A very general notion of QFT

What is a QFT?

In a world without quantum physics it could have been discovered purely mathematically:

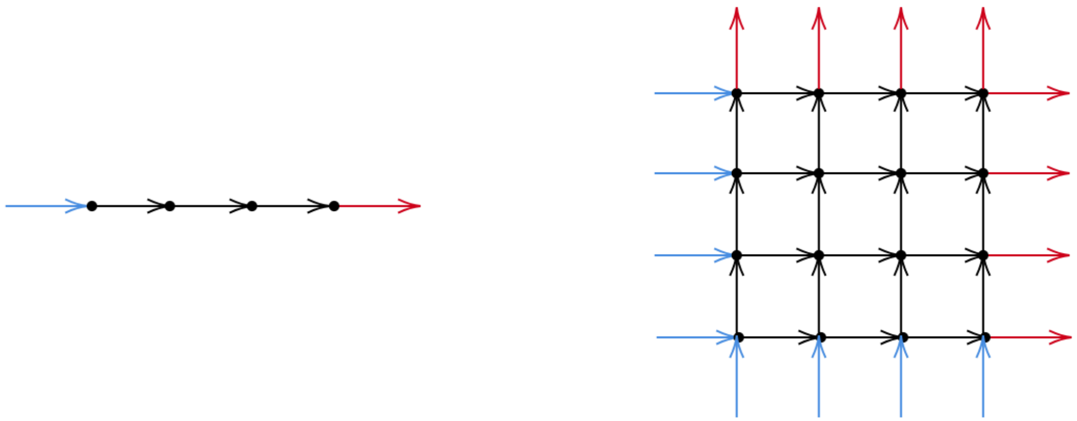
Fix an integer $d \geqslant 1$.
A translation-invariant lattice model on $\mathbb{Z}^d$ is determined by a collection of d finite-dimensional complex vector spaces $V_1, \dots V_d$ and a linear map $R : V_1 \otimes \cdots \otimes V_d \to V_1 \otimes \cdots \otimes V_d$.
If we choose a basis for each of the spaces V_i, i.e identify it with $\mathbb{C}^{n_i}$ for some $n_i \in \mathbb{N}$ for each $i = 1, \dots, d$, then R becomes a tensor with d upper and d lower indices:

$$R = (R^{i_1, \dots i_d}_{j_1, \dots j_d} \in \mathbb{C}) \quad \text{where } 1 \leqslant i_1, j_1 \leqslant n_1, \dots, 1 \leqslant i_d, j_d \leqslant n_d$$

For any $N_1, \dots, N_d \geqslant 1$ we can partially contract indices for $\prod_{i=1}^d N_i$ copies of tensor R and get tensor $R_{[N_1, \dots N_d]}$ with

$$2 \sum_{i=1}^{d} \Big(\prod_{j \neq i} N_j \Big)$$

indices, by stacking copies of R inside integer points in d-dimensional rectangular $N_1 \times \cdots \times N_d$ box:



If $n_1 = \cdots = n_d = n > 1$ and $N_1 = \cdots = N_d = N$ then we get an element of a vector space of huge dimension (if $d \geqslant 2$):

$$R_{[N,\ldots,N]} \in \mathbb{C}^{n^{2N^{d-1}}}, \qquad \text{in the case } d = 1 : \quad R_N \in \mathbb{C}^{n^2} = \operatorname{End}(V_1) = \operatorname{Mat}(n \times n, \mathbb{C}) \ \forall N \geqslant 1$$

Real part of free energy: (more precisely, *negative of the real part of free-energy density per site*), we define number $f_{\mathbb{R}} = f_{\mathbb{R}}(R) \in \mathbb{R} \cup \{-\infty\}$ as the limit:

$$f_{\mathbb{R}} := \lim_{N_1,\ldots,N_d \to +\infty} \frac{1}{N_1 \ldots N_d} \log \left|R_{[N_1,\ldots,N_d]}\right|^2$$

A version of Van Hove theorem: ***limit exists.***

We obtain a $\mathbb{R}\cup\{-\infty\}$ - valued plurisubharmonic function on the complex *parameter space* $\mathbb{C}^{\prod_{i=1}^d n_i^2}$, invariant under the action of $\prod_{i=1}^d \mathrm{Aut}(V_i)\simeq\prod_{i=1}^g \mathrm{GL}_{d_i}(\mathbb{C})$.
Why is it plurisubharmonic? - for finite $N_1,\dots N_d$ this is (up to rescaling) the logarithm of a finite sum $\sum_\alpha |g_\alpha|^2$ of squares of norms of holomorphic functions.

Phase transition: points where $f_\mathbb{R}$ is ***not*** locally pluriharmonic (i.e. locally is not $\Re f_\mathbb{C}$ where $f_\mathbb{C}$ is a holomorphic function defined up to a purely imaginary constant).

In an open domain in parameter space $f_\mathbb{R}$ is pluriharmonic. In the closure we get a non-negative closed $(1,1)$-current $\partial\bar\partial f_\mathbb{R}$. It can be thought of as a degenerate Kähler metric, which is zero in some direction, infinity some other directions, and genuinely interesting metric in intermediate directions (generalizes Zamolodchikov metric on the moduli space of CFTs in 2 dimension).

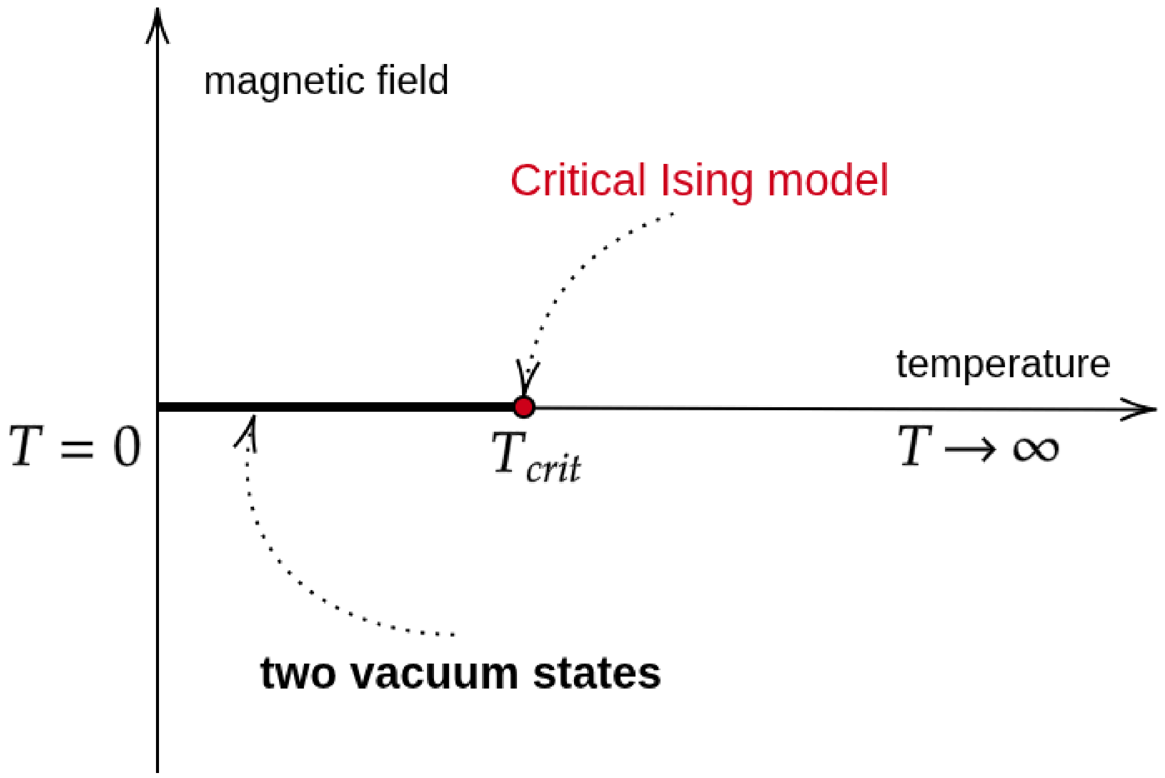
Case $d=1$, - nothing truly interesting happens

$$R=(R^i_j)\in\mathrm{End}(V_1)\simeq\mathrm{Mat}(n_1\times n_1,\mathbb{C}),\qquad f_\mathbb{R}(R)=\max_\alpha\log|\lambda_\alpha|,\quad \lambda_\alpha\in\mathrm{Spec}(R)$$

All "phase transitions" are of "first-order": several eigenvalues compete for the maximal norm, locally

$$f_\mathbb{R}=\max_\alpha(\Re f_{\mathbb{C},\alpha})\quad + \text{a bit more involved picture for Jordan blocks}$$

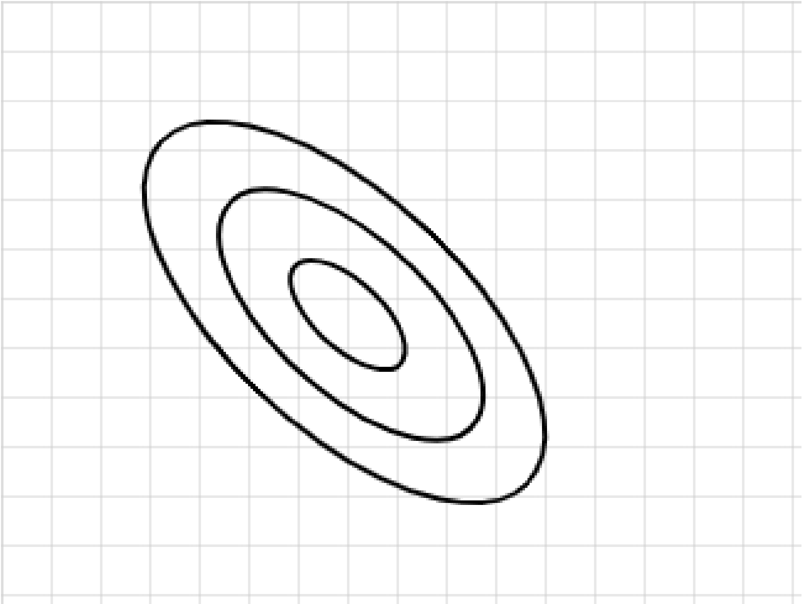
Case $d=2$: main example: Ising model (reformulated so that spins are associated to *edges* and not vertices of the square grid),
we get the usual phase diagram in *real* coupling constants (temperature, magnetic field)



Similar picture in complex domain.

real codimension 1: competition between two leading eigenvalues, first-order phase transition

real codimension 2: second-order phase transition, we get *universally* critical Ising model (A.Zamolodchikov: small deformations give massive theory with 8 particles, masses $\sim$ components of the Perron-Frobenius eigenvector of the E_8 Cartan matrix, + *emergent* rotational symmetry (equivalently, an emergent conformal structure):



Lesson: rotational symmetry is not fundamental, or even natural in lattice models, it could be emergent (or it could be not!). In general, interesting phase transitions signal not a conformal field theory but only a scale-invariant theory (we always have *translation invariance*).

My paper with Graeme Segal: "Wick rotation and the positivity of energy in quantum field theory", a variation of the idea of rotational symmetry: consider *complex-valued* metrics.
Goal of today's lecture: strip away Euclidean symmetry (or Lorentz symmetry as a limiting case of so-called "allowed" complex-valued metrics), as well as any positivity constraint.

Positivity: two meanings:

1. probabilistic models: all coefficients $R^{i_1,\dots,i_d}_{j_1,\dots,j_d}$ are in $\mathbb{R}_{\geqslant 0}$ then $f_{\mathbb{R}}$ coincides with the usual free energy

2. reflection-positive models, e.g. $R^{i_1,i_2,...,i_d}_{j_1,j_2,...,j_d} = \overline{R}^{j_1,i_2,...,i_d}_{i_1,j_2,...,j_d}$ gives Hilbert space associated with the reflection with respect to the hyperplane $\{x_1 = \frac{1}{2}\} = \mathbb{R}^{d-1} \subset \mathbb{R}^d$.

We would like to strip off the positivity condition (in either sense) as well.

Goal: ***non-unitary, non-probabilistic physics***

Examples in any d : dimer models ("free fermionic lattice theories"), - here we need a little generalization of the lattice model formulation: each $V_i, i = 1, \ldots, d$ is now a *super-vector space* $V_i = V_{i,+} \oplus V_{i,-}$ (and tensor $R = (R^{i_1,...i_d}_{j_1,...j_d} \in \mathbb{C})$ of "Boltzmann weights" is *even*).
A class of dimer models (quadratic action in fermionic variables):

$$f_{\mathbb{R}} = \frac{1}{(2\pi)^d} \int_{(S^1)^d} \log |\operatorname{Pfaff} P(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_d})| \, d\theta_1 \ldots d\theta_d$$

where $P(z_1, \ldots, z_d) \in \operatorname{Mat}(2M \times 2M, \mathbb{C}[z_1^{\pm 1}, \ldots z_d^{\pm 1}])$ is a skew-symmetric matrix with coefficients in Laurent polynomials in d variables $z_1, \ldots, z_d$.

Observation in the fermionic case: f_R could be *not pluriharmonic* in an *open* domain in the parameter space (case $d \geqslant 2$).
Indeed, generically complex valued polynomial $\operatorname{Pfaff} P(z_1, \ldots, z_d)$ vanishes on real-codimension 2 subvariety in d-dimensional torus

$$(S^1)^d = \{(z_1, \ldots, z_d) | \, \forall \, i = 1, \ldots d : |z_i| = 1\} \subset (\mathbb{C}^\times)^d$$

which ensures that $f_{\mathbb{R}}$ is not pluriharmonic. This leads to a pathology: 2-point functions become *singular* outside of the diagonal. Indeed, in the momentum space it is singular in real-codimension 2 subspace, like $(p_1 + \mathrm{i}p_2)^{-1}$ as a function in $(p_1, \dots, p_d) \in \mathbb{R}^d$, and its Fourier transform is distribution $\sim (x_1 - \mathrm{i}x_2)^{-1}$ with singularities in $\mathbb{R}^2 \subset \mathbb{R}^d$.

Maybe in general, for fermionic models one should restrict ourselves only to phase transition points in the parameter space which lie in the *closure* of the "trivial" (pluriharmonic) points.

Lesson from dimer models: the pluriharmonic case $\iff$ complex-valued Laurent polynomial $\mathrm{Pfaff}\ P(z_1, \dots, z_d)$ does not vanish on the torus $(S^1)^d \subset (\mathbb{C}^\times)^d$. Limits of such polynomials: functions vanishing exactly at one point (say, at $(1, \dots, 1) \in (S^1)^d \subset (\mathbb{C}^\times)^d$ corresponding to $\theta_1 = \dots = \theta_d = 0$ in the integral formula for $f_{\mathbb{R}}$). Then the leading term in Taylor expansion gives homogeneous matrix-valued polynomial on momentum space $\mathbb{R}^d$ which is *invertible* at all points outside of $0 \in \mathbb{R}^d$. This is exactly the *ellipticity* condition for the symbol of matrix-valued differential operator.

Side remark: the only elliptic operators used in practice are of second order (Laplacian Δ) or of first order (Dirac $\not{\partial}$). This is a corollary of the unitarity and local Euclidean/Lorentz invariance paradigm. More general elliptic operators and their zeta-functions (R.Seeley,...) are almost forgotten.

Suppose now that we are given a very general QFT defined on a d-dimensional manifold X, which is asymptotically scale-invariant near each point $x \in X$.

Axioms (*very weak*): we have a $\mathbb{C}$-vector bundle Φ_X on X with countably infinite-dimensional fibers (the fiber Φ_x is the space of local observables at x) which is filtered by positive real numbers (scaling dimensions, possibly depending smoothly on $x \in X$)

$$\Phi_x = \cup_{\Delta \in \mathbb{R}_{\geqslant 0}} \Phi_x^{\leqslant \Delta} \qquad \dim_{\mathbb{C}} \Phi_x^{\leqslant \Delta} < \infty \qquad \Phi_x^{\leqslant 0 = \Delta_{0,x}} \subset \Phi_x^{\leqslant \Delta_{1,x}} \subset \dots, \quad \lim_i \Delta_{i,x} = +\infty$$

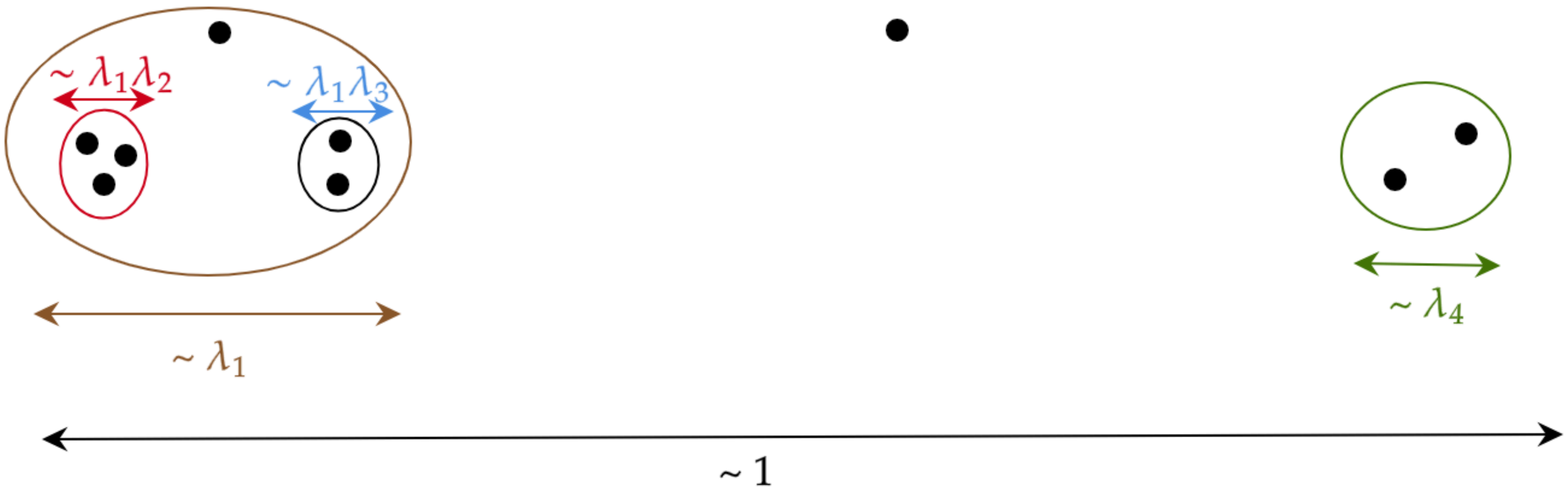
1. System of correlators

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle \in \mathbb{C}, \quad (x_1, \dots, x_n) \in X^n - \mathrm{Diag}, \quad \phi_i(x_i) \in \Phi_{x_i}^{\leqslant \Delta_i}$$

which can be understood as a C^∞-section of the external product of dual bundles $\boxtimes_{i=1}^n \left(\Phi^{\leqslant \Delta_i}\right)^*$ on the configuration space $\mathrm{Conf}_n(X) := (X^n - \mathrm{Diag}) \subset X^n$, such that when points collide we have an asymptotic expansion in local coordinates

$$\langle \phi_1(\vec{x} + \lambda \vec{v}_1) \dots \phi_n(\vec{x} + \lambda \vec{v}_n) \rangle \sim \sum_{\Delta_{\alpha,x}} \sum_{k=0}^{k^{max}_{\Delta_{\alpha,x}}} c_{\Delta_{\alpha,x},k} \cdot \lambda^{\Delta_{\alpha,x} - \sum_i \Delta_{i,x}} (\log \lambda)^k$$

where $\vec{v}_1, \dots \vec{v}_n \in \mathbb{R}^d \simeq T_x X$ are *distinct* vectors, and a similar condition for the multi-scale collision:



2. *Operator Product Expansion (OPE)*: (formulation in the case when a group of points collapses to one point, the more general case is similar):
   when we consider configurations (in local coordinates near $x \in X$)

   $$x_i = \vec{x} + \lambda \vec{v}_i, \quad i = 1, \ldots, n$$

   approaching the same point $x \in X$, where $\vec{v}_1, \ldots \vec{v}_n \in T_{\vec{x}}\mathbb{R}^d \simeq T_x X$ are *distinct* tangent vectors, then we have an infinite sequence of fields *at* point x such that

   $$\phi_1(\vec{x} + \lambda \vec{v}_1) \ldots \phi_n(\vec{x} + \lambda \vec{v}_n) \sim \sum_{\Delta_{\alpha,x}} \sum_{k=0}^{k^{max}_{\Delta_{\alpha,x}}} \phi_{\Delta_{\alpha,x},k}(\vec{x}) \cdot \lambda^{\Delta_{\alpha,x} - \sum_i \Delta_{i,x}} (\log \lambda)^k$$

   in the sense that for *arbitrary* collection of distinct points $x_{n+1}, \ldots, x_{n+m}$ in $X - \{x\}$ and any local observables $\phi_{n+1}(x_{n+1}), \ldots, \phi_{n+m}(x_{n+m})$ we have

$$\langle \phi_1(\vec{x}+\lambda\vec{v}_1)\dots\phi_n(\vec{x}+\lambda\vec{v}_n)\prod_{j=1}^{m}\phi_{n+j}(x_{n+j})\rangle \sim \sum_{\Delta_{\alpha,x}}\sum_{k=0}^{k^{max}_{\Delta_{\alpha,x}}}\langle \phi_{\Delta_{\alpha,x},k}(\vec{x})\prod_{j=1}^{m}\phi_{n+j}(x_{n+j})\rangle\cdot\lambda^{\Delta_{\alpha,x}-\sum_i\Delta_{i,x}}(\log\lambda)^k$$

Moreover, OPE should satisfy an associativity axiom, which is automatically satisfied if any observable at a point is uniquely characterized by its correlators with an observable in some other point.

One can think of OPE as a kind of theory of algebras (and the structure is *local* in the space-time manifold X), whereas the system of correlators as a kind of trace functional.

Example: for the free scalar boson theory in dimension $d \geqslant 3$, the space of local observables is the space of normal-ordered products of derivatives $:\prod_\alpha \partial^{I_\alpha}\phi:$ where $I_\alpha \in \mathbb{Z}^d_{\geqslant 0}$ are multi-indices, i.e. correspond to differential polynomials in the basic field ϕ.
Here scaling dimension of ϕ is $\frac{d-2}{2} > 0$ and the scaling dimension of the composite field $:\prod_\alpha \partial^{I_\alpha}\phi:$ is the sum

$$\sum_\alpha \Big(\frac{d-2}{2} + \#I_\alpha\Big) \geqslant 0$$

We assume that ϕ satisfies the classical equation of motion, i.e. $\sum_{i=1}^d \partial_i^2\phi(x) = 0$ so the derivatives $\partial^{I_\alpha}\phi$ are not independent.

Next, I propose a *universal* formula for the deformation of a given system of correlators.
Main thesis: deformations of theory not destroying scale invariance at short distances are governed by *densities* on X with values in the bundle $\Phi^{\leqslant d}$ (think about the renormalizable perturbation of the Lagrangian).

The naive formula is the following:
for compact X and "modification of Lagrangian"

$$\delta\mathcal{L} \in \Gamma(X, |\Omega_X^d| \otimes \Phi^{\leqslant d})$$

the perturbed theory "with Lagrangian $\mathcal{L} + \varepsilon \cdot \delta L$" is a QFT formally depending on ε with the *same* observables and new correlators given by

$$\langle \phi_1(x_1) \dots \phi_d(x_d) \rangle_{\text{new}} := \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} \int_{y_1} \dots \int_{y_m} \langle \phi_1(x_1) \dots \phi_d(x_d)\, \delta\mathcal{L}(y_1) \dots \delta\mathcal{L}(y_m) \rangle\, d^d y_1 \dots d^d y_m \qquad (*)$$

This is the standard lore: imagine that the correlator is written as Feynman integral

$$\langle \phi_1(x_1) \dots \phi_d(x_d) \rangle = \int \prod_i \phi_i(x_i) \cdot e^{S(\phi)} \mathcal{D}\phi$$

Now modify the action functional $S(\phi) \rightsquigarrow S_{\text{new}}(\phi) = S(\phi) + \varepsilon \int_{y \in X} \delta\mathcal{L}(y) d^d y$, and expand exponent into Taylor series in ε.

The trouble with the naive formula $(*)$ is that the integrals over running points $y_1, \dots, y_m$ are in general *divergent* near the diagonals or when some points y_j approach some of points x_i.

Here is the general recipe, which depends on two choices:

1. for a function F of positive real variable $r > 0$ which admits an asymptotic expansion

$$F(r) \sim \sum_{\Delta_i \geqslant -\,\mathrm{const}} \sum_{k=0}^{k_i^{max}} c_{\Delta_i,k} r^{\Delta_i} (\log r)^k$$

   we want to define its "regularized limit" at $r = 0$ which is linear in F and coincides with the usual limit if it exists. For example, one can make the most obvious choice

$$\lim^{reg}_{r \to +0} F(r) := c_{0,0}$$

2. we also choose Riemannian metric on X (this is a purely technical choice, it *does not* mean that we return to the usual situation of local rotational invariance!), hence we can speak about *distance*.

The corrected version of $(*)$ is the following, we replace each integral by its *regularized value* which is defined as

$$\Big[\int_{y_1} \cdots \int_{y_m}\Big]_{reg} \langle \phi_1(x_1) \ldots \phi_d(x_d)\, \delta\mathcal{L}(y_1) \ldots \delta\mathcal{L}(y_m)\rangle d^d y_1 \ldots d^d y_m :=$$

$$= \lim^{reg}_{r_1 \to +0} \int_{\substack{y_1 \in X \\ \forall i : |y_1 - x_i| > r_1}} \left(\lim^{reg}_{r_2 \to +0} \int_{\substack{y_2 \in X \\ \forall i : |y_2 - x_i| > r_2 \\ |y_2 - y_1| > r_2}} \Big(\ldots \Big(\lim^{reg}_{r_m \to +0} \int_{\substack{y_m \in X \\ \forall i : |y_m - x_i| > r_m \\ |y_m - y_1| > r_m \\ \ldots \\ |y_m - y_{m-1}| > r_m}} \langle \phi_1(x_1) \ldots \phi_d(x_d)\, \delta\mathcal{L}(y_1) \ldots \delta\mathcal{L}(y_m)\rangle d^d\right.$$

One can show that different choices of the functional $\lim^{reg}_{r \to +0}$ as well as different choices of distance function give in a sense equivalent prescriptions: one obtains the *same* formal germ in the space of theories, the only thing which will change is the parametrization by perturbations of the action.

Moreover, this modification procedure makes sense already for OPE algebras, and is local/sheaf-theoretic, so that X need not be compact, and there is no question of convergence of integrals at infinity (infrared divergence).

I claim that the whole story can be reduced to the special case of translation-invariant theories.

Conjecture in any $\dim > 1$ there exists a countable union of **finite-dimensional spaces**: moduli space $\mathcal{T}_d$ of logarithmic scale-invariant and translation-invariant theories in $\mathbb{R}^d$, with the action of $GL(d,\mathbb{R})$ on $\mathcal{T}_d$ (rescaling: fixed points).

More precise picture: $\mathcal{T}_d$ is not a manifold, but locally a closed subset in a complex-analytic manifold, for which we take *formal neighborhood*, i.e. consider functions which are formal series in local complex coordinates, satisfying asymptotic expansion constraints at any point.
The closed subset should be invariant under Renormalization Group flow, and consist (roughly) of points for which the orbit has limit (i.e. theory).
The RG flow should act on an infinite-dimensional manifold.

By translation-invariance we identify fibers of $\Phi_{\mathbb{R}^d}$ and get just one countable-dimensional space $\Phi = \Phi_{\vec{0}} = \cup_{\Delta_\alpha \geqslant 0} \Phi^{\leqslant \Delta_\alpha}$ endowed with the action of rescaling, with weights $\Delta_\alpha \geqslant 0$. On associated graded pieces $\Phi^{\leqslant \Delta_\alpha} / \Phi^{< \Delta_\alpha}$ the Lie generator of rescaling acts with eigenvalue Δ_α. We *modify* the action of this generator by

$$\Delta_\alpha \rightsquigarrow (\Delta_\alpha - d) \in [-d, +\infty]$$

The shift comes from the scaling dimension of the volume element $d^d x$.

The infinitesimal generator of RG flow will have the linearized action at 0 given by the modified action, so that the formal tangent space to $\mathcal{T}_d$ will correspond to the attracting directions and the center manifold, i.e. to $\Phi^{\leqslant d}$ (marginal and relevant perturbations).

The only additional structure: group $\mathrm{GL}_d(\mathbb{R})$ acts on $\mathcal{T}_d$, and the action means that any element of Lie algebra gives a first-order deformation of the theory. Now, according to the principle that deformations are governed by local fields, this should correspond to certain fields, forming a *stress-energy tensor* $T_i^j \in \Phi$, $1 \leqslant i, j \leqslant d$ (no longer symmetric, just operator-valued).

The theories with *varying* coupling constants can be understood (at least on OPE level) as deformations of translation-invariant theories, using the same master formula as before.

Clean examples of OPE algebras: free bosonic and fermionic theories with elliptic symbols, especially for an operator of degree $< d$. A version: consider arbitrary elliptic operators of order $\geqslant d$ but instead of all differential polynomials we keep only the normal-ordered products of sufficiently high derivatives of the fundamental fields, thus getting only *positive* scaling dimensions for non-trivial observables.
