Modular invariant vertex operator algebras.

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§1. **Definitions**

Let V be a vector space over $\mathbb{C}$. A series $a(z) = \sum\limits_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, where $a_{(n)} \in End\ V$, is called a quantum field if $a_{(n)} v = 0$ for $n >> 0$ for each $v \in V$.

The data of a vertex algebra consists of the space of states V (a vector (super)space over $\mathbb{C}$), the vacuum vector $|0\rangle \in V$, the translation operator $T \in End\ V$, and a collection of quantum fields $\mathfrak{F} = \{a^j(z) = \sum\limits_{n \in \mathbb{Z}} a^j_{(n)} z^{-n-1}\}_j$, subject to the following four axioms:

(vacuum) $\quad T|0\rangle = 0;$

(translation covariance) $\quad [T, a^j(z)] = \frac{d}{dz} a^j(z);$

(locality) $\quad (z-w)^{N_{jk}} [a^j(z), a^k(w)] = 0 \quad$ for some $N_{jk} \in \mathbb{Z}_{\geq 0}$;

(completeness) $\quad \text{Span} \left\{ a^{j_1}_{(n_1)} \ldots a^{j_s}_{(n_s)} |0\rangle \right\} = V.$

An equivalent Definition:

A vertex algerbra is a vector superspace over $\mathbb{C}$, $V = V_{\bar{0}} \oplus V_{\bar{1}}$, with a vacuum vector $\mathbf{1} \in V_{\bar{0}}$ and a translation operator $T \in (\mathrm{End}\, V)_{\bar{0}}$, endowed with a product with values in $V((z))$, written as a quantum field $Y(a,z) = \sum\limits_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, where $a_{(n)} \in \mathrm{End}\ V$, satisfying the following axioms $(a, b \in V)$:

(vacuum) $\quad Y(\mathbf{1}, z) = \mathrm{I}_V, \quad Y(a,z)\mathbf{1} \in a + zV[[z]];$

(translation invariance) $\quad [T, Y(a,z)] = \frac{d}{dz} Y(a,z);$

(locality) $\quad (z-w)^N [Y(a,z), Y(b,w)] = 0$ for some $N \in \mathbb{Z}_{\geq 0}$.

Note that $a_{(n)}b$ is a bilinear product $V \otimes V \to V$ for each $n \in \mathbb{Z}$, such that $a_{(n)}b = 0$ for $n >> 0$.

The product $a_{(-1)}b$ is denoted by :ab: and is called the normally ordered product. One has:

$a_{(-n-1)}b = \frac{1}{n!} : (T^n a)b:$ for $n \geq 0$.

Define the λ-bracket $V \otimes V \to V[\lambda]$ by

$[a_\lambda b] = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\lambda^n}{n!}(a_{(n)}b)$.

It satisfies the axioms of a Lie conformal superalgebra:

(Sesquilinearity) $[Ta_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda Tb] = (T + \lambda)[a_\lambda b];$

(Skew-symmetry) $[b_\lambda a] = -(-1)^{p(a)p(b)}[a_{-\lambda-T}b];$

(Jacobi identity) $[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c] + (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]].$

Computations in a vertex algebra $(V, \mathbf{1}, T, Y(a,z))$ can be performed using the third equivalent definition, in terms of the quintuple

$(V, \mathbf{1}, T, [._\lambda.], ::)$, where

(1) $(V, T, [\cdot_\lambda \cdot])$ is a Lie conformal superalgebra;

(2) $(V, \mathbf{1}, T, ::)$ is a unital differential superalgebra with derivation T, satisfying

(quasi commutativity) $:ab: - (-1)^{p(a)p(b)} :ba: = \int_{-T}^{0} [a_\lambda b]\, d\lambda$;

(quasi associativity)

$::ab:c: - :a:bc:: = :(\int_0^T d\lambda\ a)[b_\lambda c]: + (-1)^{p(a)p(b)} a \leftrightarrow b$;

(3) (quasi Wick formula)

$[a_\lambda :bc:] = :[a_\lambda b]c: + (-1)^{p(a)p(b)} :b[a_\lambda c]: + \int_0^\lambda [[a_\lambda b]_\mu c]\, d\mu$.

[BK03]

Given a Lie conformal superalgebra R, its universal enveloping vertex algebra $(V(R), \varphi)$ is defined in the same way as for Lie superalgebras:

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & V(R) \arrow[d, dashed] \\
& V
\end{tikzcd}$$

PBW Theorem. Let $a_1, a_2, \ldots$ be a basis of R over $\mathbb{C}$. Then the ordered monomials $: a_{i_1} a_{i_2} \cdots a_{i_s} :$ form a basis of the $V(R)$ over $\mathbb{C}$. Here, the normally ordered products are taken from right to left, and $i_r < i_{r+1}$ if both a_{i_r} and a_{ir+1} are odd, and $\leq$ otherwise.

Remark 1. If R has a central element K, then for each $k \in \mathbb{C}$, we get the universal enveloping VA of level k:

$V^k(R) = V(R)/(K - k\mathbf{1})$.

Remark 2. Lie conformal superalgebras encode an important class of infinite-dimensional Lie superalgebras. Namely, given a Lie conformal superalgebra R, the corresponding Lie superalgabra Lie_R is

$span\{a_{(n)} \ \mid a \in R, \ n \in \mathbb{Z}\}/span\{(Ta)_{(n)} + na_{(n-1)}\}$

with the (well-defined) bracket:

$[a_{(m)}, b_{(n)}] = \sum\limits_{j \in \mathbb{Z}_{\geq 0}} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}$.

The universal enveloping vertex algebra $V^k(R)$ can be constructed explicitly by inducing to Lie_R from the 1-dimensional module over the subalgebra

$span\{a_{(n)} | a \in R, n \in \mathbb{Z}_{\geq 0}\} \oplus \mathbb{C}K$, given by $a_{(n)}\mathbf{1} = 0$, $K\mathbf{1} = k\mathbf{1}$.

Then $\mathbf{1}$ is the vacuum vector, T acts in the obvious way, and the generating quantum fields are $Y(a, z) = \sum_{n \in \mathbf{Z}} a_{(n)} z^{-n-1}$.

§2. Basic examples

Example a) Let A be a vector superspace over $\mathbb{C}$ with a non-degenerate super skew-symmetric bilinear form $< \cdot, \cdot >$.

The fermionic Lie conformal superalgebra is $R_A = (\mathbb{C}[T] \otimes A) \oplus \mathbb{C}K$ with the λ-brackets

$[\varphi_\lambda \psi] =< \varphi, \psi > K$ for $\varphi, \psi \in A$, K central.

Its universal enveloping VA $F^1\ (A)$ is called the fermionic VA based on A. It is simple.

The corresponding Lie superalgebra Lie_{R_A} has basis $\varphi^i_{(n)}$, where $\{\varphi^i\}$ is a basis of A, $n \in \mathbb{Z}$, and K, with the brackets:

$$[\varphi^i_{(m)}, \varphi^j_{(n)}] =< \varphi^i, \varphi^j > \delta_{m,-n-1} K, \ K \text{ central.}$$

Example b) The Virasoro Lie conformal algebra is

$\mathrm{Vir} = \mathbb{C}[T] \otimes L + \mathbb{C}\, C$

with λ-brackets

$[L_\lambda L] = (T + 2\lambda)L + \frac{\lambda^3}{12}\, C$, C central.

The corresponding universal enveloping VA of level c, called the central charge $c \in \mathbb{C}$, is denoted by V^c. It is not simple, but has a unique maximal ideal, the quotient by which is denoted by V_c. It is well known that V^c is not simple if and only if

$$c = 1 - \frac{6(p-q)^2}{pq}, \text{ where } p, q \in \mathbb{Z}\ , p > q \geq 2\ , gcd(p, q) = 1.$$

The corresponding Lie algebra Lie_{Vir} is the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}C,$$

where $L_m = L_{(m+1)}$, C central, $m, n \in \mathbb{Z}$.

Definition. An element L of a vertex algebra V is called a Virasoro vector (or conformal vector, or energy-momentum vector) if:

(i) $\mathbb{C}[T]L + \mathbb{C}\mathbf{1}$ with the λ-bracket from V is a Virasoro Lie conformal algebra;

(ii) $L_{-1} = T$;

(iii) L_0 is diagonalizable.

The central charge of (V, L) is the central charge of (i).

Definition. A vertex algebra V endowed with a Virasoro vector L is called a VOA (or conformal VA) if the eigenspace decomposition of V with respect to L_0 is of the form

$$V = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_+} V_j, \text{ where } \dim V_j < \infty \text{ and } V_0 = \mathbb{C}\mathbf{1}.$$

Continuation of Example a). $F^1(A)$ is a VOA with Virasoro vector

$$L = \frac{1}{2} \sum_i :(T\varphi_i)\varphi^i:,$$

where $\{\varphi_i\}$ and $\{\varphi^i\}$ are dual bases of A, the central charge being $-\frac{1}{2}\,\mathrm{sdim}\, A$.

Continuation of Example b). The universal Virasoro vertex algebra V^c and its simple quotient are VOA's with Virasoro vector L of central charge c.

Example c) Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, and normalize the invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ by the condition $(\alpha|\alpha) = 2 \quad$ if α is a long root.

The current Lie conformal algebra is

$\mathrm{Cur}\,\mathfrak{g} = (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus \mathbb{C}K$

with the λ-brackets

$[a_\lambda b] = [a, b] + \lambda(a|b)K,$ for $a, b \in \mathfrak{g}, K$ central.

Its universal enveloping vertex algebra of level k is called the universal affine VA of level k, and is denoted by $V^k(\mathfrak{g})$.

If $k \neq -h^\vee$, where $h^\vee$ = dual Coxeter number ($\frac{1}{2}$ eigenvalue of the Casimir operator on $\mathfrak{g}$), then $V^k(\mathfrak{g})$ has a unique simple quotient $V_k(\mathfrak{g})$.

The corresponding to Cur $\mathfrak{g}$ Lie algebra $Lie_{Cur\mathfrak{g}}$ is the affine Lie algebra $\widehat{\mathfrak{g}} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K$ with the brackets $[a_{(m)}, b_{(n)}] = [a, b]_{(m+n)} + \delta_{m,-n} (a|b)\ K$, where $a_{(m)} = t^m \otimes\ a,\ a \in \mathfrak{g}, K$ central.

The vertex algebras $V^k(\mathfrak{g})$ and $V_k(\mathfrak{g})$ with the Virasoro vector

$$L_{Sug} = \frac{1}{2(k+h^\vee)} \sum_i :a_i a^i:,$$

where $\{a_i\}$ and $\{a^i\}$ are dual bases of $\mathfrak{g}$, are VOA's, provided that $k \neq -h^\vee$, of central charge $c_k(\mathfrak{g}) = \frac{k \dim \mathfrak{g}}{k+h^\vee}$.

Example d) Let $\mathfrak{s} = \langle e, 2x, f\rangle$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, and let

$\mathfrak{g} = \bigoplus\limits_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ be the (ad x)-eigenspace decomposition.

The corresponding

Quantum Hamiltonian Reduction (QHR) is constructed as follows ([FF90], principal $\mathfrak{s}$; [KRW03] arbitrary $\mathfrak{s}$).

Consider the following vertex algebra ($k \in \mathbb{C}$):

$$C^k(\mathfrak{g}, \mathfrak{s}) = V^k(\mathfrak{g}) \otimes \underbrace{F^{\mathrm{ch}} \otimes F^{\mathrm{ne}}}_{\text{ghosts}},$$

where $F^{\mathrm{ch}} = F^1(\Pi(\mathfrak{g}_{<0} + \mathfrak{g}_{>0}))$, $F^{ne} = F^1(\mathfrak{g}_{\frac{1}{2}})$,

the corresponding skew-symmetric bilinear forms being the pairing between $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{>0}$ given by $(\cdot|\cdot)$, and $< a, b >^{ne} = (f|[a, b])$ on $\mathfrak{g}_{\frac{1}{2}}$, respectively.

Define $\mathbb{Z}-$ grading $C^k(\mathfrak{g},\mathfrak{s}) = \bigoplus\limits_{j\in\mathbb{Z}} C^k_j$ by

$$\deg V^k(\mathfrak{g}) = 0 = \deg F^{\mathrm{ne}}, \deg \mathfrak{g}_{>0} = -\deg \mathfrak{g}_{<0} = 1,$$

and let

$$d = \sum_{j\in S_{>0}} (:\varphi^j u_j: + (f|u_j)\varphi^j) + \sum_{j\in S_{\frac{1}{2}}} :\varphi^j \Phi_j: \\ + \frac{1}{2}\sum_{i,j\in S_{>0}} :\varphi^i\varphi^j\varphi_{[u_j,u_i]}:,$$

where $\{u_j\}_{j\in S_{>0}}$ is a basis of $\mathfrak{g}_{>\mathfrak{o}}$, compatible with ad x-grading, $\{u^j\}_{j\in S_{>0}}$ is its dual basis of $\mathfrak{g}_{<0}$, $\{\varphi_j\}$ and $\{\varphi^j\}$ are the corresponding bases of $\Pi\mathfrak{g}_{>0}$ and $\Pi\mathfrak{g}_{<0}$, and $\{\Phi_j\}_{j\in S_{\frac{1}{2}}}$ is the corresponding basis of $\mathfrak{g}_{\frac{1}{2}}$.

The element d is an odd element of the vertex algebra $C^k(\mathfrak{g},\mathfrak{s})$ of degree -1, and a direct calculation gives:

$$[d_\lambda d] = 0.$$

In particular $d_{(0)}d = 0$, hence $d_{(0)}^2 = 0$. But for any a in VA, $a_{(0)}$ is its derivation, hence $(C^k(\mathfrak{g}, \mathfrak{s}), d_{(0)})$ is a homology complex, whose homology is a vertex algebra.

The 0^{th} homology of this complex, denoted by $W^k(\mathfrak{g}, \mathfrak{s})$, is called the universal quantum affine W-algebra, associated to $(\mathfrak{g}, \mathfrak{s}, k)$. If $k \neq -h^\vee$, this vertex algebra has a unique maximal ideal, and the quotient by this ideal is the simple quantum affine W-algebra, denoted by $W_k(\mathfrak{g}, \mathfrak{s})$.

Theorem 1 [KRWO3], [KW04]. Assume that $k \neq -h^\vee$.

(a) The vertex algebra $C^k(\mathfrak{g}, \mathfrak{s})$ is a VOA with the Virasoro element

$L = L_{Sug} + Tx + L^{ch} + L^{ne}$,

of central charge

$c_k(\mathfrak{g}, x) = \dim \mathfrak{g}_0 - \frac{1}{2} \dim \mathfrak{g}_{\frac{1}{2}} - \frac{12}{k+h^\vee} |\rho - (k + h^\vee)x|^2$.

Also $d_{(0)}L = 0$, hence $W^k(\mathfrak{g}, \mathfrak{s})$ and $W_k(\mathfrak{g}, \mathfrak{s})$ are VOA's of central charge $c_k(\mathfrak{g}, x)$.

(b) For $a \in \mathfrak{g}$ let

$J^{(a)} = a + \sum_{j \in S_{>0}} : \varphi^j \varphi_{[u_j, a]} :\in C^k(\mathfrak{g}, \mathfrak{s})$.

Then for each $a \in \mathfrak{g}_{-\mathfrak{j}}^{\{\mathfrak{f}\}}$ with $j \geq 0$, there exists a $d_{(0)}$-closed element $J^{\{a\}} \in C^k(\mathfrak{g}, \mathfrak{s})$ of conformal weight $1 + j$, such that $J^{\{a\}} - J^{(a)}$ is a linear combination of normally ordered products of the elements $J^{(b)}$ with $b \in \mathfrak{g}_{-s}$, $0 \leq s < j$, the elements $\Phi_j, j \in S_{\frac{1}{2}}$, and of their derivatives (i.e., their images under powers of T).

(c) The homology classes of the elements $J^{\{a_i\}}$, where $\{a_i\}$ is a basis of $\mathfrak{g}^f$ consisting of eigenvectors of ad x, strongly generate the VOA $W^k(\mathfrak{g}, \mathfrak{s})$ and obey the PBW theorem.

(d) $H_j(C^k(\mathfrak{g}, \mathfrak{s}), d_{(0)}) = 0 \quad \text{if } j \neq 0.$

Remark 3. The simplest W-algebra $W^k(s\ell_2, s\ell_2)$ is isomorphic to the Virasoro vertex algebra V^c with $c = 1 - 6\frac{(k+1)^2}{k+2}$.

The λ-brackets between generators are known for minimal W-algebras $W^k(\mathfrak{g}, \mathfrak{s})$, where $f \in \mathfrak{s}$ has the adjoint orbit in $\mathfrak{g}$ of minimal non-zero dimension. This VOA is denoted by $W^k_{min}(\mathfrak{g})$, and its simple quotient by $W_k^{min}(\mathfrak{g})$.

§3 Conjectures on modular invariant VOA's.

Let V be a VOA with a Virasoro vector L of central charge c, and let

$$q = e^{2ni\tau}, \text{ where } \tau \in \mathbb{C}, Im\, \tau > 0.$$

Conjecture 1. Assume that V is of strong CFT type, i.e. for $v \in V$ of L_0-eigenvalue 1, one has $L_1 v = 0$.

If the series $tr_V q^{L_0 - a}$, with $a \in \mathbb{C}$, converges to a modular function, then $a = \frac{c}{24}$.

Definition. A VOA V is called modular invariant if the series (normalized character of V)

$$ch_V(\tau) = tr_V q^{L_0 - \frac{c}{24}}$$

converges to a modular function on a congruence subgroup of $SL_2(\mathbb{Z})$.

By [Zhu96], any rational VOA with the spectrum of L_0 in $\mathbb{Z}$, is modular invariant.

[DR18],[KW25] Conjecture 2. If V is a modular invariant VOA, which is lisse, i.e. the Poisson algebra $P(V) = V/(V_{(-2)}V)$ is finite-dimensional, then V is rational. (The structure of a Poisson algebra on $P(V)$ is given by $ab = a_{(-1)}b$ and $\{a, b\} = a_{(0)}b$.)

[KW25] Conjecture 3. A modular invariant VOA is simple.

Example a) $V = F^1(A)$, where A is a finite-dimensional superspace. This VOA is modular invariant if and only if A is purely odd vector superspace.

Then $V \simeq F^1(\Pi\mathbb{C})^{\otimes d}$, where $d = \dim A$, so the problem reduces to the VOA $F = F^1(\Pi\mathbb{C})$, one fermion. Then we have:

$$ch_F(\tau) = q^{-\frac{1}{48}} \prod_{n\in\mathbb{Z}_{\geq 0}} (1 + q^{\,n+\frac{1}{2}}) = \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)},$$

$$sch_F(\tau) = q^{-\frac{1}{48}} \prod_{n\in\mathbb{Z}_{\geq 0}} (1 - q^{\,n+\frac{1}{2}}) = \frac{\eta(\frac{\tau}{2})}{\eta(\tau)},$$

$$ch_{F_R}(\tau) = q^{\frac{1}{24}} \prod_{n\in\mathbb{Z}_{\geq 0}} (1 + q^{\,n+1}) = \frac{\eta(2\tau)}{\eta(\tau)}.$$

Hence $F^{\otimes d}$ is a modular invariant VOA and , moreover, d-th powers of these three modular functions span an $SL_2(\mathbb{Z})$-invariant space, in agreement with [v E 13] , which generalizes [Zhu 96] [DLM98].

Example b) $V = V_c$, simple Virasoro VOA of central charge c, is modular invariant if and only if V^c is not simple, i.e.
$c = 1 - \frac{6(p-q)^2}{pq}, p, q \in \mathbb{Z}, p > q \geq 2, gcd(p, q) = 1.$
All these VOA's are rational, even regular.

Remark 4. The triplet VOA, whose central charge is the above c with q = 1, is lisse, but not modular invariant, according to [AM95], hence not rational.

Example c) $V = V_k(\mathfrak{g}), k \neq -h^\vee$, simple affine VOA. There are two series of levels k, for which $V_k(\mathfrak{g})$ is modular invariant:

(principal admissible k) $k = -h^\vee + \frac{p}{q}$, where p and q are coprime positive integers, q is coprime with lacety $r^\vee$ of $\mathfrak{g}$, and $p \geq h^\vee$;

(subprincipal admissible k) $k = -h^\vee + \frac{p}{q}$, where p and q are coprime positive intergers, q is divisible by $r^\vee > 1$, and $p \geq h$, the Coxeter number of $\mathfrak{g}$.

Formulas for normalized characters of these VOA for k principal admissible:

$$ch_{V_k(\mathfrak{g})}(\tau) = \sum_{\gamma \in pQ^\vee} d(\gamma) q^{|\rho+\gamma|^2/2(k+h^\vee)} / \eta(\tau)^{\dim \mathfrak{g}},$$

where

$$d(\gamma) = \prod_{\alpha\ \in \Delta_+} \frac{(\gamma + \rho|\alpha)}{(\rho|\alpha)}.$$

For k subprincipal admissible the same formula holds with $Q^\vee$ (co-root lattice) replaced by Q (root lattice).

[KW25] Conjecture 4. There are no other modular invariant simple affine VOA.

However, there are some very interesting examples that come close ("Deligne series"), for which $ch_{V_k(\mathfrak{g})}$ is a ratio of a quasi-modular form $\Omega(\tau)$ by $\eta(\tau)^{\dim \mathfrak{g}}$.

Namely, for $\mathfrak{g}$ of type $D_n (n \geq 4)$, and E_6, E_7, E_8, let $b = 2$, and $= \frac{h^\vee}{6} + 1$ respectively. Then for any negative integer $k \geq -b$ there exists a unique positive root α, such that $(\rho|\alpha) = h^\vee + k := p$.

Then one can show that

(!) $ch_{V_k(\mathfrak{g})}(\tau) = \frac{1}{2p} \sum\limits_{\gamma \in p\,Q} (\alpha|\gamma)\, d(\gamma)\, q^{\frac{|\rho+\gamma|^2}{2p}} / \eta(\tau)^{\dim \mathfrak{g}}$.

The proof is based on the following observations.

Let $\hat{\mathfrak{g}}$ be an affine Lie algebra and let Λ be a weight of level $k > -h^\vee$. Then

(*) $\hat{R}\, chL(\Lambda) = \sum_{w \in \hat{W}} c_\Lambda(w) e^{w(\Lambda+\hat{\rho})}$.

If, in addition, Λ is a quasi-dominant integral weight (i.e. its restriction to $\mathfrak{g}$ is dominant integral), then

(**) $c_\Lambda(ut_\gamma) = \epsilon(u) c_\Lambda(t_\gamma)$ for $u \in W, \gamma \in Q^\vee$.

Proposition. [KW18]. Let $\hat{\mathfrak{g}}$ and $k \in \mathbb{Z}_{<0}$ be as above, let Λ be a quasi-dominant integral weight of level k, for which there exists a root α of $\mathfrak{g}$, such that $(\Lambda + \hat{\rho}, \delta - \alpha) = 0$, and any $\beta \in \hat{\Delta}_+$, orthogonal to $\Lambda + \hat{\rho}$, is equal to $\delta - \alpha$. Then, assuming that,

(***) $c_\Lambda(t_\gamma)$ is a linear function in $\gamma \in Q^\vee$, we have

(****) $c_\Lambda(t_\gamma) = \frac{1}{2}(\alpha, \gamma)$, $\gamma \in Q^\vee$.

Using (**) and (****), provided that (***) holds, and specializing the resulting formula (*), it is not difficult to deduce the character formula (!). This is the easy part of the proof.

Of course, the constants $c_\Lambda(t_\gamma)$ can be computed inductively on $\ell(t_\gamma)$ via Kazdan-Lusztig polynomials [KT1998], but this will lead to very complicated expressions, and would not help in proving (****).

In some cases, including $\Lambda = k\Lambda_0$, this formula was proved in [BKK24], using the description of the corresponding module over the affine Hecke algebra in terms of the equivariant K-theory of the Springer resolution, which is the hard part of the proof of (!).

This leads to the following open problem:

For which Λ the coefficients $c_\Lambda(t_\gamma)$ are bounded (resp. linear) functions in $\gamma \in Q^\vee$? For dominant integral, and, more generally, admissible Λ, the answer is trivial:

$$c_\Lambda(t_\gamma) = 1 \text{ for all } \gamma \in Q^\vee.$$

One other example: for $L(-\Lambda_0)$ over $\hat{s\ell}_n$ and $\hat{sp}_n$, one has an explicit character formula [KW18], which implies that

$$c_{-\Lambda_0}(t_\gamma) = 1 \textit{ or } 0.$$

§4 Modular and quasi-modular forms and functions

Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup, commensurable with $SL_2(\mathbb{Z})$, and let w and m be non-negative integers. A holomorphic function $f(\tau)$, $Im\,\tau > 0$, is called a quasi-modular form of weight w and depth $\leq m$ on Γ if there exist holomorphic functions $f_1, \ldots, f_m$, such that

$$\frac{1}{(c\tau+d)^w} f\left(\frac{a\tau+b}{c\tau+d}\right) = f_0(\tau) + f_1(\tau)\frac{c}{c\tau+d} + \cdots + f_m(\tau)\left(\frac{c}{c\tau+d}\right)^m$$

for all τ with $Im\,\tau > 0$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and LHS is bounded for $Im\,\tau \to \infty$.

Example 1 If $m = 0$, a quasi-modular form $f(\tau)$ is a modular form of weight w, and $f'(\tau)$ is a quasi-modular form of weight $w+2$ and depth ≤ 1:

$$\frac{1}{(c\tau+d)^{w+2}} f'\left(\frac{a\tau+b}{c\tau+d}\right) = f'(\tau) + w\frac{c}{c\tau+d} f(\tau).$$

Example 2 (Eisenstein series). Let $G_{2k}(\tau) = \sum'_{m,n\in\mathbb{Z}} \frac{1}{(m\tau+n)^{2k}}$, $k = 1, 2, \ldots$ Usually this function is normalized by a constant factor to make the constant term equal 1, to get

$$E_{2k}(\tau) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n\geq 1} \sigma_{2k-1}(n) q^n,$$

where B_k are Bernoulli numbers ($B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, \ldots$), and $\sigma_j(n) = \sum_{d|n} d^j$. The functions $E_{2k}(\tau)$ with $k > 1$ are holomorphic modular forms on $SL_2(\mathbb{Z})$ of weight $2k$. However,

$$\frac{1}{(c\tau+d)^2} E_2\left(\frac{a\tau+b}{c\tau+d}\right) = E_2(\tau) + \frac{6}{i\pi}\frac{c}{c\tau+d}\ ,$$

hence $E_2(\tau)$ is a quasi-modular form of weight 2 and depth ≤ 1.

Theorem (a) $\mathbb{C}[E_4, E_6, E_8, \dots] = \mathbb{C}[E_4, E_6]$ is the space of all holomorphic modular forms on $SL_2(\mathbb{Z})$.

(b) (Kaneko-Zagier) $\mathbb{C}[E_2, E_4, E_6]$ is the algebra of all holomorphic quasi-modular forms on $SL_2(\mathbb{Z})$. If coincides with $\displaystyle\sum_{m\geq 0}\left(\frac{d}{d\tau}\right)^m \mathbb{C}[E_4, E_6]$.

(c) In the definition of a quasi-modular form f, the functions f_k, are uniquely defined quasi-modular forms of weight $w-2k$ and depth $\leq m-k$, and $f_0 = f$.

Definition The modular (resp. quasi-modular) function is a function of the form

$$f(\tau) = \frac{\Omega(\tau)}{\phi(\tau)},$$

where $\Omega(\tau)$ is a holomorphic modular (resp. quasi-modular) form of weight w and $\phi(\tau)$ is a nowhere vanishing modular function of the same weight.

Remark 5 For $k = -b$, completely different formulas have been found in [AK 18] by making use of modular differential equations:

$ch_{V_{-2}(D_4)}(\tau) = \frac{E_4'(\tau)}{240\eta(\tau)^{10}}$;

$ch_{V_{-3}(E_6)}(\tau) = -\frac{1}{462}\left(\frac{E_6(\tau)E_4'(\tau)}{240\,\eta(\tau)^{22}} - \eta(\tau)\right)$;

$ch_{V_{-4}(E_7)}(\tau) = \frac{\eta(\tau)^{24}}{204204}\left(P_2\left(\frac{E_6(\tau)}{\eta(\tau)^{12}}\right)\frac{E_4'(\tau)}{240\eta(\tau)^{34}} - \frac{E_6(\tau)}{\eta(\tau)^{34}}\right)$;

where $P_2(x) = x^2 + 462$;

$ch_{V_{-6}(E_8)}(\tau) =$ very complicated.

Conjecture 5. If V is a VOA, for which $ch_V(\tau)$ is a modular or quasi-modular function, then V is quasi-lisse (i.e. the Poisson algebra $P(V)$ has finitely many symplectic leaves [AK18]).

Conjecture (Dražen) If $\mathfrak{g} = D_n (n \geq 4)$, or $E_{6,7,8}$, and k is a non-negative integer, then $W_k^{\min}(\mathfrak{g})$ is not rational (though it is lisse [AM18]), in a sharp contrast to the AKC conjecture (stated below).

Open Problem. Find all $k \in \mathbb{C}$, for which $ch_{V_k(\mathfrak{g})}(\tau)$ is a quasi-modular function.

Remark 6. If $\sum\limits_{\gamma \in Q} c(\gamma) q^{|\gamma+\lambda|^2/2p}$ is a modular form on Γ, then $\sum\limits_{\gamma \in Q} (\alpha, \gamma) c(\gamma) q^{|\gamma+\lambda|^2/2p}$ is a quasi-modular form on Γ for any $\alpha \in \mathbb{Q}Q$.

Example d) Let M be a $V^k(\mathfrak{g})$-module, then

$M \otimes F^{ch} \otimes F^{ne}$ is a $C^k(\mathfrak{g}, \mathfrak{s})$-module, and we get the homology complex $(M \otimes F^{ch} \otimes F^{ne}, d_{(0)})$.

Its homology $H(M)$ is a module over the W-algebra $W^k(\mathfrak{g}, \mathfrak{s})$. By [A06], this module is either irreducible or 0, if M is irreducible , provided that $\mathfrak{s}$ is the minimal $s\ell_2$-subalgebra of $\mathfrak{g}$.

For arbitrary $\mathfrak{s}$ this is still an open problem.

In the special case of $M = L(k\Lambda_0)$, the homology $H(L(k\Lambda)_0)$ is either 0 or a quotient of the W-algebra $W^k(\mathfrak{g}, \mathfrak{s})$, which is a VOA, denoted by $\widetilde{W}_k(\mathfrak{g}, \mathfrak{s})$. If $\mathfrak{s}$ is minimal in $\mathfrak{g}$, then, by [A06] , $\widetilde{W}_k(\mathfrak{g}, \mathfrak{s}) = W_k^{\min}(\mathfrak{g})$ is simple if k is not in $\mathbb{Z}_+$.

[KW25] Theorem 2

(a) The simple W-algebra $W_k^{min}(\mathfrak{g})$ for $\mathfrak{g} = D_n (n \geq 4)$ or $E_{6,7,8}$, and k a negative integer ≥ -2 or $\geq -\frac{h^\vee}{6} - 1 = -b$ (b is the length of the longest leg in the extended Dynkin diagram), respectively, is a modular invariant VOA. Since it is also lisse by [AM 18], Conjecture 2 implies that this W-algebra is rational ([ACK24] conjecture).

(b) Let $k = -h^\vee + \frac{p}{q}$ be an admissible level.

Then $\tilde{W}_k(\mathfrak{g}, \mathfrak{s})$ is a modular invariant vertex algebra if $u > \theta(x)$ for k principal admissible, and $q > \theta_s(x)$ if k is subprincipal admissible, where θ and θ_s denote the highest and the highest short root respectively, and 0 for all other admissible k.

Consequently, by Conjecture 3, $\widetilde{W}_k(\mathfrak{g}, \mathfrak{s}) = W_k(\mathfrak{g}, \mathfrak{s})$ for these k (and 0 for all other admissible k [A15], [KW25]).

Open problem. For which k the VOA $W_k(\mathfrak{g}, \mathfrak{s})$ is modular invariant?

Conjecture 6. Quantum Hamiltonian reduction of a quasi-modular invariant affine VOA $V_k(\mathfrak{g})$ is either 0 or a modular invariant W-algebra.