^2 - j \mid \mathbf{p} \mid^2)\,(p_0^2 - j^2 \mid \mathbf{p} \mid^2), \tag{49}$$

each of which can be split again into two linear factors, thus giving SIX different homogeneous solutions, which can be arranged in three pairs, the first one as in (49) above, the next two pairs given by

$$\delta(p_0 - j \mid \mathbf{p} \mid) + \delta(p_0 + j \mid \mathbf{p} \mid), \text{ and } \delta(p_0 - j^2 \mid \mathbf{p} \mid) + \delta(p_0 + j^2 \mid \mathbf{p} \mid), \tag{50}$$

and we should remember another set of six delta-functions with negative p_0, corresponding to negative frequencies, all in all twelve independent solutions of the homogeneous system.

One could be puzzled by the appearance of delta-functions with complex arguments, which do not seem to be well defined. However, they correspond to complex mass shells (in the massive case), or the Z_3 complex generalizations of the light-cone. They correspond to twelve exponential solutions obtained by direct computation, with frequencies taking values $\pm\omega$, $\pm j\omega$, $\pm j^2\omega$ combined with complex wave vectors $\mathbf{k}$, $j\mathbf{k}$ and $j^2\mathbf{k}$, displayed in matrices **(66, 67)** and **(68)**.

We have now two different ways to treat the problem. Either we decompose the inverse of **(49)** into simple fractions, with six first-order poles, as follows:

$$\frac{1}{p_0^6 - \mid \mathbf{p} \mid^6} = \frac{1}{3 \mid \mathbf{p} \mid^4} \left[\frac{1}{p_0^2 - \mid \mathbf{p} \mid^2} + \frac{j}{p_0^2 - j \mid \mathbf{p} \mid^2} + \frac{j^2}{p_0^2 - j^2 \mid \mathbf{p} \mid^2} \right], \tag{51}$$

(with each of the three second-order inverses split later into a sum of two first-order inverses, displaying explicitly six simple poles at $\pm \mid \mathbf{p} \mid, \; \pm j \; \mid \mathbf{p} \mid$ and $\pm j^2 \mid \mathbf{p} \mid$. But we get a fourth-order singular factor in front of these, $\mid \mathbf{p} \mid^{-4}$, which may make the Fourier integration problematic.

Or we can keep the decomposition into product of three factors, each of them an inverse of a second-order expression, which will remove the problem of the $|\,\mathbf{p}\,|^{-4}$ formfactor in front of the sum **(51)** and the research of its inverse Fourier image.

What remains now is the integration over $d\mathbf{p}$ and dp_0. The integral over dp_0 is taken first, and evaluated by extension to the complex domain. The first factor G_1 has two poles on the real line, $p_0 = \pm\,|\,\mathbf{p}\,|$, and is evaluated as a principal value. The final result is the well known Green's function of the d'Alembertian. The remaining two factors are integrated over dp_0 even more easily, because their poles are found off the real axis, as shown in the following figure:

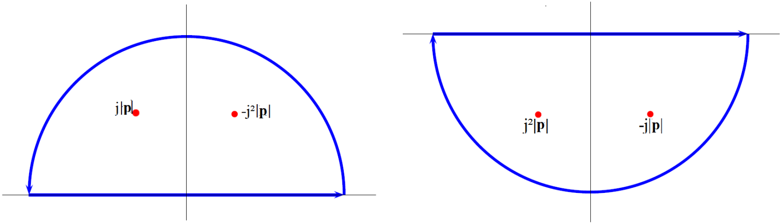
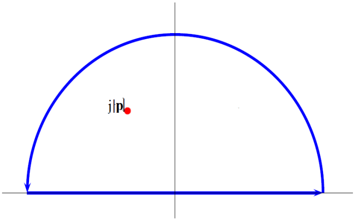


Figure: Left: The upper contour, containing the poles at $j \mid \mathbf{p} \mid$ and $-j^2 \mid \mathbf{p} \mid$, for $t < 0$; Right: The lower contour, containing the poles at $-j \mid \mathbf{p} \mid$ and $j^2 \mid \mathbf{p} \mid$, for $t > 0$.

The Green functions corresponding to the two remaining factors, with two alternative couples of poles, can be obtained by similar integration performed along the real axis, without being forced to avoid the real poles: the four remaining ones are found over or under the real axis, as shown in the figures below:



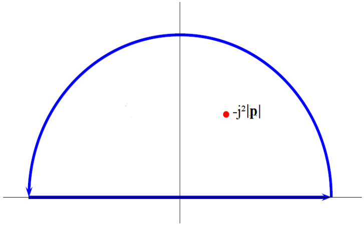


Figure: Two upper contours, containing the poles at $j \mid \mathbf{p} \mid$ and $-j^2 \mid \mathbf{p} \mid$, for $t > 0$;

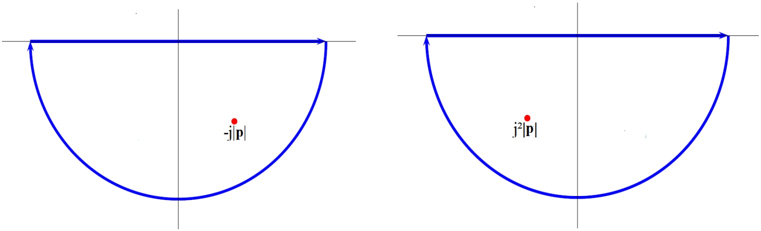


Figure: Two lower contours with poles at $-j \mid \mathbf{p} \mid$ and at $j^2 \mid \mathbf{p} \mid$, for $t < 0$.

The two poles in the upper half of the complex plane, $j \mid \mathbf{p} \mid$ and $-j^2 \mid \mathbf{p} \mid$, belong to two different terms of the decomposition into three fractions, the second and the third one, and so do the two poles $-j \mid \mathbf{p} \mid$ and $j^2 \mid \mathbf{p} \mid$ found in the lower half of the complex plane. The results of integration in the complex p_0-plane should be appropriately combined in order to get simple and elegant and real form.

Here are the two resulting combinations: the upper plane integrals performed for $t > 0$, the lower plane integrals performed for $t < 0$ are

$$2\pi Y(t)\,\frac{1}{\mid \mathbf{r} \mid}\; e^{-\frac{\sqrt{3}}{2}\omega t}\;\sin\frac{\omega t}{2} \quad \text{and} \quad -2\pi Y(-t)\,\frac{1}{\mid \mathbf{r} \mid}\; e^{\frac{\sqrt{3}}{2}\omega t}\;\sin\frac{\omega t}{2}, \tag{52}$$

which combine the results of complex integrations of two inverses, $(p_0^2 - j \mid \mathbf{p} \mid^2)^{-1}$ and $(p_0^2 - j^2 \mid \mathbf{p} \mid^2)^{-1}$.

The final results for each of the factors become, as could be expected, as follows: besides the usual retarded Green function $G_1 = Y(ct)\, \frac{\delta(kct-kr)}{r}$, two extra ones must be added,

$$G_2 = Y(ct)\, \frac{\delta(kct - j^2\, kr)}{r}, \quad G_3 = Y(ct)\, \frac{\delta(kct - j\, kr)}{r}.$$

and the Green's function of the sixth-order massless operator can be obtained by the convolution of the three functions: $G(x^\mu) = G_1 * G_2 * G_3$. The Dirac δ-**functions** vanish everywhere except for the light cone $kct - kr = 0$ in G_1, and for the complex-valued wave vectors k such that for $G_2(x^\mu)$ k must be proportional to j, and in the case of $G_3(x^\mu)$ k must be aligned along the j^2 axis.

The massless case - with decomposition in simple fractions

Performing a triple convolution with Heaviside and Dirac delta distributions depending on complex arguments poses some problems. Let us try an alternative approach instead: decomposing the Fourier image of the inverse operator in the massless case into a sum of simpler operators multiplied by a common formfactor, and try to find the original of the $\mid p \mid^4$ formfactor separately, and after that perform the convolution.

We have to find out the general form of the Green's function whose Fourier image is given by

$$\hat{G}(p) = \frac{1}{p_0^6 - \mid \mathbf{p} \mid^6} = \frac{1}{3 \mid \mathbf{p} \mid^4} \left[\frac{1}{p_0^2 - \mid \mathbf{p} \mid^2} + \frac{j}{p_0^2 - j \mid \mathbf{p} \mid^2} + \frac{j^2}{p_0^2 - j^2 \mid \mathbf{p} \mid^2} \right]$$

The presence of the factor $\mid \mathbf{p} \mid^{-4}$, strongly diverging at $\mid \mathbf{p} \mid \rightarrow 0$ (an infrared divergence), seems to pose a problem, as compared with the mild factor $\mid \mathbf{p} \mid^{-1}$ in a similar decomposition in the Klein-Gordon equation of 2-**nd** order. However, we shall show that it can be circumvented using a well known regularization technique. Before applying it to the common formfactor $\mid \mathbf{p} \mid^{-4}$, let us find the spacetime original of the sum of three fractions:

$$\left[\frac{1}{p_0^2 - \mid \mathbf{p} \mid^2} + \frac{j}{p_0^2 - j \mid \mathbf{p} \mid^2} + \frac{1}{p_0^2 - j^2 \mid \mathbf{p} \mid^2} \right] \tag{53}$$

What remains now is to find the spacetime original of the formfactor $\frac{1}{|\mathbf{p}|^4}$. Let us make an “educated guess” that the result of the inverse Fourier integral transforming $\hat{G}(p)$ into $G(r)$ is a function of r of the form

$$V(r) = \alpha \, r^{\beta}, \tag{54}$$

with α and β unknown real constants. In order to avoid divergence of the integral, let us multiply this function by a Yukawa-like potential $e^{-\lambda r}$, with some real positive parameter λ, which will be made tend to zero after integral is performed.

The volume element in thee dimensions is given by $r^2 \sin\theta dr\, d\theta d\varphi$; integration over $d\varphi$ from 0 to 2π will yield the factor 2π; the integration with respect to the polar angle θ from 0 to π can be replaced by integration over $u = \cos\theta$ because $\sin\theta\, d\theta = -d(\cos\theta)$. After this standard procedure, the integral to be taken is given by:

$$2\pi\alpha \int_0^{\infty} r^2 dr \int_{-1}^{1} du\; r^{\beta}\; e^{-ikru} e^{-\lambda r} \tag{55}$$

The integral over du gives:

$$\int_{-1}^{1} e^{-ikru}\, du = \frac{e^{-ikr} - e^{ikr}}{-ikr} = \frac{2\sin(kr)}{kr}, \tag{56}$$

What is left now is the integration over dr:

$$2\pi\alpha \int_0^\infty r^{2+\beta-1} e^{-\lambda r} \frac{2\sin(kr)}{k} dr. \tag{57}$$

A short Mathematica program gives the answer according to three possible choices of the exponent β (up to a numerical factor in front of the result):

$$\beta = -1 : \quad \frac{\alpha}{k^2+\lambda^2} \quad \text{(Coulomb potential when } \lambda \to 0, \text{)}$$

$$\beta = 0 : \quad \frac{\lambda}{(\lambda^2+k^2)^2} \quad \text{(going to zero with } \lambda \to 0, \text{)} \tag{58}$$

$$\beta = 1 : \quad \frac{2(k^2-3\lambda^2)}{(\lambda^2+k^2)^3} \quad \text{(linear potential } V \simeq r.\text{)}$$

- The original of the formfactor $\mid \mathbf{p} \mid^{-4}$ in its most general form is a superposition of two effective potentials, $V_1 \simeq r^{-1}$ **and** $V_2 \simeq r$, the form is often claimed for the effective interaction between the quarks.

- The original of the formfactor $\mid \mathbf{p} \mid^{-4}$ in its most general form is a superposition of two effective potentials, $V_1 \simeq r^{-1}$ **and** $V_2 \simeq r$, the form is often claimed for the effective interaction between the quarks.
- In order to get the propagator in the space-time representation, we should perform the convolution of one of these functions tensorised with $\delta(ct)$ (keeping in mind that the convolution concerns all four space-time variables) with *retarded* propagators obtained above, all of them containing the **Heaviside function** $Y(t)$ as common factor, because a convolution is well defined only for distributions with compact support, or with supports bounded from below.

The result of convolution with respect to the time variable of $(\alpha(r) \otimes \delta(ct)$ with any distribution depending on $[ct, x, y, z]$ will just reproduce the dependence on ct without any modification, $\delta(ct)$ acting as unit element in the convolution algebra; what remains then is the triple convolution with respect to space variables (x, y, z):

$$G(x^\mu) \simeq (r) * \left[Y(t) \left(\frac{\delta(ct - r)}{r} + [j \frac{\delta(ct - jr)}{r} + j^2 \frac{\delta(ct - j^2 r)}{r}] \right) \right] \tag{59}$$

Solutions

The 6-th order characteristic equation $E^6 - \mid \mathbf{p} \mid^6 c^6 = m^6 c^{12}$ yields the following 6-th order differential equation after applying the quantum corespondence principle: $E \rightarrow -i\hbar\partial_t, \;\; p_k \rightarrow -i\hbar\partial_k$.

$$-\hbar^6 \frac{\partial^6}{\partial t^6} = -\hbar^6 \Delta^6 c^6 + m^6 c^{12}. \tag{60}$$

Any system of linear equations with constant coefficients has solutions expressible in terms of exponential functions :

$$f(t,x) \simeq C \, e^{\omega t - \mathbf{k}\cdot\mathbf{r}}, \quad \text{with} \quad \mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z. \tag{61}$$

provided that the frequency ω and the wave vector k satisfy the dispersion relation $\omega^6 - c^6 \mid \mathbf{k} \mid^6 = \frac{m^6 c^{12}}{\hbar^6}$.

In the case of real positive mass the sixth-order characteristic equation $E^6 = \mid \mathbf{p} \mid^6 c^6 = m^6 c^{12}$ yields the following sixth-order differential equation after applying the quantum corespondence principle according to which
$E \rightarrow -i\hbar\partial_t, \;\; p_k \rightarrow -i\hbar\partial_k, \;\; k = 1, 2, 3,$

$$-\hbar^6 \frac{\partial^6}{\partial t^6} = -\hbar^6 \Delta^6 c^6 + m^6 c^{12}. \tag{62}$$

In order to ensure positive sixth-order derivatives on both sides, with the same separation of variables as in the massless case, imaginary values of ω and k must be used as solutions, with possibility of multiplying them, as previously, by -1, $\pm j$ and $\pm j^2$.

Let us chose a Cartesian frame in which the **x-axis** is aligned with the wave vector **k**, so that the scalar product $\mathbf{k} \cdot \mathbf{r}$ is reduced to the expression kx. All combinations of ω, $j\omega$ and $j^2\omega$ with k, jk and j^2k are displayed in the 3×3 matrix below:

$$\begin{pmatrix} e^{i(\omega t - kx)} & e^{i(j\omega t - kx)} & e^{i(j^2\omega t - kx)} \\ e^{i(\omega t - jkx)} & e^{i(j\omega t - jkx)} & e^{i(j^2\omega t - jkx)} \\ e^{i(\omega t - j^2 kx)} & e^{i(j\omega t - j^2 kx)} & e^{i(j^2\omega t - j^2 kx)} \end{pmatrix} \tag{63}$$

We can easily check that not only the matrix **(63)** is singular, but all its 2×2 **minors** have vanishing determinant. This means that only six out of nine solutions displayed here are linearly independent. The choice of six out of nine is arbitrary, but for the symmetry sake we shall suppress the three diagonal entries, which by themselves can form a diagonal and non-singular matrix, which we shall denote by S_{0+}:

$$\begin{pmatrix} e^{i(\omega t - kx)} & 0 & 0 \\ 0 & e^{i(j\omega t - jkx)} & 0 \\ 0 & 0 & e^{i(j^2\omega t - j^2 kx)} \end{pmatrix} \tag{64}$$

Without diagonal terms we have the following non-singular matrix of solutions:

$$\begin{pmatrix} 0 & e^{i(j\omega t-kx)} & e^{i(j^2\omega t-kx)} \\ e^{i(\omega t-jkx)} & 0 & e^{i(j^2\omega t-jkx)} \\ e^{i(\omega t-j^2kx)} & e^{i(j\omega t-j^2kx)} & 0 \end{pmatrix} \tag{65}$$

Another set of solutions is readily generated by taking complex conjugates of all the entries in (65);

$$\begin{pmatrix} 0 & e^{-i(j^2\omega t-kx)} & e^{-i(j\omega t-kx)} \\ e^{-i(\omega t-j^2kx)} & 0 & e^{-i(j\omega t-j^2kx)} \\ e^{-i(\omega t-jkx)} & e^{-i(j^2\omega t-kx)} & 0 \end{pmatrix} \tag{66}$$

The determinants of both matrices (65) and (66) are equal to 2. **Let us separate them in two matrices, each of them having determinant** 1

$$S_1 = \begin{pmatrix} 0 & e^{i(j\omega t - kx)} & 0 \\ 0 & 0 & e^{i(j^2\omega t - jkx)} \\ e^{i(\omega t - j^2 kx)} & 0 & 0 \end{pmatrix} \tag{67}$$

and the complex-conjugate one:

$$S_2 = \begin{pmatrix} 0 & 0 & e^{i(j^2\omega t - kx)} \\ e^{i(\omega t - jkx)} & 0 & 0 \\ 0 & e^{i(j\omega t - j^2 kx)} & 0 \end{pmatrix} \tag{68}$$

with $\det\,(S_1) = \det\,(S_2) = 1.$

The six independent solutions are obtained by changing the sign of ω while keeping everything else unchanged (which corresponds to introducing solutions with negative frequencies, or with the reversed arrow of time).

Any linear combination of the above set od solutions will satisfy the sixth-order equation; therefore, combining solutions with their complex conjugates, we can produce a basis of real solutions, containing sine and cosine functions multiplied by real exponential ones. Here are two 3×3 matrices with two types of independent solutions, with cosine and sine functions;

Let us denote them by M_{1+} and M_{2+}, the matrix M_{1+} being defined as

$$\begin{pmatrix} 0 & e^{-\frac{\sqrt{3}}{2}\omega t}\cos(\frac{\omega}{2}t + kx) & 0 \\ 0 & 0 & e^{\frac{\sqrt{3}}{2}(\omega t + kx)}\cos(\frac{\omega}{2}t - \frac{k}{2}x) \\ e^{-\frac{\sqrt{3}}{2}kx}\cos(\omega t + \frac{k}{2}x) & 0 & 0 \end{pmatrix} \tag{69}$$

and the matrix M_{2+} defined as

$$\begin{pmatrix} 0 & 0 & e^{\frac{\sqrt{3}}{2}\omega t}\sin(\frac{\omega}{2}t + kx) \\ e^{\frac{\sqrt{3}}{2}kx}\sin(\omega t + \frac{k}{2}x) & & \\ 0 & e^{-\frac{\sqrt{3}}{2}(\omega t + kx)}\sin(\frac{\omega}{2}t - \frac{k}{2}x)) & 0 \end{pmatrix} \tag{70}$$

The six independent solutions displayed in the two matrices M_{1+} and M_{2+} should be completed by another set of six independent solutions which are obtained by changing the sign of ω (negative frequency solutions), or, eqivalently, by changing the sign of t, which is related to particle-antiparticle symmetry of the Dirac equation. We can display again these solutions in two 3×3 matrices, and denote them by M_{1-} and M_{2-}. Obviously, these matrices are also non-singular, and contain similar exponentially damping of exponentially growing factors.

Let us show that there exist some cubic or quadratic combinations of the apparently unphysical solutions that result in finite cosine of sine functions of t and x, whose linear combinations can represent freely propagating waves.

Upon a closer glance at the matrices we notice that the three real exponents sum up to zero in each of them. And indeed, taking the determinants confirms this: let us start with det $(M_{1+}$ for example. The result contains only finite trigonometric functions:

$$\det(M_{1+}) = \cos(\frac{\omega}{2}t - kx)\ \cos(\frac{\omega}{2}t - \frac{k}{2}x)\ \cos(\omega t + \frac{k}{2}x). \tag{71}$$

Using the standard trigonometric formulas, we get the linearized expression:

$$\det(M_{1+}) = \frac{1}{4}\ [\cos(2\omega t + kx) + \cos(\omega t + 2kx) + \cos(\omega t - kx) + 1] \tag{72}$$

In addition to the three running waves we get a constant term $\frac{1}{4}$ which is obviously unphysical. But we can easily get rid of it by substracting det (M_{1-}) from the above expression: as we saw, it can be obtained from the expression (72) just by replacing ω by $-\omega$; therefore, we get, using the fact that $\cos\alpha = \cos(-\alpha)$:

$$\det(M_{1+}) - \det(M_{1-}) = \frac{1}{4}\left[\cos(2\omega t + kx) - \cos(2\omega t - kx)\right]$$

$$+\frac{1}{4}\left[\cos(\omega t + 2kx) - \cos(\omega t - 2kx) + \cos(\omega t + kx) - \cos(\omega t - kx)\right]. \tag{73}$$

The determinants of the M_{2+} and M_{2-} matrices display a similar structure, with cubic expressions of sine functions in place of the cosine ones, but with the same arguments and the same real exponentials that cancel each other in the determinant. The final result is:

$$\det(M_{2+}) - \det(M_{2-}) = \frac{1}{4}\left[\sin(\omega t - kx) + \sin(\omega t + kx)\right]$$

$$+\frac{1}{4}\left[\sin(\omega t - 2kx) + \sin(\omega t + 2kx) - \sin(2\omega t - kx) - \sin(2\omega t + kx)\right]. \tag{74}$$

The same expressions appear in the third powers of matrices, which turn out to be proportional to the 3×3 unit matrix. multiplied by the corresponding determinant. We have therefore:

$$(M_{1+})^3 = \det(M_{1+})\mathbb{1}_3, \quad (M_{2+})^3 = \det(M_{2+})\mathbb{1}_3,$$
$$(M_{1-})^3 = \det(M_{1-})\mathbb{1}_3, \quad (M_{2-})^3 = \det(M_{2-})\mathbb{1}_3, \tag{75}$$

which is not surprising given the form of these off-diagonal traceless matrices. It can be easily checked that the relation **(75)** between the determinant and the third power of any matrix of this kind hold automatically.

The mass shell condition

$$E^6 - c^6 \mid \mathbf{p} \mid^6 = m^6 c^{12} \qquad (76)$$

can be decomposed into the usual relativistic Klein-Gordon invariant multiplied by a strictly positive factor:

$$C_6 = p_0^6 - \Omega^6 = (p_0^2 - \mid \mathbf{p} \mid^2)(p_0^4 + p_0^2 \mid \mathbf{p} \mid^2 + \mid \mathbf{p} \mid^4) = m^6 c^6, \qquad (77)$$

The sixth-order polynomial C_6 can be further decomposed into the product of the following three second-order polynomials,

$$C_6 = \overset{(0)}{C_2} \, \overset{(1)}{C_2} \, \overset{(2)}{C_2} \,, \qquad (78)$$

$$\text{with} \quad \overset{(0)}{C_2} = p_0^2 - \mathbf{p}^2, \quad \overset{(1)}{C_2} = j \, p_0^2 - \mathbf{p}^2, \quad \overset{(2)}{C_2} = j^2 \, p_0^2 - \mathbf{p}^2. \qquad (79)$$

The mass shell condition

$$E^6 - c^6 \mid \mathbf{p} \mid^6 = m^6 c^{12} \tag{80}$$

can be decomposed into the usual relativistic Klein-Gordon invariant multiplied by a strictly positive factor:

$$C_6 = p_0^6 - \Omega^6 = (p_0^2 - \mid \mathbf{p} \mid^2)(p_0^4 + p_0^2 \mid \mathbf{p} \mid^2 + \mid \mathbf{p} \mid^4) = m^6 c^6, \tag{81}$$

The sixth-order polynomial C_6 can be further decomposed into the product of the following three second-order polynomials,

$$C_6 = \overset{(0)}{C_2} \; \overset{(1)}{C_2} \; \overset{(2)}{C_2} \,, \tag{82}$$

$$\text{with} \quad \overset{(0)}{C_2} = p_0^2 - \mathbf{p}^2, \quad \overset{(1)}{C_2} = j \; p_0^2 - \mathbf{p}^2, \quad \overset{(2)}{C_2} = j^2 \; p_0^2 - \mathbf{p}^2. \tag{83}$$

Let us denote by superscripts (0), (1) and (2) the four-momenta with quadratic invariants given by $\overset{(0)}{C_2}$, $\overset{(1)}{C_2}$ and $\overset{(2)}{C_2}$. We get explicitly

$$(p_0)^2 - (\overset{(0)}{\mathbf{p}})^2 = \overset{(0)}{C_2} \, ,$$

$$(\overset{(1)}{p_0})^2 - (\overset{(1)}{\mathbf{p}})^2 = \overset{(1)}{C_2} \, ,$$

$$(\overset{(2)}{p_0})^2 - (\overset{(2)}{\mathbf{p}})^2 = \overset{(2)}{C_2} \, , \tag{84}$$

Subsequently, we get the following triplet of homogeneous transformations:

$\overset{(0)}{L}_{00}\,,\ \overset{(0)}{L}_{11}$ and $\overset{(0)}{L}_{22}$:

$$\overset{(0)}{L}_{00}(u) = \begin{pmatrix} \mathrm{ch}u & \mathrm{sh}u \\ \mathrm{sh}u & \mathrm{ch}u \end{pmatrix},\ \overset{(0)}{L}_{11}(u) = \begin{pmatrix} \mathrm{ch}u & j^2\mathrm{sh}u \\ j\mathrm{sh}u & \mathrm{ch}u \end{pmatrix},\ \overset{(0)}{L}_{22}(u) = \begin{pmatrix} \mathrm{ch}u & j\mathrm{sh}u \\ j^2\mathrm{sh}u & \mathrm{ch}u \end{pmatrix} \tag{85}$$

preserving respectively the bilinear forms $\overset{(r)}{C}_2$.

- **The matrices (85) are self-adjoint:**

$$\overset{(0)\dagger}{L}_{00} = \overset{(0)}{L}_{00}\,,\quad \overset{(0)\dagger}{L}_{11} = \overset{(0)}{L}_{11}\,,\quad \overset{(0)\dagger}{L}_{22} = \overset{(0)}{L}_{22} \tag{86}$$

- **The matrices (85) are self-adjoint:**

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- **The generalized Lorentz boosts (85) conserve the group property: the product of two Lorentz boosts acting in the r-th sector is a boost of the same type. Indeed, we see from (85) that the product of two boosts acting in the r-th sector ($r = 0, 1, 2$) looks as follows (no summation over r!):**

$$\overset{(0)}{L}_{rr}(u) \cdot \overset{(0)}{L}_{rr}(v) = \overset{(0)}{L}_{rr}(u+v). \tag{87}$$

If we look at three 4-dimensional Lorentz boost transformations on planes $(0, i)$, $i = 1, 2, 3$, the respective set of three independent "classical" Lorentz boosts belonging to $\overset{(0)}{L_{00}}$ requires the introduction of three 4×4 matrices with three independent parameters u, v, w:

$$\begin{pmatrix} \mathrm{ch}u & \mathrm{sh}u & 0 & 0 \\ \mathrm{sh}u & \mathrm{ch}u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{ch}v & 0 & \mathrm{sh}v & 0 \\ 0 & 1 & 0 & 0 \\ \mathrm{sh}v & 0 & \mathrm{ch}v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{ch}w & 0 & 0 & \mathrm{sh}w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{sh}w & 0 & 0 & \mathrm{ch}w \end{pmatrix} \tag{88}$$

Next, let us consider the general set of matrices transforming the s-th sector into the r-th one,

$$\overset{(r)}{p}{}'_\mu = (\overset{(r-s)}{L_{rs}})_\mu{}^\nu \, \overset{(s)}{p}_\nu , \quad r, s = 0, 1, 2, \quad r \neq s. \tag{89}$$

There are two types of such matrices: raising and lowering the Z_3-grade by 1. For the sake of simplicity, let us firstly consider the two-dimensional case (i.e. $\mu, \nu = 0, 1$ in (89)).

The 2×2-**matrices** raising the Z_3 **index** (r) of the generalized four-momenta $(\overset{(r)}{p}_\mu \to \overset{(r+1)}{p}_\mu$ are:

$$\overset{(1)}{L}_{10} = \begin{pmatrix} j^2 \mathrm{ch} u & j^2 \mathrm{sh} u \\ \mathrm{sh} u & \mathrm{ch} u \end{pmatrix}, \ \overset{(1)}{L}_{21} = \begin{pmatrix} j^2 \mathrm{ch} u & j \mathrm{sh} u \\ j \mathrm{sh} u & \mathrm{ch} u \end{pmatrix}, \ \overset{(1)}{L}_{02} = \begin{pmatrix} j^2 \mathrm{ch} u & \mathrm{sh} u \\ j^2 \mathrm{sh} u & \mathrm{ch} u \end{pmatrix} \tag{90}$$

The determinants of the matrices **(90)** are equal to j^2.

The matrices lowering the Z_3 index by one (or increasing it by 2, what is equivalent from the point of view of the Z_3-grading) are:

$$\overset{(2)}{L}_{01} = \begin{pmatrix} j\,\mathrm{ch}u & \mathrm{sh}u \\ j\,\mathrm{sh}u & \mathrm{ch}u \end{pmatrix}, \quad \overset{(2)}{L}_{12} = \begin{pmatrix} j\,\mathrm{ch}u & j^2\mathrm{sh}u \\ j^2\mathrm{sh}u & \mathrm{ch}u \end{pmatrix}, \quad \overset{(2)}{L}_{20} = \begin{pmatrix} j\,\mathrm{ch}u & j\,\mathrm{sh}u \\ \mathrm{sh}u & \mathrm{ch}u \end{pmatrix} \tag{91}$$

The determinants of the matrices (91) are equal to j.

The above two sets of three matrices each are mutually **Hermitean-adjoint**:

$$\overset{(1)\dagger}{L_{01}} = \overset{(2)}{L_{10}}\,, \quad \overset{(1)\dagger}{L_{12}} = \overset{(2)}{L_{21}}\,, \quad \overset{(1)\dagger}{L_{20}} = \overset{(2)}{L_{02}} \tag{92}$$

We recall that the superscript over each matrix $\overset{(t)}{L_{rs}}$ is equal to the difference of its lower indices, i.e. $(t) = (r - s)$.

The matrices $\overset{(1)}{L}_{rs}$ and $\overset{(2)}{L}_{rs}$ $(r, s = 0, 1, 2)$ raising or lowering respectively the Z_3-grade of the four-momentum vectors $\overset{(r)}{p}_{\mu}$ do not form a Lie group.

However, together with matrices $\overset{(0)}{L}_{rs}$ they can be used as building blocks in bigger 12×12 matrices forming a Z_3-graded generalization of the Lorentz group.

This construction is possible due to the chain rule obeyed by these matrices, which due to the definition display the group property. We have:

$$\overset{(r-s)}{L_{rs}}\ (p_0, p_1;\ u)\ \overset{(s-t)}{L_{st}}\ (p_0, p_1;\ v) = \overset{(r-t)}{L_{rt}}\ (p_0, p_1;\ (u+v)). \quad (93)$$

In order to pass to arbitrary four-momentum vectors $\overset{(r)}{p}_{\mu}$, $\mu = 0, 1, 2, 3$ one should embed the 2×2 matrices **(90 - 91)** into 4×4 matrices in a way analogous to passing from the 2×2 boost matrices $\overset{(0)}{L_{00}}$ to the triplet of boosts in planes $(0, i), i = 1, 2, 3$ described by the 4×4 matrices (88).

If we write a Z_3-extended four-momentum vector $(\overset{(0)}{p^\mu}, \overset{(1)}{p^\mu}, \overset{(2)}{p^\mu})^T$ as a column with 12 entries, we can introduce three boost sectors $\overset{(r)}{\Lambda}$, $(r = 0, 1, 2)$ of the generalized Z_3-graded Lorentz group as 12×12 matrices as follows:

$$\overset{(0)}{\Lambda} : \begin{pmatrix} \overset{(0)}{L}_{00} & 0 & 0 \\ 0 & \overset{(0)}{L}_{11} & 0 \\ 0 & 0 & \overset{(0)}{L}_{22} \end{pmatrix} \overset{(1)}{\Lambda} : \begin{pmatrix} 0 & 0 & \overset{(1)}{L}_{02} \\ \overset{(1)}{L}_{10} & 0 & 0 \\ 0 & \overset{(1)}{L}_{21} & 0 \end{pmatrix} \overset{(2)}{\Lambda} : \begin{pmatrix} 0 & \overset{(2)}{L}_{01} & 0 \\ 0 & 0 & \overset{(2)}{L}_{12} \\ \overset{(2)}{L}_{20} & 0 & 0 \end{pmatrix}. \tag{94}$$

In each of the 12×12 **matrices** $\overset{(r)}{\Lambda}$, $r = 0, 1, 2$ the triplets of 4×4 **matrices** $\overset{(r-s)}{L_{rs}}$ are obtained from the standard classical Lorentz boosts by using the definition, i.e. each $\overset{(r)}{\Lambda}$-matrix depends exclusively on three parameters defining three independent classical Lorentz boosts.

One can show that our matrices display the following Z_3-**graded** multiplication rules:

$$\overset{(0)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r)}{\boldsymbol{\Lambda}}, \quad \overset{(1)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r+1)}{\boldsymbol{\Lambda}}, \quad \overset{(2)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r+2)}{\boldsymbol{\Lambda}}, \tag{95}$$

where $\overset{(r)}{\boldsymbol{\Lambda}}$ $(r = 0, 1, 2)$ denote the Z_3-**graded** sectors of the full set of 12×12 matrix Lorentz group which includes also the Z_3-**graded** $O(3)$ spatial rotations.

The Z_3-graded Lorentz group should also act on the coloured Dirac spinors through certain linear action, which should be realized as a generalized spinorial representation..

Such construction was introduced in our most recent publication:

R. Kerner and J. Lukierski, *Internal quark symmetries and colour $SU(3)$ entangled with Z_3-graded orentz algebra*, *Nuclear Physics B*, Vol. 972, (November 2021), 115529

which we shall briefly present in what follows.

What we want to find is a set of 12×12 matrices $J_i^{(r)}$, and $K_j^{(r)}$, $i, j.. = 0, 1, 2$ satisfying the same Z_3-graded commutation relations as their vectorial counterparts:

$$\left[J_i^{(r)}, J_k^{(s)}\right] = \epsilon_{ikl} J_l^{(r+s)}, \quad \left[J_i^{(r)}, K_k^{(s)}\right] = \epsilon_{ikl} K_l^{(r+s)},$$
$$\left[K_i^{(r)}, K_k^{(s)}\right] = -\epsilon_{ikl} J_l^{(r+s)}. \tag{96}$$

The spinor representation of the zeroth sector $L^{(0)}$ of the Z_3-**graded Lorentz algebra** is obtained in a simplest possible manner, by tensorising the spinorial generators of the usual representation on Dirac spinors by the unit 3×3 **matrix**:

$$J_l^{(0)} = -\frac{i}{2} \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K_i^{(0)} = -\frac{1}{2} \mathbb{1}_3 \otimes \sigma_1 \otimes \sigma_i. \tag{97}$$

satisfying classical Lorentz algebra commutation relations:

$$\begin{aligned} &\left[J_i^{(0)}, J_k^{(0)}\right] = \epsilon_{ikl} J_l^{(0)}, \quad \left[J_i^{(0)}, K_k^{(0)}\right] = \epsilon_{ikl} K_l^{(0)}, \\ &\left[K_i^{(0)}, K_k^{(0)}\right] = -\epsilon_{ikl} J_l^{(0)}. \end{aligned} \tag{98}$$

The two extra Lorentz sectors, $L^{(1)}$ and $L^{(2)}$, are constructed as the following 12×12 matrices:

$$J^{(1)}_l = -\frac{i}{2}\, Q_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K^{(1)}_i = -\frac{1}{2}\, Q_3 \otimes \sigma_1 \otimes \sigma_i. \tag{99}$$

$$J^{(2)}_i = -\frac{i}{2}\, Q^{\dagger}_3 \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K^{(2)}_m = -\frac{1}{2}\, Q^{\dagger}_3 \otimes \sigma_1 \otimes \sigma_m. \tag{100}$$