Analytic Approximations for Fermionic Preheating

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Based on ArXiv: 2604.07326 (submitted to PRD)
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Light Dark World 2026

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Introduction

- Could dark matter be made up of a sea of cold degenerate fermions? - A promising candidate!

  M. Carena, N. M. Coyle, Y.-Y. Li, S. D. McDermott, and Y. Tsai, 2108.02785

- How can such particles be produced? - From a coherently oscillating scalar field like the inflaton, producing particles out of thermal equilibrium in the early universe.

  P. B. Greene and L. Kofman, hep-ph/9807339, hep-ph/0003018;

  G. F. Giudice, M. Peloso, A. Riotto, and I. Tkachev, hep-ph/9905242; J. Garcia-Bellido, S. Mollerach, and E. Roulet, hep-ph/0002076

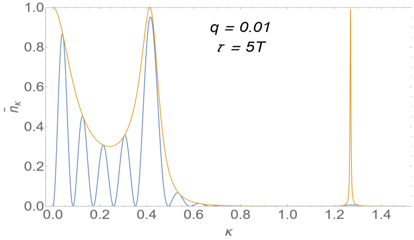
Production of Fermions during Preheating

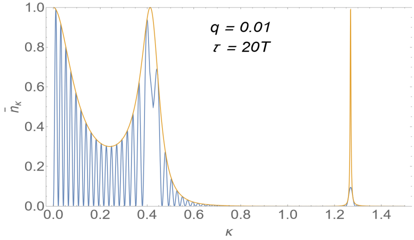
time

momentum

$$\mathcal{L} \supset -\frac{\lambda}{4}\,\phi^4 - h\,\phi\,\overline{\psi}\,\psi\;; \qquad \boxed{q = h^2/\lambda}, \qquad \Omega_\kappa(q,\ \tau) = \sqrt{\kappa^2 + q\,\varphi^2(\tau)}$$

coherently oscillating scalar (inflaton), freq. $\omega_\varphi = 2\pi/T$

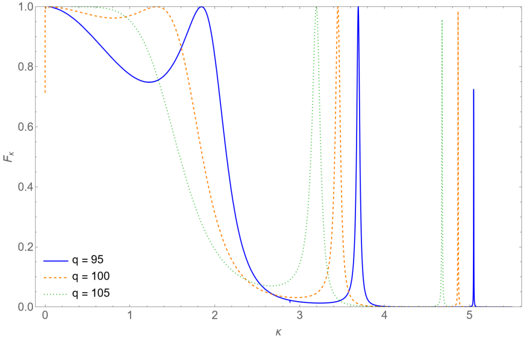




$$\bar{n}_\kappa(\tau) \overset{\tau \gg T}{\longrightarrow} \langle F_\kappa \sin^2(\nu_\kappa \tau)\rangle = F_\kappa/2$$

Filling of κ Modes

F_κ is not fully degenerate! Greene, Kofman (1999, 2000)



'Bulk region' for small κ, resonance peaks for higher κ.

Peak positions evolve with q.

Outline

This work:

1. Can we predict the locations of the resonance peaks?
2. Can we get simple approximations for the total number density?
3. What are the implications for degenerate dark matter?

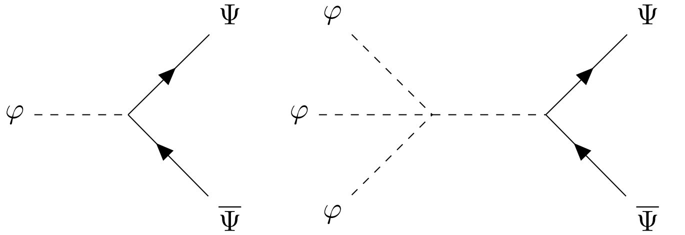
Resonance Peaks for Small $q = h^2/\lambda$

For $q \ll 1$, resonance peaks are observed at: Greene, Kofman (1999)

$$\kappa = \frac{\pi}{T}(2l+1), \text{ where } l = 0, 1, 2, 3, \ldots$$

Energy-conservation in $n\varphi \to \Psi\overline{\Psi}$:

$$\int d^4x f(\tau)^n e^{i(\kappa^\mu_\Psi + \kappa^\mu_{\bar{\Psi}})x_\mu} \sim \delta^{(3)}\left(\boldsymbol{\kappa}_\Psi + \boldsymbol{\kappa}_{\overline{\Psi}}\right) \int d\tau e^{-i\boxed{n\omega_\varphi}\tau} e^{i\boxed{2\kappa}\tau}$$

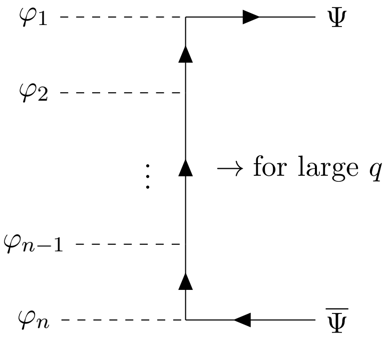


n must be odd

Resonance Peaks for All $q = h^2/\lambda$

$$\overline{\Omega}_\kappa(q) \equiv \frac{1}{T}\int_0^T d\tau \boxed{\sqrt{\kappa^2 + q f^2(\tau)}} \to \Omega_\kappa(q,\tau),$$

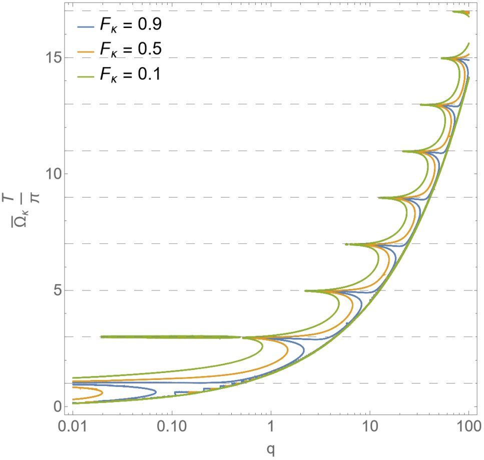
$\overline{\Omega}_\kappa(q) \approx \kappa$ for small q. Accumulated phase in $\Psi\overline{\Psi}$ pair: $2\overline{\Omega}_\kappa(q)\tau$



$$\therefore \quad n\omega_\varphi = 2\overline{\Omega}_\kappa(q)$$
$$\Longrightarrow \overline{\Omega}_\kappa(q) = \frac{\pi}{T} n.$$

Amplitude is zero for even n from Dirac eq. for $m_\psi \approx 0$.

Resonance Peaks for All $q = h^2/\lambda$



$$\boxed{\overline{\Omega}_\kappa(q) = \frac{\pi}{T}(2l+1),}$$

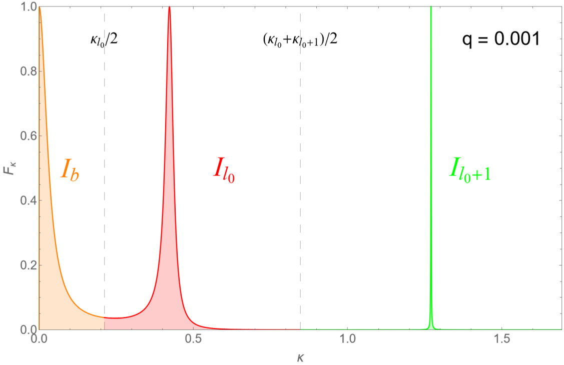
where:

$$l = l_0,\ l_0+1,\ l_0+2,\ldots$$

Total Number Density

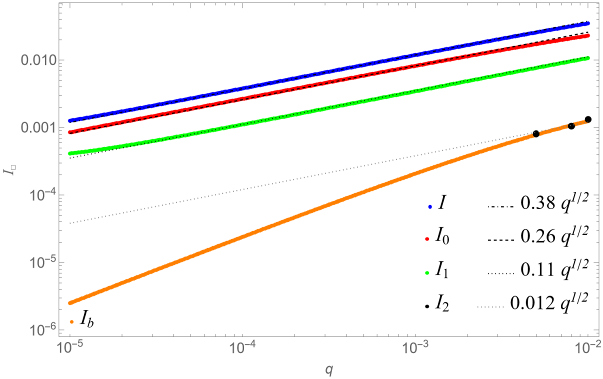
$$a^3 n \propto \int_0^\infty d\kappa \kappa^2 \frac{F_\kappa}{2} = \frac{I\ (q)}{2}$$

approximations of parts



Small $q = h^2/\lambda$

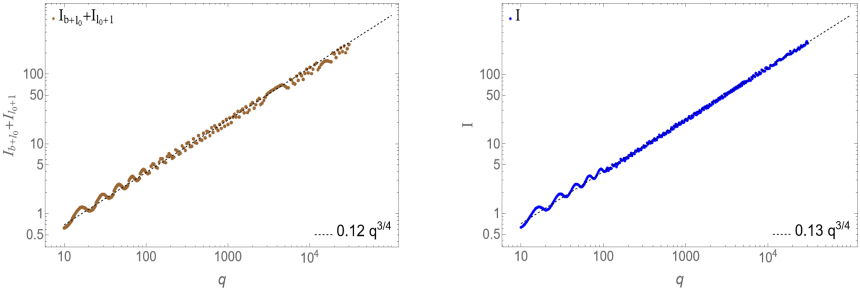
$l_0 = 0$ for all small q values; $q^{1/2}$ power-laws:



95% is made up of $l = 0$ ($\sim 60\%$) and $l = 1$ ($\sim 35\%$) shells.

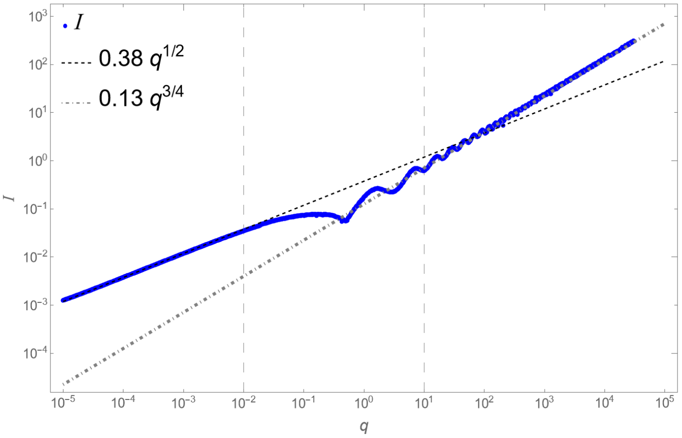
Large $q = h^2/\lambda$

$q^{3/4}$ power-laws (as expected from bulk domination, Greene, Kofman (1999, 2000)):



About 90% is from bulk + first two peaks.

All $q = h^2/\lambda$



Degenerate Fermion Dark Matter

Zero temperature degenerate fermions are relativistic for

$$p_F \gtrsim m_\psi$$

If sufficiently light, some of these fermions are relativistic at the time of structure formation, which sets a bound on their mass:

$$m_\psi > 2 \text{ keV} \quad \text{(Majorana fermions, } N_f = 1\text{)}$$

M. Carena, N. M. Coyle, Y.-Y. Li, S. D. McDermott, and Y. Tsai, 2108.02785

We adapt this limit for our case of non-perturbatively produced Dirac fermions, $N_f = 2$.

Dark Matter Mass Bound for Small $q = h^2/\lambda$

Taking constraints from first peak at $\kappa_0 = \pi/T$:

$$m_\psi > 2\ \text{keV} \left(\frac{2\kappa_0^3}{3N_f I_0(q)}\right)^{1/4} \approx 2\ \text{keV} \left(\frac{0.56}{q^{1/8}}\right)$$

Taking constraints from second peak at $\kappa_1 = 3\pi/T$:

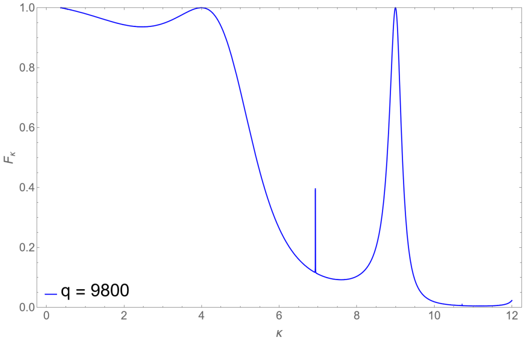
$$m_\psi > 2\ \text{keV} \left(\frac{2\kappa_1^3}{3N_f I(q)}\right)^{1/4} \approx 2\ \text{keV} \left(\frac{1.2}{q^{1/8}}\right)$$

$$\therefore \quad q = 10^{-5} \implies m_\psi \gtrsim 5 - 10\ \text{keV}$$

$$\therefore \quad q = 0.01 \implies m_\psi \gtrsim 2 - 4\ \text{keV}$$

Dark Matter Mass Bound for Large $q = h^2/\lambda$

$$\bar{n}_\kappa(\tau) \overset{\tau \gg T}{\rightarrow} F_\kappa/2$$



Approx. half-filled Fermi sphere, $N_f = 2 \implies m_\psi \gtrsim 2$ keV unchanged

Conclusions and Results

- ⋆ Fermionic dark matter produced in preheating;
  small q: resonant momentum shells,
  large q: approximately half-filled Fermi sphere.
- ⋆ Formula for locations of resonance peaks (essentially from energy conservation).
- ⋆ Power-law fits for the total number density of fermions:
  $$\propto q^{1/2} \text{ for } q \lesssim 0.01, \qquad \propto q^{3/4} \text{ for } q \gtrsim 10.$$
- ⋆ Applied lower bounds on mass of fermionic dark matter.

Back-Up Slides

Back-up Slide, Back-reaction

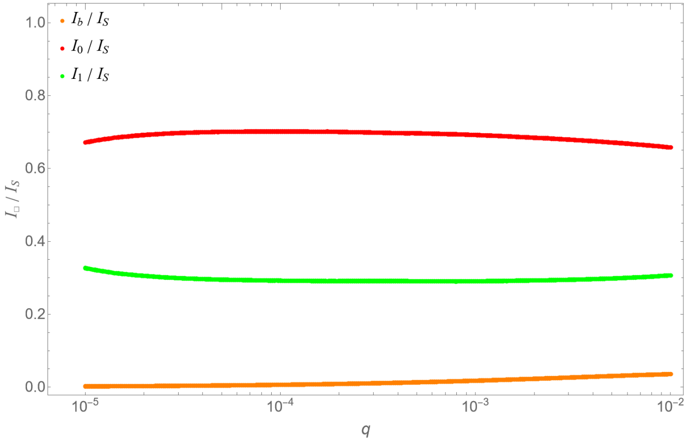
Ratio of energy in fermions compared to inflaton:

$$\frac{\rho_\psi}{\rho_\varphi} = \frac{4\lambda}{\pi^2} \int d\kappa \kappa^2 \overline{\Omega}_\kappa F_\kappa \approx (4.0 \cdot 10^{-15}) \times \left(\frac{\lambda}{10^{-13}} \right) \times q^{1.2}.$$

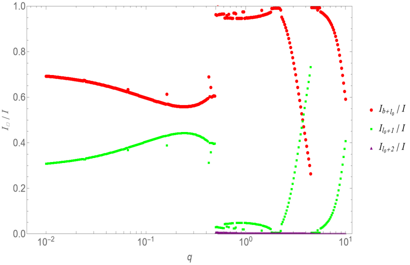
Less than 1% of the energy of the inflaton is transferred into the fermions for $q \lesssim 10^{10}$, assuming $\lambda = 10^{-13}$.

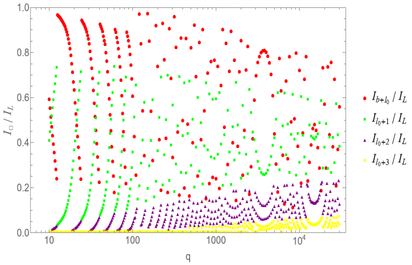
We focus on much smaller q values in our analysis, justifying neglecting back-reaction.

Back-up Slide, Small q Fractions

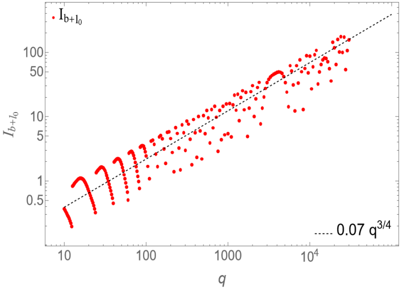


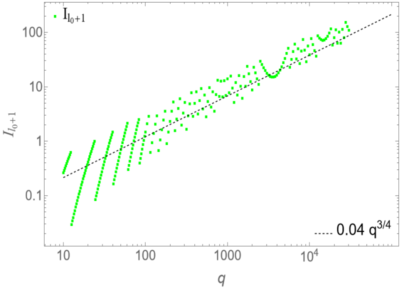
Back-up Slide, Larger q Fractions





Back-up Slide, Large q Approx.

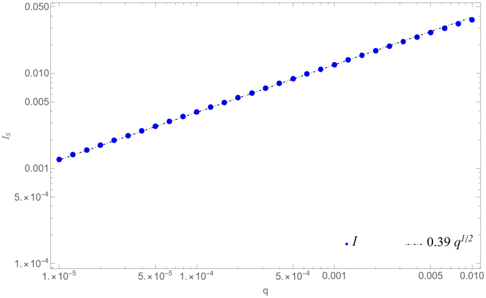




Back-up Slide, with $m^2\phi^2$ Potential

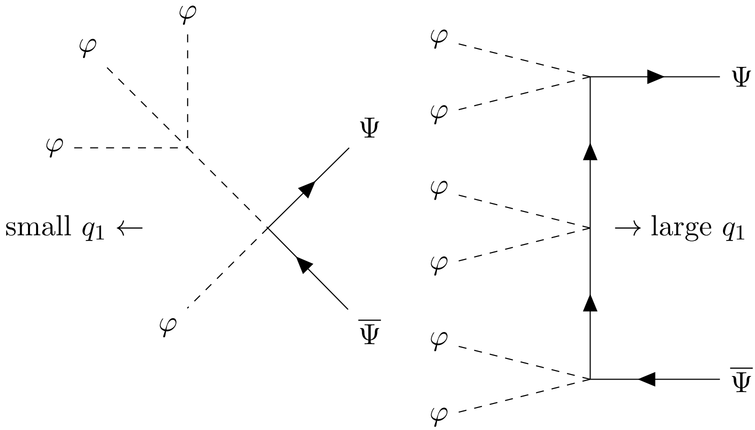
Identical fit for small $q = h^2\varphi_0^2/m^2$, almost all from $l = 0$ peak:

$$I_S(q) = \int_0^{(\kappa_1+\kappa_2)/2} d\kappa\kappa^2 F_\kappa \approx 0.39 \times q^{1/2}.$$



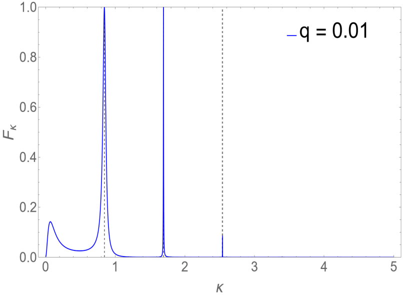
Leads to same dark matter mass bounds as κ_0 case for $\lambda\phi^4$.

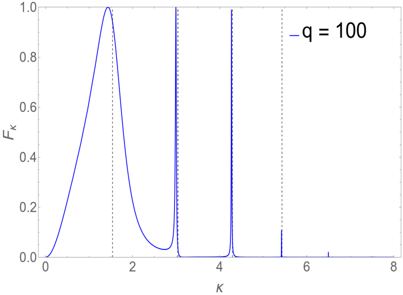
Back-up Slide, with a $\phi\phi\overline{\psi}\psi$ Coupling



$q_1 = h_1^2\varphi_0^2/\lambda;\qquad \overline{\Omega}_\kappa(q_1) = (\pi/T)(2l)$, where $l = l_0, \ldots$

Back-up Slide, with a $\phi\phi\overline{\psi}\psi$ Coupling





Back-up Slide, $\lambda\phi^4$ Inflaton Oscillations

$$\phi''(t) + 3\frac{a'(t)}{a(t)}\phi'(t) + \lambda\phi^3 = 0$$

Oscillations of the inflaton field independent of the initial amplitude. Conformal variables i.e. $\varphi = a\phi$, $\eta = \int dt/a(t)$ and then rescale i.e. $f(\tau) = \varphi(\tau)/\varphi_0$, $\tau = \sqrt{\lambda}\varphi_0\eta$:

$$\varphi'' + \lambda\varphi^3 = 0 \rightarrow \ddot{f} + f^3 = 0$$

Solution is a Jacobi Elliptic Cosine:

$$f(\tau) = cn\left(\tau - \tau_1, m\right); m = 1/2$$

$\tau_1 = 0$ to ensure that $f(\tau)$ is symmetric around $\tau = 0$.
$f(\tau)$ is periodic with time period $T \approx 7.4163$.

Back-up Slide, Inflation Oscillations

The general $cn(u, m)$ function can be represented as a series expansion. For the case of $m = 1/2$, it is written as:

$$cn(u, 1/2) = \frac{8\pi\sqrt{2}}{T} \sum_{n=1}^{\infty} \frac{e^{-\pi(n-1/2)}}{1 + e^{-2\pi(n-1/2)}} \cos\left(\frac{2\pi(2n-1)u}{T}\right).$$

The period for $m = 1/2$ given by an elliptic integral as:

$$T = 4 \int_0^{\pi/2} \frac{d\theta}{\left(1 - \frac{1}{2}\sin^2\theta\right)^{1/2}} \approx 7.4163.$$

Back-up Slide, Mode Equation

Dirac equation:

$$\left(i\gamma^{\mu}\partial_{\mu} - h\varphi(\eta) - m_{\psi}a(\eta)\right)\Psi = 0.$$

Ansatz to solve the Dirac equation:

$$\Psi = \left(i\gamma^{\mu}\partial_{\mu} + h\varphi(\eta) + m_{\psi}a(\eta)\right)X(\mathbf{x}, \eta).$$

Using above, model-independent mode equation:

$$X_k''(\eta) + \left[k^2 + \left(h\varphi(\eta) + m_{\psi}a(\eta)\right)^2 - i\left(h\varphi'(\eta) + m_{\psi}a'(\eta)\right)\right]X_k(\eta) = 0.$$

Transformations to dimensionless variables:

$$\kappa^2 = \frac{k^2}{\lambda\varphi_0^2}, \quad q = \frac{h^2}{\lambda}, \quad \tau = (\sqrt{\lambda}\varphi_0)\eta, \quad X_\kappa = (\sqrt{\lambda}\varphi_0)X_k,$$

$$f(\tau) = \varphi(\tau)/\varphi_0, \quad \Omega_\kappa(\tau) = \Omega_k(\eta)/\sqrt{\lambda}\varphi_0.$$

Back-up Slide, Diagonalizing Hamiltonian

Hamiltonian:

$$H_D = \int d^3x \left(i\Psi^\dagger \partial_\eta \Psi \right).$$

$$\Psi(\eta, \mathbf{x}) = \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3} \left(\hat{a}_{k,s} u_{k,s}(\eta) e^{+i\mathbf{k}\cdot\mathbf{x}} + \hat{b}^\dagger_{k,s} v_{k,s}(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right).$$

Bogoliubov transformation:

$$\hat{a}_{k,+} = \alpha_1^* \hat{c}_{k,+} - \beta_1 \hat{d}^\dagger_{k,+}, \quad \hat{a}_{k,-} = \alpha_2^* \hat{c}_{k,-} - \beta_2 \hat{d}^\dagger_{k,-},$$
$$\hat{b}_{-k,+} = \alpha_2^* \hat{d}_{k,-} + \beta_2 \hat{c}^\dagger_{k,-}, \quad \hat{b}_{-k,-} = \alpha_1^* \hat{d}_{k,+} + \beta_1 \hat{c}^\dagger_{k,+}.$$

$$|\alpha_j|^2 + |\beta_j|^2 = 1, \qquad j = 1, 2.$$

We see: $\alpha_1 = \alpha_2 = \alpha, \quad \beta_1 = \beta_2 = \beta.$ $\boxed{n_k(\eta) = |\beta|^2}$

Back-up Slide, $n_\kappa(\tau)$

$$n_\kappa(\tau) = \frac{\Omega_\kappa(\tau) - E_\kappa(\tau)}{2\Omega_\kappa(\tau)},$$

where:

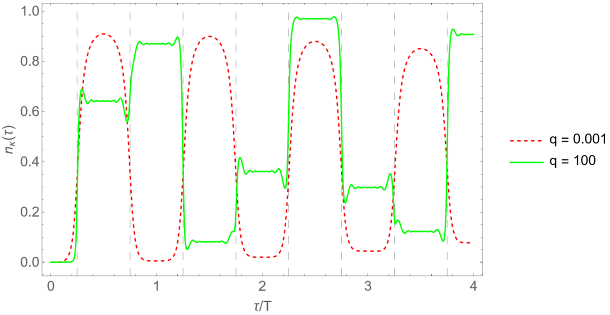
$$E_\kappa(\tau) = \sqrt{q} f(\tau)(|\dot{X}_\kappa|^2 + \Omega_\kappa^2(\tau)|X_\kappa|^2) + 2\Omega_\kappa^2(\tau)\mathrm{Im}(X_\kappa \dot{X}_\kappa^*).$$

Initial conditions:

$$X_\kappa(0) = [2\Omega_\kappa(0)\,[\Omega_\kappa(0) + \sqrt{q} f(0)]]^{-1/2}\,,$$
$$\dot{X}_\kappa(0) = -i\Omega_\kappa(0) X_\kappa(0).$$

Back-up Slide, Number Density per Mode $n_\kappa(\tau)$

$n_\kappa(\tau)$ is evaluated using Bogoliubov coefficients and depends on numerical solutions $X_\kappa(\tau)$. Initial conditions ensure $n_\kappa(0) = 0$.



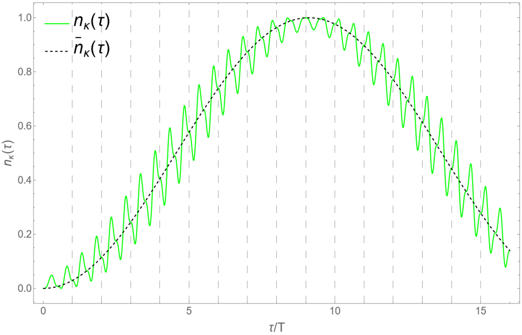
$n_\kappa(\tau)$ is smooth for small q, while it varies rapidly around the zero-crossings of $f(\tau)$ for large q.

Back-up Slide, Smoothed Number Density

V. M. Mostepanenko, V. M. Frolov, Yad. Fiz. **19**, 885-896(1974)

With symmetric $f(\tau)$, $\bar{n}_\kappa(\tau) = F_\kappa \sin^2(\nu_\kappa \tau)$ smooths over $n_\kappa(\tau)$, where F_κ and ν_κ are from the mode equation.

For $q = 1$, $\kappa = 1.05236$:



Back-up Slide, Small q Approx.

$$I(q) = \int_0^{(\kappa_1+\kappa_2)/2} d\kappa \kappa^2 F_\kappa + \cdots = \underbrace{\int_0^{\kappa_0/2} d\kappa \kappa^2 F_\kappa}_{\text{Bulk, } I_b}$$

$$+ \underbrace{\int_{\kappa_0/2}^{(\kappa_0+\kappa_1)/2} d\kappa \kappa^2 F_\kappa}_{l=0 \text{ Peak, } I_0} + \underbrace{\int_{(\kappa_0+\kappa_1)/2}^{(\kappa_1+\kappa_2)/2} d\kappa \kappa^2 F_\kappa}_{l=1 \text{ Peak, } I_1} + \ldots.$$

$I(q)$ follows $q^{1/2}$ power-law behaviour for $q \lesssim 0.01$.

Back-up Slide, Large q Non-Adiabaticity

Non-adiabaticity condition:

$$\left|\frac{d\Omega_\kappa}{d\tau}\right| \gtrsim \Omega_\kappa^2 \implies \kappa \lesssim \sqrt{\left(qf\dot{f}\right)^{2/3} - qf^2},$$

Particle production for large q occurs whenever the effective fermion mass $\sqrt{q}f(\tau)$ crosses zero (for $m_\psi \approx 0$).

The boundary of the bulk:

$$\kappa_* \approx 0.52q^{1/4} \quad \implies \quad I(q) \propto q^{3/4},$$

if the major contribution to the fermion number density comes from the bulk region.