Gradient Flow in Lattice QCD: Recent Applications and Developments

Andrea Shindler

RWTH Aachen University

shindler@physik.rwth-aachen.de

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Bridge Between QCD and Experiment

Experiments measure things like

- Cross sections and differential distributions
- Decay rates and lifetimes
- Asymmetries, EDM signals, form factors, etc.

$$\mathcal{A} \sim \sum_i C_i(\mu) \langle H | O_i(\mu) | H \rangle$$

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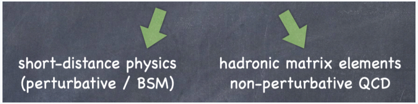

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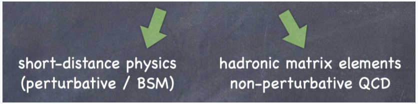


- Hadron structure and precision QCD
  PDFs, GPDs $\rightarrow$ Higgs & BSM searches
- Flavor physics (Belle II, LHCb)
  $\Delta F = 1, 2$ operators: four-fermion, chromomagnetic, ...
- Fundamental symmetry tests
  CP-odd matrix elements $\rightarrow$ CP-violating source / new-physics scale

- Bridge between theory and data
- Limiting factor in many key observables
- Handle on new-physics scales

What Are the Non-Perturbative Tools?

Can we determine hadronic matrix elements non-perturbatively?

In principle yes – several frameworks exist

- Lattice QCD (fully non-perturbative, systematically improvable)
- QCD sum rules (analytic constraints + non-perturbative inputs)
- Large N_c expansions (useful organizing principles)

But if lattice QCD is the ideal tool...
do we actually know how to compute these matrix elements
and with the precision phenomenology requires?

Scope of the Talk

- The gradient flow is now a large subject in lattice field theory, with many papers and QCD applications
- Examples include scale setting, renormalization, running couplings, thermodynamics, topology, smearing, and matrix elements relevant for flavor physics, hadron structure, and symmetry tests
- The main theme is flow time as an auxiliary regulator scale that preserves continuum symmetries
- Focus: hadronic matrix elements where the flow either overcomes standard lattice difficulties or provides a complementary strategy to established methods
- Given the breadth of the field, some relevant work may have been unintentionally omitted. I would be grateful for suggestions and will gladly add them to the proceedings

Outline

1. Flowed gauge and fermion fields
2. The short-flow-time expansion and matching
3. Gauge-field example: the strong coupling
4. Fermionic correlators and normalization
5. Hadron structure from flowed local operators
6. Other applications
7. Flow time as a factorization scale?

Flowed Fields

Definition and properties

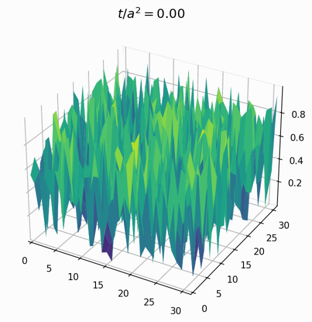
Gauge Fields at Positive Flow Time

$$A_\mu(x) \xrightarrow{\text{gradient flow}} B_\mu(t,x)$$

R. Narayanan and H. Neuberger, JHEP 03 (2006) 064
M. Lüscher, JHEP 08 (2010) 071
M. Lüscher and P. Weisz, JHEP 02 (2011) 051

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x),$$
$$B_\mu(0,x) = A_\mu(x)$$

- Diffusion-like evolution in field space
- Gaussian damping at large momenta
- Smoothing over a range $\sqrt{8t}$



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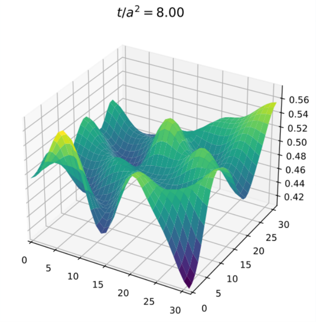

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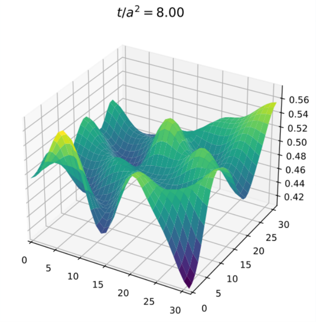
$\langle \cdots B_\mu(t,x) \cdots \rangle$ finite for $t > 0$



Fermion Flow and Normalization

M. Lüscher, JHEP 04 (2013) 123

$$\psi(x) \xrightarrow{\text{gradient flow}} \chi(t,x)$$

$$\partial_t \chi(t,x) = \Delta \chi(t,x), \quad \chi(0,x) = \psi(x),$$
$$\partial_t \bar{\chi}(t,x) = \bar{\chi}(t,x) \overleftarrow{\Delta}, \quad \bar{\chi}(0,x) = \bar{\psi}(x).$$

- Same smoothing radius $\sqrt{8t}$
- Flowed fermions still require a universal normalization
- Composite operators inherit one factor per flowed fermion field

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Field normalization

$$\chi_R(t,x) = Z_\chi^{1/2} \chi(t,x),$$
$$\bar{\chi}_R(t,x) = Z_\chi^{1/2} \bar{\chi}(t,x)$$

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$$\bar{\chi}_R(t,x) = Z_\chi^{1/2} \bar{\chi}(t,x)$$

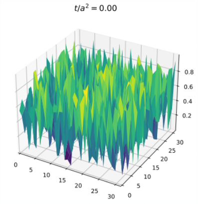
Composite operators

$$\mathcal{O}_R^{(n,\bar{n})}(t,x) = Z_\chi^{(n+\bar{n})/2}\, \mathcal{O}^{(n,\bar{n})}(t,x),$$
$$\mathcal{O}(t,x) = \bar{\chi}(t,x)\,\Gamma(t)\,\chi(t,x),$$
$$\mathcal{O}_R(t,x) = Z_\chi\, \mathcal{O}(t,x)$$

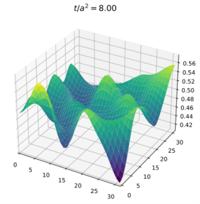
Question

Finite t simplifies renormalization

QCD matrix elements are defined at $t = 0$. How do we get back?



$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x)$$
$$\partial_t \chi(t,x) = \Delta\chi(t,x)$$



Back to $t = 0$

Short-flow-time expansion, continuum perturbation theory, and the lattice

Short-Flow-Time Expansion

M. Lüscher and P. Weisz, JHEP 02 (2011) 051
M. Lüscher, PoS LATTICE2013 (2014) 016
H. Suzuki, PTEP 2013 (2013) 083B03

$$\underbrace{O_i(t,x)}_{\text{lattice at } t>0} = \sum_j \zeta_{ij}(t,\mu)\, \underbrace{O_j^{\overline{\mathrm{MS}}}(\mu,x)}_{\text{QCD at } t=0} + O(t).$$

$$\zeta_{ij}(t,\mu) = \delta_{ij} + \frac{\alpha_s(\mu)}{4\pi}\left[\gamma_{ij}^{(0)} \log(\mu^2 t) + B_{ij}\right] + \mathcal{O}(\alpha_s^2)$$

Why finite t helps

- New UV regulator preserving continuum symmetries
- Matching separated from the hard lattice cutoff
- New strategies for hadronic matrix elements

Three steps

- Field normalization: remove $Z_\chi^{(n+\bar{n})/2}$, or use ratios
- Operator matching: compute $\zeta_{ij}(t,\mu)$, mixing, and possible $1/t^p$ terms
- Correlator matching: relate finite-t lattice ratios to $t=0$ matrix elements

What Changes from Case to Case?

The usefulness of the flow depends on the lattice renormalization problem and on the structure of the SFTX.

LQCD case	What the flow changes	Observables
Case 1 no power divergences multiplicative renormalization, or continuum-like mixing	Matching is like continuum/$\overline{\mathrm{MS}}$ renormalization	$G_\pi \ldots$ $B_K \ldots$
Case 2 power divergences lattice breaks continuum symmetries	Matching becomes continuum/$\overline{\mathrm{MS}}$-like: a clear improvement	PDF moments...
Case 3 power divergences any cutoff scheme	Flow improves mixing, but power divergences remain as $1/t^p$ terms	BSM EDM operators... B-meson lifetimes...

Perturbative SFTX Matching Coefficients

Observable class	Order	References
$E(t)$, GF coupling, quark masses, $\langle \bar{\chi}\chi \rangle$, $\langle \bar{\chi} \not{D} \chi \rangle$	NLO; NNLO; finite volume; quark-mass effects	Lüscher, JHEP 08 (2010) 071 Lüscher, JHEP 04 (2013) 123 Harlander, Neumann, JHEP 06 (2016) 161 Dalla Brida, Lüscher, EPJC 77 (2017) 308 Takaura, Harlander, Lange, JHEP 09 (2025) 025 Harlander, Mason, JHEP 05 (2026) 245
Energy-momentum tensor	NLO; NNLO	Suzuki, PTEP 2013 (2013) 083B03 Makino, Suzuki 1404.2758 [hep-lat] Harlander, Kluth, Lange, EPJC 78 (2018) 944
HVP and current correlators	NNLO	Harlander, Lange, Neumann, JHEP 08 (2020) 109
Local quark bilinears	NLO; NNLO	Hieda, Suzuki, MPLA 31 (2016) 1650214 Borgulat, Harlander, Kohnen, Lange, JHEP 05 (2024) 179
PDF moments	NLO for any n; NNLO up to $n = 6$	A.Sh., PRD 110 (2024) L051503 Harlander, Kohnen, A.Sh., EPJC 86 (2026) 439
Wilson-line bilinears	NLO	Monahan, Phys. Rev. D 97 (2018) 054507 Brambilla, Wang, JHEP 06 (2024) 210
LEFT operators	NLO	Crosas, Stoffer, JHEP 07 (2026) 047
EDM operators	NLO	Rizik, Monahan, A.Sh., PRD 102 (2020) 034509 Mereghetti, Monahan, Rizik, A. Sh., Stoffer, JHEP 04 (2022) 050 Crosas, Monahan, Rizik, A. Sh., Stoffer, PLB 847 (2023) 138301
Four-fermion operators	NNLO; $\Delta F = 2$; $\Delta F = 0$ non-singlet	Harlander, Lange, Phys. Rev. D 105 (2022) L071504 Black, Harlander, Lange, Rago, A.Sh., Witzel, 2603.28517

The Strong Coupling

A precision gauge-field application

Strong Coupling

- The energy density $E(t) = \frac{1}{4} G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x)$ gives a natural definition of a renormalized coupling

$$\langle E(t) \rangle = \frac{1}{2} g_0^2 \frac{N_c^2 - 1}{(8\pi t)^{D/2}} (D-1) \left[1 + c_1 g_0^2 + \mathcal{O}(g_0^4) \right]$$

Lüscher, JHEP 08 (2010) 071

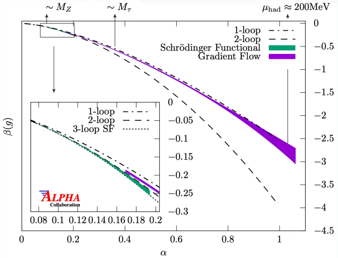
- Define the coupling in finite volume: a practical RG ladder

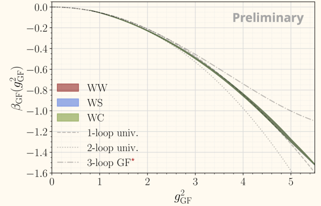
$$\bar{g}^2_{\mathrm{GF}}(L) = \left[\mathcal{N}^{-1}(c, T/L, x_4/T)\, t^2 \langle E(t, x_4) \rangle \right]_{t = c^2 L^2/8}$$

Lüscher, Weisz, Wolff, Nucl. Phys. B 359 (1991) 221–243
Fritzsch, Ramos, JHEP 10 (2013) 008
Dalla Brida, Höllwieser, Knechtli, Korzec, Ramos, Sint, Sommer [ALPHA Collaboration], Nature 642 (2026) 366–371

- At each step the continuum limit can be taken, even as the energy scale becomes very large
- Normalization dictated by the finite-volume tree level
- $\mathcal{O}(10^2 - 10^3)$ less expensive at $g \sim 4$
- Finite variance when $a \to 0$: $\nu \sim \langle E^2(t) \rangle - \langle E(t) \rangle^2$
- Statistical precision independent of coupling value
- Adjust c to minimize cutoff effects

Talk S. Sint





311. Determination of the strong coupling from gradient flow
Curtis Peterson (Michigan State University)
7/28/26, 2:40 PM
4 Standard-Model parame... Contributed talk Standard-Model parame...

Fermionic Correlators

Normalization, continuum limits, and $t \to 0$

Flowed Fermions: One Remaining Normalization

OpenLat: Francis, Fritzsch, Kim, Pederiva, Pefkou, Rago, A. Sh., Walker-Loud, Zafeiropoulos

- OpenLat: $SU(3)$-flavor symmetric point: $m_u = m_d = m_s$, $m_\pi \simeq 410\,\mathrm{MeV}$
- Stabilized Wilson fermions + Lüscher-Weisz gauge action
- 4 lattice spacings ($0.12 \to 0.064\,\mathrm{fm}$)

- $\chi(t,x)$ requires one multiplicative factor Z_χ
- normalized local flowed bilinears need no further UV renormalization

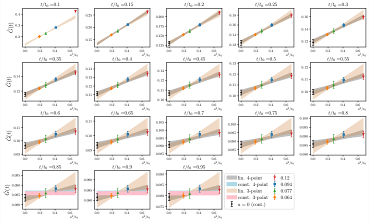
$$\mathring{O}_\Gamma(t,x) = \mathring{\bar{\chi}}(t,x)\Gamma\mathring{\chi}(t,x) = \mathring{Z}_\chi\, O_\Gamma(t,x)$$

Ringed scheme

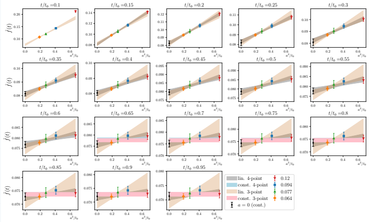
$$\mathring{Z}_\chi = -\frac{N_c}{(4\pi t)^2}\left[\left\langle \bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle\right]^{-1}$$

- local and state independent
- natural for perturbative SFTX matching

$$\mathring{G}_\pi(t) = \lim_{a\to 0}\mathring{Z}_\chi\, G_\pi(t)$$



$$\mathring{f}_\pi(t) = \lim_{a\to 0}\mathring{Z}_\chi\, f_\pi(t)$$



Kim, Pefkou, A. Sh., Walker-Loud,... arXiv:260X.XXXX in preparation

Fixing Z_χ in the Caron Scheme

$$P^{rs}(x) = \bar{\psi}^s(x)\gamma_5\psi^r(x), \qquad V_\mu^3 = \frac{1}{2}\left(V_\mu^{11} - V_\mu^{22}\right)$$

$$\partial_\mu^y \langle P^{21}(x)V_\mu^3(y)P^{12}(0)\rangle = \left[\delta^{(4)}(y) - \delta^{(4)}(y-x)\right] \langle P^{21}(x)P^{12}(0)\rangle$$

$$\frac{\int d^3x\, d^3y\, \langle P^{21}(x)V_{4,R}^3(y)P^{12}(0)\rangle}{\int d^3x\, \langle P^{21}(x)P^{12}(0)\rangle} = 1$$

Caron scheme

$$\check{V}_\mu^3(t,y) = \check{Z}_\chi\, V_\mu^3(t,y), \qquad \boxed{\check{Z}_\chi = \frac{2m_\pi}{\langle\pi(\mathbf{0})|V_4^3(t)|\pi(\mathbf{0})\rangle}}$$

Absolute nonperturbative normalization from the unit pion charge

Related approaches:

Monahan, Orginos, PoS Lattice2013 (2014) 443
Carosso, Hasenfratz and Neil, Phys. Rev. Lett. 121 (2018) 201601
Hasenfratz, Monahan, Rizik, A. Sh. and Witzel, PoS LATTICE2021 (2022) 155
Borgulat, PhD thesis, RWTH Aachen University, 03/2026
Black, Hasenfratz and Witzel, arXiv:2607.00493

17. Gradient flow renormalization for composite fermion operators
Anna Hasenfratz
7/28/26, 2:00 PM
★ 6 Standard-Model parame... Standard-Model parame...

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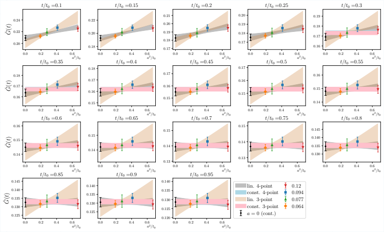

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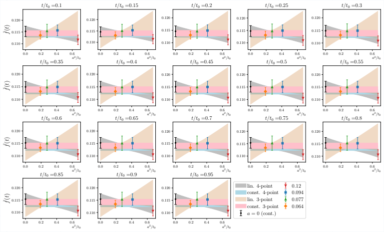


Kim, Pefkou, A. Sh., Walker-Loud,... arXiv:260X.XXXX in preparation

From Flowed Data to Physical G_π and f_π

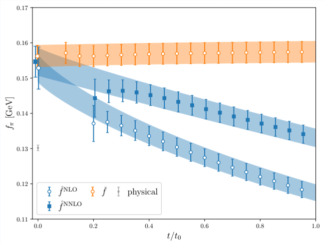
- SFTX relates finite-t matrix elements to $\overline{\mathrm{MS}}$ observables

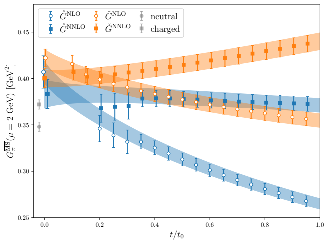
$$\mathring{G}_\pi(\mu) = \left[\mathring{\zeta}_P(t,\mu)\right]^{-1} \mathring{G}_\pi(t) + \mathcal{O}(t),$$
$$\mathring{f}_\pi(\mu) = \left[\mathring{\zeta}_A(t,\mu)\right]^{-1} \mathring{f}_\pi(t) + \mathcal{O}(t)$$

$$\check{G}_\pi(\mu) = \left[\check{\zeta}_P(t,\mu)\right]^{-1} \check{G}_\pi(t) + \mathcal{O}(t),$$
$$f_\pi = \check{f}_\pi(t) + \mathcal{O}(t) \qquad \text{(chiral-symmetric scheme)}$$

- $\mathring{f}_\pi$: residual μ-dependence mostly reflects perturbative truncation in $\mathring{\zeta}_A$
- Caron scheme: determination of f_π without Z_A and c_A for Wilson-Clover fermions
- The continuum limit remains $\mathcal{O}(a)$ improved

SFTX coefficients known to NNLO
Borgulat, Harlander, Kohnen and Lange, JHEP 05 (2024) 179





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Question

Can flow open access to difficult matrix elements?

Hadron structure gives a demanding test case.

Hadron Structure

Flowed local operators for PDF moments

PDF and Lattice QCD

Direct lattice calculations of higher PDF moments are notoriously difficult

$$\langle x^{n-1}\rangle_q^h(\mu) = \int_0^1 dx\, x^{n-1}\left[q^h(x,\mu) + (-1)^n \bar{q}^h(x,\mu)\right]$$

Curci, Furmanski, Petronzio, Nucl. Phys. B 175 (1980) 27
Collins, Soper, Nucl. Phys. B 194 (1982) 445

PDF and Lattice QCD

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$$\langle h(p)|\widehat{O}^{q}_{\{\mu_1\cdots\mu_n\}}|h(p)\rangle = 2\,\langle x^{n-1}\rangle^{h}_{q}(\mu)\big(p_{\mu_1}\cdots p_{\mu_n} - \text{traces}\big)$$

$$\widehat{O}^{q}_{\{\mu_1\cdots\mu_n\}}(x) = \bar{\psi}^{q}(x)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\psi^{q}(x) - \text{traces}$$

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- Calculate matrix elements using lattice QCD
  - Rotational symmetry breaks to the hypercubic group $H(4)$

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  - Irreps of $H(4)$ allow mixing with lower-dimensional operators and complicate mixing with operators of the same dimension

Beccarini et al., Nucl. Phys. B 456 (1995) 271
Göckeler et al., Phys. Rev. D 54 (1996) 5705

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  - Irreps of $H(4)$ allow mixing with lower-dimensional operators and complicate mixing with operators of the same dimension
- Operators with different index combinations belong to different irreps of $H(4)$

Beccarini et al., Nucl. Phys. B 456 (1995) 271
Göckeler et al., Phys. Rev. D 54 (1996) 5705

Davoudi, Savage, Phys. Rev. D 86 (2012) 054505

Strategy

A. Sh., Phys. Rev. D 110 (2024) 5, L051503

- Consider flowed operators
$$\mathcal{O}_n^{rs}(x,t) = \bar{\chi}^r(t,x)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\chi^s(t,x)$$
- Renormalize flowed operators $\rightarrow$ renormalization is always multiplicative
$$\mathring{\mathcal{O}}_n^{rs}(t) = \lim_{a\to 0}\mathring{Z}_n\mathcal{O}_n^{rs}(t), \qquad Z_n = Z_\chi$$

Continuum limit is finite for any n

- Perform a short-flow-time expansion $\rightarrow$ use $O(4)$ irreps $\rightarrow$ traceless operators (twist-2)

$$\widehat{\mathring{\mathcal{O}}}_n^{rs}(x,t) = \mathring{\bar{\chi}}^r(x,t)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\mathring{\chi}^s(x,t) - \text{terms with } \delta_{\mu_i\mu_j}$$

$$\langle h(p)|\widehat{\mathring{\mathcal{O}}}_n(t)|h(p)\rangle = 2\left(p_{\mu_1}\cdots p_{\mu_n} - \text{trace terms}\right)\langle\mathring{x}^{n-1}\rangle_h(t)$$

- Matching is multiplicative for traceless operators

$$\widehat{\mathring{\mathcal{O}}}_n^{rs}(t) = \mathring{\zeta}_n(t,\mu)\,\widehat{O}_n^{rs,\overline{\mathrm{MS}}}(\mu) + O(t)$$

At finite t: power-divergence renormalization and signal-to-noise issues are, in principle, resolved

$\Rightarrow$ **no boost needed**

Matching Coefficients

A. Sh., Phys. Rev. D 110 (2024) 5, L051503

Matching equations

$$\left\langle \psi^r \widehat{\mathring{O}}_n^{rs}(t)\bar{\psi}^s \right\rangle = \mathring{\zeta}_n(t,\mu) \left\langle \psi^r \widehat{O}_n^{rs,\overline{\mathrm{MS}}}(t=0,\mu)\bar{\psi}^s \right\rangle$$

- Expand integrands of loop integrals in all scales excluding t
  - Analytic structure altered $\to$ distortion of IR structure
  - In the matching equation the IR modification drops out in the difference
  - Expanding loop integrals in the RHS vanish in DR $\to$ UV and IR are identical
  - The LHS is UV-finite, beside renormalization of bare parameters and flowed fermion fields
  - The IR singularities on the LHS exactly match the UV $\overline{\mathrm{MS}}$ counterterms

$$\zeta_n(t,\mu) = 1 + \frac{\bar{g}^2(\mu)}{(4\pi)^2}\zeta_n^{(1)}(t,\mu) + \mathcal{O}(\bar{g}^4), \qquad \zeta_n^{(1)}(t,\mu) = C_F\left[\gamma_n \log(8\pi\mu^2 t) + B_n\right], \qquad \gamma_n = 1 + 4\sum_{j=2}^{n}\frac{1}{j} - \frac{2}{n(n+1)}$$

Gross and Wilczek, Phys. Rev. D 9 (1974) 980

$$B_n = \frac{4}{n(n+1)} + 4^{\,n-1}\log 2 + \frac{2-4n^2}{n(n+1)} - \frac{2}{n(n+1)}\psi(n+2) + \frac{4}{n}\psi(n+1) - 4\psi(2) - 4\sum_{j=2}^{n}\frac{1}{j(j-1)}\,2^j\phi(1/2,1,j) - \log(432)$$

$n=2$: Makino and Suzuki, PTEP 2014 (2014) 063B02

$$\phi(z,s,a) = \sum_{k=0}^{\infty}\frac{z^k}{(k+a)^s}$$

Cutoff Effects

- With chiral symmetry expect leading effects of $O(a^2)$
- The flow does not spoil automatic $O(a)$ improvement for Wtm — A. Sh., Nucl. Phys. B 881 (2014) 71
- For clover formulations twist-2 fields are affected by specific $O(a)$ effects that depend on n
  - Improvement coefficients are known only for $n = 2$ and only in perturbation theory

Flowed twist-2 operators

- Hadronic matrix elements, beside the $O(a)$ from the lattice theory, are only affected by $O(am)$
- Short-distance $O(a)$ effects are negligible at large physical distances
- The $O(am)$ effects are independent of n and depend only on the fermion content
- Renormalized flowed fields, ringed or Caron, or ratios of moments have $O(a^2)$ cutoff effects
- Wilson-clover fermions are back in the game

$\Rightarrow$ **Finite continuum limit and $O(a)$ improved for every n**

Test Case: Heavy Pion at $SU(3)$ Flavor-Symmetric Point

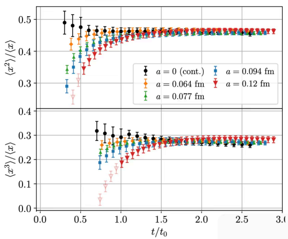
- $SU(3)$-flavor symmetric point: $m_u = m_d = m_s$, $m_\pi \simeq 410\,\mathrm{MeV}$
- Stabilized Wilson fermions + Lüscher-Weisz gauge action — Francis, Fritzsch, Lüscher and Rago Comput. Phys. Commun. 255 (2020)
- 4 lattice spacings ($0.12 \to 0.064\,\mathrm{fm}$)
- 400–800 configurations per ensemble
- NLO calculation for any n — A. Sh., Phys. Rev. D 110 (2024) 5, L051503
- NNLO calculation for $n = 2, \ldots, 6$ — Harlander, Kohnen and A. Sh. Eur. Phys. J. C 86 (2026) 4, 439

OPEN LATtice initiative

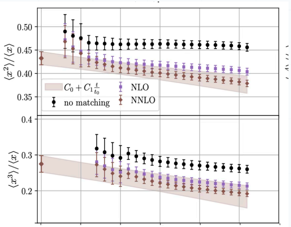
openlat1.gitlab.io

Phys. Rev. Lett. 136 (2026) 17, 171903
Phys. Rev. D 113 (2026) 7, 074520

Flowed moments



$t \to 0$ matching (NLO/NNLO)

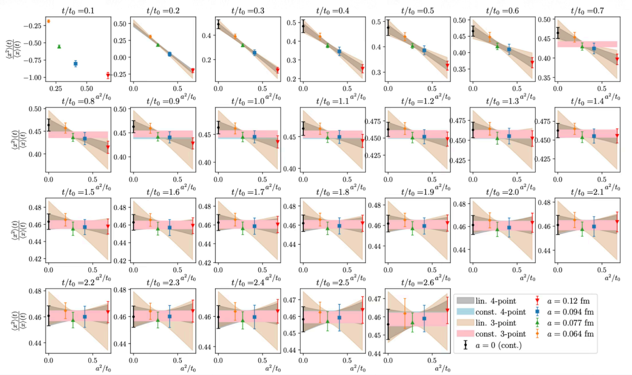


Continuum Limit at Fixed Flow Time

- Take $a \to 0$ at fixed physical flow time
- Repeat the continuum extrapolation for each t/t_0
- Compare constant and linear fits in a^2/t_0
- Use 3-point and 4-point fit variants as a systematic check

OPEN LATtice initiative

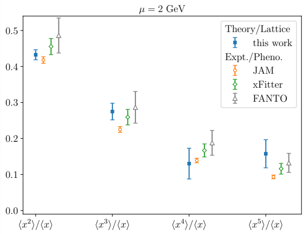
Phys. Rev. Lett. 136 (2026) 17, 171903
Phys. Rev. D 113 (2026) 7, 074520

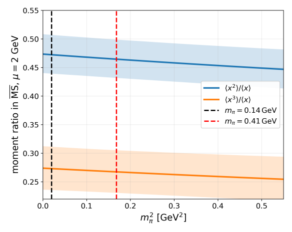


Pion Moments and Phenomenology

- Continuum-extrapolated pion moment ratios at $\mu = 2\,\mathrm{GeV}$
- Computed at $m_\pi \simeq 411\,\mathrm{MeV}$; comparison is to physical-point phenomenology
- JAM: global QCD fit with threshold resummation
- xFitter: charged-pion PDFs from Drell-Yan and leading-neutron DIS
- FANTO: flexible Bézier parametrizations with epistemic and nuclear uncertainties
- Ratios are compatible with the pheno extractions within present uncertainties
- Older lattice data suggest mild m_π^2-dependence for $\langle x^2\rangle/\langle x\rangle$ and $\langle x^3\rangle/\langle x\rangle$
- Physical-point confirmation still requires lighter pion masses and controlled chiral fits

OpenLat: Francis et al., PRL 136 (2026) 171903; PRD 113 (2026) 074520
JAM: Barry, Ji, Sato and Melnitchouk, PRL 127 (2021) 232001
xFitter: Novikov et al., PRD 102 (2020) 014040
FANTO: Kotz, Courtoy, Nadolsky and Ponce-Chavez, PRD 112 (2025) L071502
Chiral: Brömmel et al., PoS LAT2005 (2006) 360; Brömmel, Ph.D. thesis (2007); Arndt and Savage, NPA 697 (2002) 429





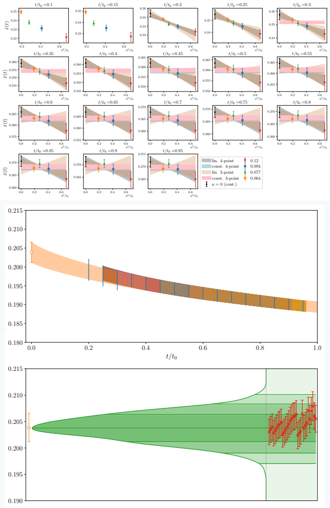
Continuum and $t \to 0$ for $\langle \check{x} \rangle_\pi$

Fixed-t continuum limit

- Take the $a \to 0$ limit at fixed physical flow time
- $\langle \check{x} \rangle_\pi$ contains a discretized operator spanning up to two lattice spacings
- Require the smoothing radius to resolve the operator: $\sqrt{8t} > 2a$
- For the continuum curve: 3-point linear fit from $t/t_0 \geq 0.2$, 4-point linear fit from $t/t_0 \geq 0.35$

$t \to 0$ extrapolation

- For $\langle \check{x} \rangle_\pi^{\overline{\mathrm{MS}}}$, A is the zero-flow-time result
- Fit variants use $A + B\, t/t_0 + C\, t/t_0 \log(t\mu^2)$
- Linear term models higher-dimensional SFTX corrections
- Log term estimates missing matching for higher-dimensional operators
- Small correlated windows: $0.3 < \Delta t/t_0 < 0.55$, SVD cut 10^{-5}
- Final value from AIC-weighted average over accepted fits



Kim, Pefkou, A. Sh., Walker-Loud,... arXiv:260X.XXXX in preparation

RG Test of Matching Uncertainties

- Missing perturbative terms leave residual dependence on the intermediate matching scale μ
- For the renormalized matrix element

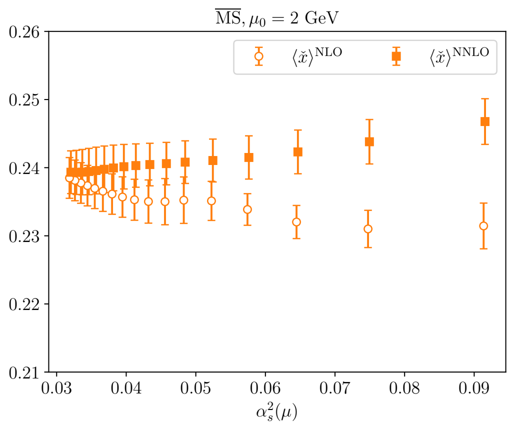
$$\mu^2 \frac{d}{d\mu^2} X^{\overline{\mathrm{MS}}}(\mu) = \gamma_O(a_s) X^{\overline{\mathrm{MS}}}(\mu), \qquad a_s = \frac{\alpha_s}{4\pi}$$

- Run each fixed-μ result back to $\mu_0 = 2$ GeV:

$$X^{S,\overline{\mathrm{MS}}}(\mu_0;\mu) = U_O(\mu_0,\mu)\, X^{S,\overline{\mathrm{MS}}}(\mu)$$

$$U_O(\mu_0,\mu) = \exp\left\{ \int_{a_s(\mu)}^{a_s(\mu_0)} da\, \frac{\gamma_O(a)}{\beta(a)} \right\}$$

- Matching scales: $\mu = 2 - 10$ GeV
- If matching and $t \to 0$ were exact, all evolved results would agree at μ_0
- Numerically relevant mainly for $\langle x \rangle_\pi$
- Additional systematic: half the spread of the evolved central values or $\alpha_s \to 0$

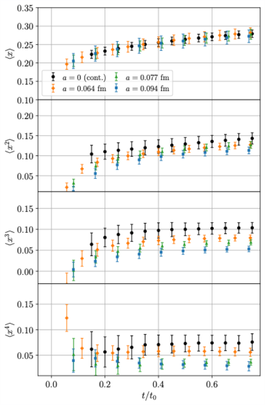


Kim, Pefkou, A. Sh., Walker-Loud,... arXiv:260X.XXXX in preparation

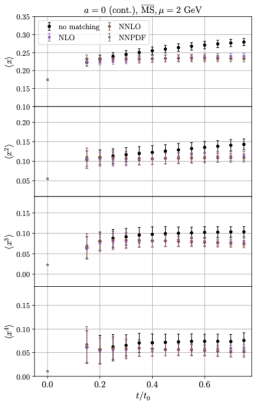
Nucleon Valence Moments $(u - d)$

- Away from the $SU(3)$-flavor symmetric point: $m_\pi = 300\,\mathrm{MeV}$
- Stabilized Wilson fermions + Lüscher-Weisz gauge action
- 3 lattice spacings ($0.094 \to 0.064\,\mathrm{fm}$)
- 100–200 configurations per ensemble
- 10 sources
- Source-sink separation $> 1.5\,\mathrm{fm}$

Flowed moments



$t \to 0$ matching



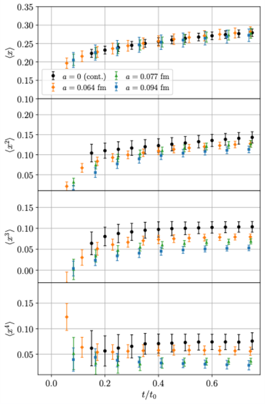
OpenLat: Francis, Fritzsch, Kim, Pederiva, Pefkou, Rago, A. Sh., Walker-Loud, Zafeiropoulos

Nucleon Valence Moments $(u - d)$

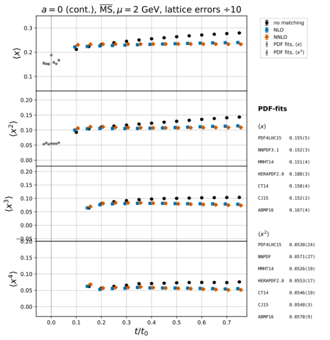
Source	$\langle x \rangle$	$\langle x^2 \rangle$	$\langle x^3 \rangle$	$\langle x^4 \rangle$
This work, NNLO, t/t0 = 0.3, errors / 10	0.44%	1.57%	2.23%	3.46%
PDF4LHC15	3.23%	4.53%	-	-
NNPDF / NNPDF3.1	1.97%	4.73%	-	-
MMHT14	2.65%	3.61%	-	-
HERAPDF2.0	1.60%	3.07%	-	-
CT14	2.53%	3.48%	-	-
CJ15	1.32%	0.55%	-	-
ABMP16	2.40%	1.56%	-	-

- State-of-the-art statistics (10^5) reach global-fit errors at $m_\pi \simeq 300\,\mathrm{MeV}$
- Preserve this precision and keep matching systematics at the same level
- From g_A: physical-point precision may require only $\sim 2-4$ more statistics, not orders of magnitude

Flowed moments



$t \to 0$ matching



Community White Paper, Prog. Part. Nucl. Phys. 100 (2018) 107–160
Hall et al., Phys. Rev. C 113 (2026) 055206
Gupta et al., Phys. Rev. D 98 (2018) 034503
Djukanovic et al., Phys. Rev. D 109 (2024) 094511

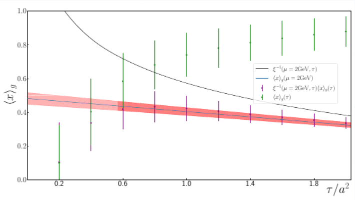
OpenLat: Francis, Fritzsch, Kim, Pederiva, Pefkou, Rago, A. Sh., Walker-Loud, Zafeiropoulos

Other Recent Applications

Gluonic Matrix Elements

Gluon Momentum Fraction

- $\langle x \rangle_g$ from flowed gauge operators, matched to $\overline{\mathrm{MS}}$ at $2\,\mathrm{GeV}$
- $N_f = 2+1$ Wilson-clover ensemble: $m_\pi = 358\,\mathrm{MeV}$, $a = 0.094\,\mathrm{fm}$

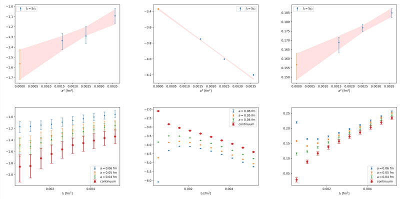


Edwards et al. (HadStruc Collaboration), Phys. Rev. D 114 (2026) 014516

Mass And Trace-Anomaly Sum Rules

$$\langle H(p)| T^{\mu\nu}_{q,g} | H(p)\rangle, \quad A_{q,g}(0), \quad \bar{C}_{q,g}(0), \quad H = \eta_c,\ J/\psi$$

- First lattice determination of $\bar{C}_{q,g}(0)$
- Consistent verification of multiple hadron-mass sum rules
- Sizable gluonic contributions to charmonium masses
- Quantitative determination of trace-anomaly contributions



Bollweg, Ding, Gao, Luo and Mukherjee, arXiv:2601.13070

Systematics

- Three lattice spacings: $a = 0.06,\ 0.05,\ 0.04\,\mathrm{fm}$
- Sizable cutoff effects; linear continuum extrapolations
- $t_f \to 0$ fits and perturbative matching uncertainty

Quark Masses

Flowed quark condensate

$$t\left\langle \mathring{\bar{\chi}}(t)\mathring{\chi}(t)\right\rangle = \mathring{\zeta}_m(t,\mu)\,m(\mu) + O(t), \qquad \zeta_m^{(0)} = -\frac{N_c}{8\pi^2}$$

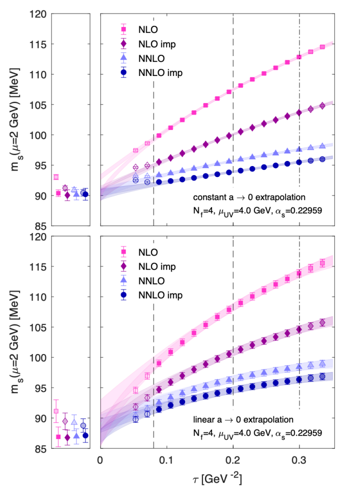
Takaura, Harlander and Lange, JHEP 09 (2025) 025
Makino and Suzuki, PTEP 2014 (2014) 063B02
Artz, Harlander, Lange, Neumann and Prausa, JHEP 06 (2019) 121

Flowed PCAC mass

$$m(t) = \frac{\left\langle \partial_\mu A_\mu(t)\, P\right\rangle}{2\left\langle P(t)\, P\right\rangle}, \qquad m(\mu) = \frac{\zeta_P(t,\mu)}{\zeta_A(t,\mu)}\, m(t) + O(t)$$

- Promising alternative renormalization strategy
- Important benchmark against RI-MOM, SF
  - final uncertainty at comparable computational cost
  - stability of perturbative matching at sub-percent precision

317. Renormalized quark masses from gradient flow using HISQ ensembles
Mingwei Dai (University of Illinois Urbana-Champaign)
7/28/26, 2:20 PM
★ 5 | Standard-Model parame... | Contributed talk | Standard-Model parame...



Black, Harlander, Hasenfratz, Rago and Witzel, Phys. Rev. D 113 (2026) 9, 094504

Neutral Heavy-Meson Mixing ($\Delta Q = 2$) and Lifetimes ($\Delta Q = 0$)

$$\mathcal{Q}_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)q)(\bar{Q}\gamma_\mu(1-\gamma_5)q)$$

$$\langle H|\mathcal{Q}_1|H\rangle = \tfrac{8}{3} f_H^2 M_H^2 \mathcal{B}_1^H$$

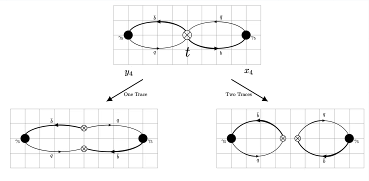
- Four-quark operators enter three-point functions
- Flowed fermions provide finite-t renormalized matrix elements
- RG-improved SFTX controls the $t \to 0$ extrapolation

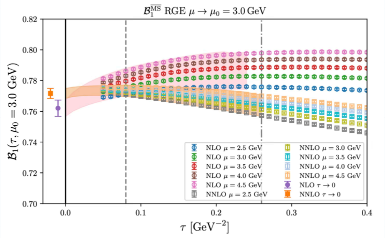
$$\mu^2 \frac{d}{d\mu^2}\zeta(t,\mu) = \zeta(t,\mu)\,\gamma(\alpha_s(\mu))$$

$$f(t,\mu) = c_0 + t\left[c_1 + c_2 \ln(\mu^2 t)\right]$$

- $\Delta B = 0$ four-quark matrix elements are the final target
- Standard renormalization introduces mixing with lower-dimensional operators: central challenge

Suzuki, Taniguchi, Suzuki and Kanaya, Phys. Rev. D 102 (2020) 034508
Harlander and F. Lange, Phys. Rev. D 105 (2022) L071504

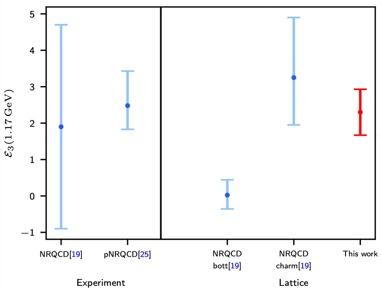


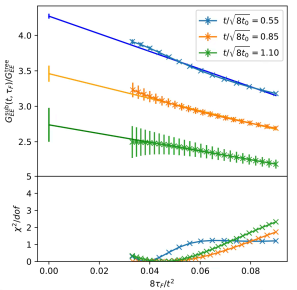


Black, Harlander, Lange, Rago, A. Sh., Witzel, 2603.28517; 2603.28516

189. Gradient flow and heavy-light physics
Oliver Witzel (University of Siegen)

Gradient flow for inclusive quarkonium decays





- pNRQCD: all P-wave widths depend on one universal $\mathcal{E}_3$
- GF regulates the Wilson-line power divergence at finite t_F
- Continuum limit at fixed t_F, then match/extrapolate to $\overline{\mathrm{MS}}$
- Direct lattice result agrees with phenomenology, with smaller uncertainty

$$\mathcal{E}_3(\Lambda) = \frac{T_F}{N_c} \int_0^\infty dt\, t^3 \, \langle 0|gE^a(t)\Phi_{ab}(t,0)gE^b(0)|0\rangle$$

131. Inclusive P-wave quarkonium decay widths in pNRQCD
Julian Mayer-Steudte
7/29/26, 11:10 AM
Hadronic and nuclear sp... | Contributed talk | Hadronic and nuclear sp...

Brambilla, Leino, Mayer-Steudte, Panaylotou, A. Sh., Vairo, Wang, 2607.13024 [hep-lat]

Charm Content of the Nucleon

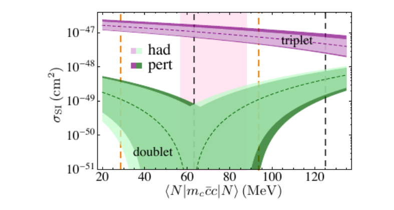
- Spin-independent WIMP-nucleon scattering is sensitive to scalar heavy-quark matrix elements
- Spin-0/spin-2 cancellations make the cross section particularly sensitive to $\langle N|m_c\bar{c}c|N\rangle$

$$S^{rs}(t,x) = \frac{\zeta_m(t,\mu)}{t} M^{rs} + \zeta_{m^3,A}(t,\mu)\,\mathrm{Tr}(M^2)\,M^{rs}$$
$$+ \zeta_{m^3,B}(t,\mu)\,(M^3)^{rs} + \zeta_S(t,\mu)\,S^{rs}(x) + O(t)$$
$$P^{rs}(t,x) = \zeta_P(t,\mu)\,P^{rs}_R(\mu,x) + O(t), \qquad \zeta_S(t,\mu) = \zeta_P(t,\mu)$$
$$S^{rs}(t,x) = \bar{\chi}^r(t,x)\chi^s(t,x), \qquad P^{rs}(t,x) = \bar{\chi}^r(t,x)\gamma_5\chi^s(t,x)$$

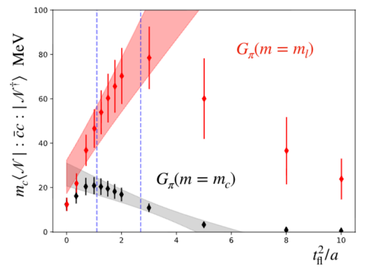
$$C^c_{\mathrm{sub}}(t,x) = \frac{G_\pi}{G_\pi(t)}\left[\left\langle N\,\bar{c}(t,x)c(t,x)\,\bar{N}\right\rangle - \left\langle\bar{c}(t,x)c(t,x)\right\rangle\left\langle N\,\bar{N}\right\rangle\right] + O(t)$$

Lüscher, PoS LATTICE2013 (2014) 016
A. Sh., de Vries and Luu, PoS LATTICE2014 (2015) 251

309. Accessing Nucleon Sigma terms with Gradient Flow
Rohith Karur (The University of California, Berkeley)
7/30/26, 4:50 PM
Structure of hadrons an... | Contributed talk | Structure of hadrons an...



Hill and Solon, Phys. Rev. Lett. 112 (2014) 211602



Karur, Kim, Pefkou, A. Sh., Walker-Loud, work in progress

Other Lattice 2026 Gradient-Flow Results

190. Renormalization and Improvement Conditions from Flowed Fermion Fields
Lukas Holan (Humboldt University of Berlin)
7/27/26, 2:20 PM
★ 3 | Standard-Model parame... | Contributed talk | Standard-Model parame...

- Flowed composite probes plus axial WIs
- Critical mass, $O(a)$ flow improvement, Z_A/Z_V, Z_S/Z_P

33. Topological susceptibility in the 2d $O(3)$ nonlinear sigma model using the tree-level improved gradient flow
Mika Lauk (HU-Berlin)
7/27/26, 3:00 PM
★ 4 | Theoretical developmen... | Contributed talk | Theoretical developmen...

- Topology in the 2d $O(3)$ model
- Improved flow/action/operator choices and finite-volume effects

52. Perturbative study of higher-moment operators in the Wilson-flow framework on the lattice
Dr Gregoris Spanoudes (University of Cyprus)
7/28/26, 5:10 PM
★ 2 | Structure of hadrons an... | Contributed talk | Structure of hadrons an...

- Lattice PT for flowed twist-2 operators
- One-loop study up to five covariant derivatives and residual $H(4)$ mixing

86. Step-scaling determination of the SU(N) Λ-parameter in the twisted gradient flow scheme
Andrea Giorgieri (University of Pisa, INFN)
7/29/26, 9:20 AM
★ 1 | Vacuum structure and c... | Contributed talk | Vacuum structure and c...

- Twisted-GF step scaling for $SU(N)$
- Large-N Λ-parameter without relying on asymptotic scaling

164. Gradient-Flow Actions, Integerness, and Gluonic–Fermionic Topology Matching in SU(3) Yang–Mills Theory
Akio Tomiya (Tokyo Woman's Christian University)
7/29/26, 9:40 AM
★ 1 | Vacuum structure and c... | Contributed talk | Vacuum structure and c...

- Flow-action dependence of topology readout
- Iwasaki/DBW2 flows improve integerness and index matching

223. Update on the gradient flow scale on CLS ensembles
Wolfgang Söldner
7/29/26, 10:20 AM
★ 2 | Hadronic and nuclear sp... | Contributed talk | Hadronic and nuclear sp...

- Updated RQCD gradient-flow scale
- 50+ CLS ensembles, six lattice spacings, baryon-octet scale setting

37. Instantons from Lattice QCD using Gradient Flow in Pion Form Factors
Vaibhav Chahar (Jagiellonian University, Poland)
7/31/26, 2:00 PM
Structure of hadrons an... | Contributed talk | Structure of hadrons an...

- Wilson flow as instanton filter
- Pion form factors: positive flow time improves high-Q^2 signal

202. Scaling and cutoff effects of the topological susceptibility with gradient flow in $SU(3)$ Yang–Mills theory on the lattice
Alessandro Conigli, Fernando Pérez Panadero (IFT-UAM/CSIC), Pietro Butti (Quantum Theory Center, IMADA and D-IAS, University of Southern Denmark)
★ 1 | Theoretical developmen... | Poster presentation

- Cutoff effects in topological susceptibility
- Disentangling action, flow-kernel and operator discretization effects

Beyond the $t \to 0$ Limit

Can flow time itself be the factorization scale?

Flow Time as a Factorization Scale

- Standard factorization separates short and long distances through μ

$$\mathcal{A}(Q) = \sum_i C_i^{\overline{\mathrm{MS}}}(Q,\mu)\,\langle H|O_i^{\overline{\mathrm{MS}}}(\mu)|H\rangle$$

- At finite flow time the smoothing radius $\sqrt{8t}$ can play the role of a physical separation scale

$$\mathcal{A}(Q) = \sum_i C_i^{\mathrm{flow}}(Q,t)\,\langle H|O_i(t)|H\rangle + \mathcal{O}(tQ^2)$$

Why this is interesting

- Flowed matrix elements are finite and lattice-regulated in a controlled way
- The flow scale is physical, not tied to a particular subtraction convention
- $\overline{\mathrm{MS}}$ may become an intermediate language, rather than the final scheme

Lüscher, PoS LATTICE2013 (2014) 016
Monahan and Orginos, Phys. Rev. D 91 (2015) 074513
Monahan and Orginos, JHEP 03 (2017) 116
Ammer and Dürr, Phys. Rev. D 110 (2024) 5, 5
Beneke and Takaura, JHEP 03 (2026) 033

Flowed Wilson Coefficients

$$O_i(t) = \sum_j \zeta_{ij}(t,\mu)\, O_j^{\overline{\mathrm{MS}}}(\mu) + \mathcal{O}(t)$$

$$C_i^{\mathrm{flow}}(Q,t) = \sum_j C_j^{\overline{\mathrm{MS}}}(Q,\mu)\left[\zeta^{-1}(t,\mu)\right]_{ji}$$

- This is the same matching information, read in the opposite direction
- Logs of $\mu^2 t$ cancel between $C^{\overline{\mathrm{MS}}}$ and ζ^{-1}
- Residual t-dependence measures truncation and higher-dimensional terms

What would be new

- Compute $C_i^{\mathrm{flow}}(Q,t)$ directly for physical observables
- Keep the final long-distance matrix elements at finite t
- Use t-evolution as the analogue of scheme/scale variation
- Avoid power-divergent lattice subtractions when the flowed OPE is favorable

Monahan, Phys. Rev. D 97 (2018) 054507
Beneke and Takaura, JHEP 03 (2026) 033

Summary

- Gradient flow tames ultraviolet fluctuations, acting as an additional regulator, restoring continuum symmetries at $t > 0$
- Rich field-theoretical framework, now established across lattice QCD, with rapidly growing lattice-perturbation theory synergy
- At positive flow time, many observables have cleaner UV behavior and improved continuum limit
- Direct impact on phenomenology: more reliable inputs for α_s, PDFs, heavy-flavor observables, scalar charges, EDMs, ...
- Matching is the essential bridge from finite-flow quantities to conventional QCD matrix elements
- Open questions: the $t \to 0$ limit and the optimal window connecting finite-flow observables to phenomenology

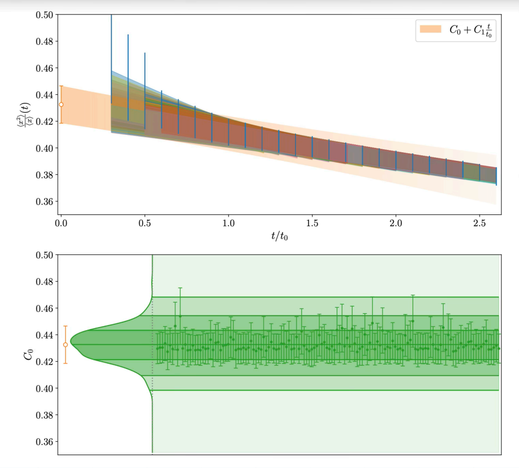
Thank you

Questions and discussion

Backup

Residual t-Dependence

- Within the current statistical precision, no clear logarithms observed for ratios
- Simple linear ansatz $C_0 + C_1\, t/t_0$
- The minimum values used are:
  $(t/t_0)_{\mathrm{min}} = [0.3,\ 0.7,\ 0.9,\ 1.3]$
  for $n = [3,\ 4,\ 5,\ 6]$
- We perform fits over all flow-time intervals with width bigger than 0.8
- Only fits with $p > 0.1$ are retained
- The final central value is from a flat average over all accepted fits
- The systematic uncertainty from the spread of the accepted fit results
- Varying the minimum and maximum flow-time cuts within reasonable ranges gives stable results



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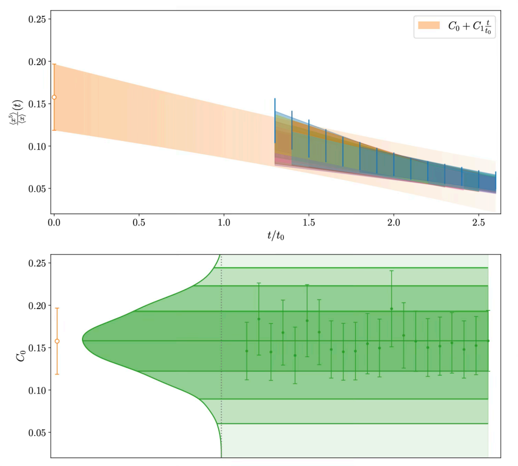
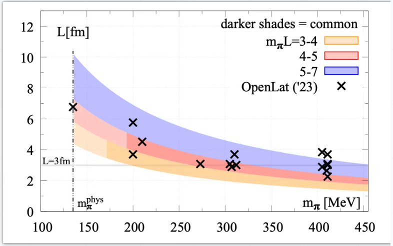
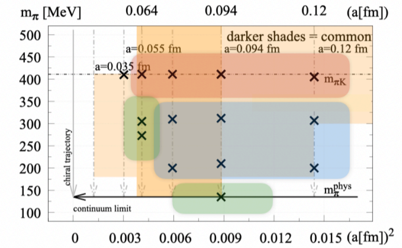


OpenLat Ensembles and Hadron Structure

- Open access to high-quality $N_f = 2+1$ QCD gauge-field ensembles
- Stabilized Wilson fermions for precision QCD
- Staged production toward the physical point and continuum limit
- Physics drivers: θEDM/CP violation and hadron structure/lattice PDFs
- Here: pion, kaon, and nucleon moments from flowed local operators

openlat.gitlab.io





Stage 1 ✓

Stage 2 WORK IN PROGRESS

Stage 3 WORK IN PROGRESS

GF Schemes for Fermion Operators

$$O_{\mathrm{GF}}^{X}(t,x) = \widetilde{Z}_{\chi}^{X}(t)\,\bar{\chi}(t,x)\Gamma\chi(t,x), \qquad X = A, V$$

$$\widetilde{Z}_{\chi}^{A}(t) = Z_A\, R_A(t), \qquad \widetilde{Z}_{\chi}^{V}(t) = Z_V\, R_V(t)$$

- Fix Z_χ from axial/vector Ward-normalized two-point functions

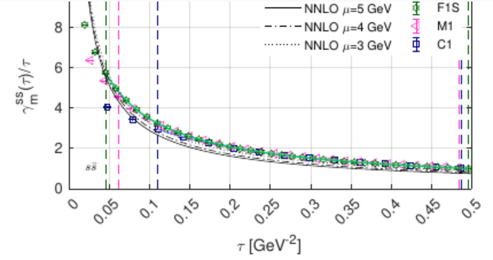
Flowed anomalous dimension

$$\widetilde{\gamma}_O^A(t) = -2t\,\frac{d}{dt}\log \widetilde{Z}_O^A(t) = -2t\,\frac{d}{dt}\log\frac{R_A(t)}{R_O(t)}$$

$$\widetilde{\gamma}_O^A(t) = \mathring{\gamma}_O(t) - \mathring{\gamma}_A(t) + O(t)$$

Concerns

- Infinite-volume C_m: lowering t at fixed a can bring back cutoff effects
- Bare $t=0$ operators are delicate: $Z_O(t,a)$ has no continuum limit; continuum matching misses lattice mixing/power divergences



Borgulat, Harlander, Kohnen and Lange, JHEP 05 (2024) 179
Carosso, Hasenfratz and Neil, Phys. Rev. Lett. 121 (2018) 201601
Hasenfratz, Monahan, Rizik, A. Sh. and Witzel, PoS LATTICE2021 (2022) 155
Hasenfratz, Neil, Shamir, Svetitsky and Witzel, Phys. Rev. D 107 (2023) 114504; Phys. Rev. D 108 (2023) L071503
Black, Hasenfratz and Witzel, arXiv:2607.00493

Gradient Flow at Finite Lattice Spacing

$$\partial_t V_\mu(t,x) = -g_0^2 \{\partial_{x,\mu} S_{\mathrm{fl}}[V]\}\, V_\mu(t,x)$$
$$V_\mu(0,x) = U_\mu(x)$$
$$\partial_t \chi(t,x) = \Delta[V]\chi(t,x), \qquad \chi(0,x) = \psi(x)$$
$$\partial_t \bar{\chi}(t,x) = \bar{\chi}(t,x)\overleftarrow{\Delta}[V], \qquad \bar{\chi}(0,x) = \bar{\psi}(x)$$
$$\Delta[V]\chi(t,x) = \sum_\mu \left[V_\mu(t,x)\chi(t,x+\hat{\mu}) + V_\mu^\dagger(t,x-\hat{\mu})\chi(t,x-\hat{\mu}) - 2\chi(t,x)\right]$$

Discretization choices

- S_{fl}: Wilson, Symanzik, Zeuthen
- $E(t)$: plaquette, clover, improved
- Lie-group Runge–Kutta integrators

Cost profile

- Gauge flow: force evaluations
- Fermion flow: covariant Laplacians
- Cost scales with V, flow steps, sources, and t's

Lüscher, JHEP 08 (2010) 071
Lüscher, JHEP 04 (2013) 123
Fritzsch, Ramos, JHEP 10 (2013) 008
Bazavov, Chuna, arXiv:2101.05320
Bazavov, BIT Numer. Math. 66 (2026) 38

Compute flowed observables at fixed physical t, then take $a \to 0$ before the SFTX

Cutoff Effects at Fixed Flow Time

- With chiral symmetry, or automatic/improved formulations, expect leading effects

$$O(t,a) = O(t,0) + \mathcal{O}(a^2)$$

- For Wilson–Clover fermions, standard action improvement is still needed: c_{sw}, b_m, b_g
- No further operator-by-operator $\mathcal{O}(a)$ improvement: the remaining terms are universal

$$b_\chi : \quad Z_\chi \longrightarrow Z_\chi\,(1 + am_q b_\chi + \ldots)$$

$$c_{\mathrm{fl}} : \quad \underbrace{\overline{\chi}(t,x)\,\chi(s,y)}$$

$$S_{\mathrm{eff}} = S_0 + aS_1 + \ldots$$

$$S_1 = \int d^4x\ \left\{c_{\mathrm{fl}}\,\overline{\lambda}\lambda + am_q b_\chi\,(\overline{\lambda}\psi + \overline{\psi}\lambda) + \ldots\right\}_{t=0}$$

Equivalent implementation:

$$\chi(0,x) = \left(1 + \frac{a}{2}c_\chi\,\not{D}\right)\psi(x)$$

$$\bar{\chi}(0,x) = \bar{\psi}(x)\left(1 - \frac{a}{2}c_\chi\,\overleftarrow{\not{D}}\right)$$

Lüscher, JHEP 04 (2013) 123

A.Sh., Nucl. Phys. B 881 (2014) 71
A.Sh., PoS LATTICE2022 (2023) 348

Short-Distance Cutoff Effects

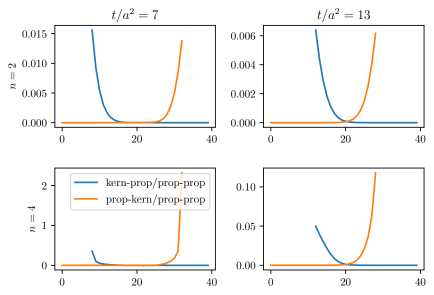
- Still represented in the Symanzik effective theory
- Modify improvement terms of local fields at $t = 0$

$$P_{\mathrm{R}} = Z_{\chi}\left(1 + am_q b_{\chi} + \ldots\right) P_{\mathrm{I}}$$

$$P_{\mathrm{I}}(0,x) = P(0,x) + a\,\widetilde{c}_P\,\widetilde{P}(0,x)$$

$$\widetilde{P}(0,x) = \overline{\lambda}(0,x)\gamma_5\psi(x) + \overline{\psi}(x)\gamma_5\lambda(0,x)$$

$$C_{P\widetilde{P}}(t) = a^3 \sum_{\mathbf{x}} \left\langle P(t,x)\,\widetilde{P}(0,\mathbf{0})\right\rangle$$



Lüscher, JHEP 04 (2013) 123
A.Sh., Nucl. Phys. B 881 (2014) 71
A.Sh., PoS LATTICE2022 (2023) 348

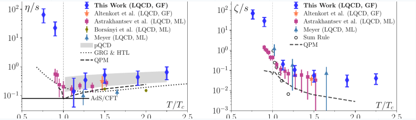
Other Gauge-Field Applications

- **Reference scales**
  $$t^2\langle E(t)\rangle \quad \Rightarrow \quad t_0,\ w_0$$
- **Energy-momentum tensor**
  $$\widetilde{O}_i(t) = \zeta_{ij}(t,\mu)\, O_j^{\mathrm{R}}(\mu) + \mathcal{O}(t)$$
- **Thermal gauge observables:** equation of state, viscosities, Polyakov loops, topology
- **Static and pNRQCD observables:** static force, E- and B-field correlators, quarkonium matrix elements

Lüscher, JHEP 08 (2010) 071; Borsanyi et al., JHEP 09 (2012) 010; Suzuki, PTEP 2013 (2013) 083B03; Makino, Suzuki, PTEP 2014 (2014) 063B02; Harlander, Kluth, Lange, Eur. Phys. J. C 78 (2018) 944; Brambilla et al., arXiv:2312.17231

Two uses of the flow

- t_0, w_0: precise lattice infrastructure
- EMT and Wilson-line observables: renormalization and matching framework



Example thermal observable: Ding, Shu, Zhang, arXiv:2601.14967

B-mesons lifetimes

- B-meson lifetimes are measured experimentally to high precision
- Key observables for probing New Physics $\to$ high precision in theory needed

$$\Gamma(H_b) = \frac{1}{\tau(H_b)} = \Gamma_b + \delta\Gamma(H_b), \qquad \delta\Gamma(H_b) \propto \mathcal{O}\left(\frac{\Lambda^2}{m_b^2}\right)$$

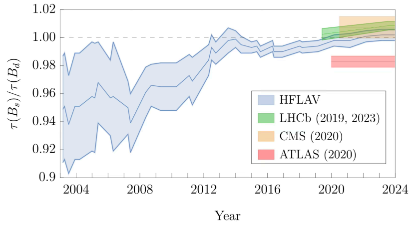
$$\frac{\tau(H_b)}{\tau(H_b')} = 1 + \frac{\delta\Gamma(H_b') - \delta\Gamma(H_b)}{\Gamma_b}$$

- For B lifetimes, we use the Heavy Quark Expansion
- $\langle O_i \rangle$ are leading uncertainties for B lifetime differences

$$\Gamma(H_b) = \Gamma_3 + \frac{\Gamma_5}{m_b^2} + \frac{\Gamma_6}{m_b^3} + 16\pi^2 \frac{\Gamma_7}{m_b^4} + \cdots$$

- Exploratory studies from 20–30 years ago $\to$ new developments for lifetime ratios
  - contributions from disconnected diagrams
  - mixing with lower-dimensional operators

Di Pierro and Sachrajda, Nucl. Phys. B 534 (1998) 373–391
Lin, Detmold, Meinel, Phys. Rev. D 107 (2023) 188
Di Pierro, Sachrajda and Michael, Phys. Lett. B 468 (1999) 143–149
Becirevic, PoS HEP2001 (2001) 098



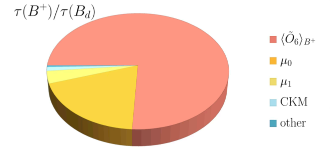


Fig. 7 Composition of the theoretical error in the HQE prediction of $\tau(B^+)/\tau(B_d)$.

Albrecht, Bernlochner, Lenz and Rusov, Eur. Phys. J. ST 233 (2024) 359–390

Matrix elements

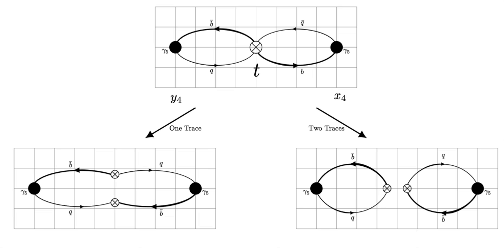
- In the Standard Model 4 operators contribute
- Four-quark operators inserted in three-point correlation functions
- Flowed fermion fields renormalized

$$O_1 = (\bar{h}_i\gamma_\mu(1-\gamma_5)q_i)(\bar{h}_j\gamma_\mu(1-\gamma_5)q_j), \qquad \langle O_1(\mu)\rangle = f_{B_q}^2 M_{B_q}^2 B_1^q(\mu),$$

$$O_2 = (\bar{h}_i(1-\gamma_5)q_i)(\bar{h}_j(1+\gamma_5)q_j), \qquad \langle O_2(\mu)\rangle = \frac{M_{B_q}^2}{(m_b+m_q)^2} f_{B_q}^2 M_{B_q}^2 B_2^q(\mu),$$

$$T_1 = (\bar{h}_i\gamma_\mu(1-\gamma_5)q_j)(\bar{h}_j\gamma_\mu(1-\gamma_5)q_i), \qquad \langle T_1(\mu)\rangle = f_{B_q}^2 M_{B_q}^2 \epsilon_1^q(\mu),$$

$$T_2 = (\bar{h}_i(1-\gamma_5)q_j)(\bar{h}_j(1+\gamma_5)q_i), \qquad \langle T_2(\mu)\rangle = \frac{M_{B_q}^2}{(m_b+m_q)^2} f_{B_q}^2 M_{B_q}^2 \epsilon_2^q(\mu)$$



Extrapolation to zero flow time

$$Q_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)q)(\bar{Q}\gamma_\mu(1-\gamma_5)q)$$

$$\langle H|Q_1|H\rangle(\mu) = \frac{8}{3} f_H^2 M_H^2 B_1^H(\mu)$$

RG improved flow-time analysis

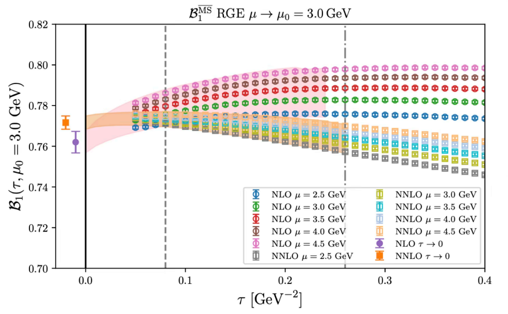
$$\mu^2 \frac{d}{d\mu^2}\zeta(t,\mu) = \zeta(t,\mu)\,\gamma(\alpha_s(\mu))$$

$$f(t,\mu) = c_0 + t\left[c_1 + c_2 \ln(\mu^2 t)\right]$$

Status and Outlook

- B-physics hadronic matrix elements are amenable to computation
- $\Delta B = 0$ four-quark matrix elements are the final target
- Standard renormalization introduces mixing with lower-dimensional operators

Black, Harlander, Lange, Rago, A. Sh., Witzel, 2603.28517; 2603.28516



Suzuki, Taniguchi, Suzuki and Kanaya, Phys. Rev. D 102 (2020) 034508
Harlander and F. Lange, Phys. Rev. D 105 (2022) L071504

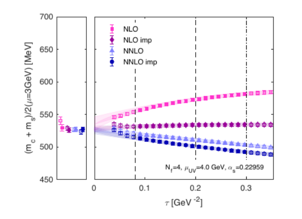
Quark Masses from Flowed PCAC

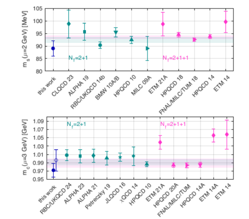
$$\overline{R}^{rs}_{AP}(t) = \sqrt{\overline{R}^{rs}_{A_0}(t)\,\overline{R}^{rs}_{P}(t)}$$

$$m^{(r)}_{\mathrm{GF}}(t) + m^{(s)}_{\mathrm{GF}}(t) = M^{(rs)}_{PS}\,\overline{R}^{rs}_{AP}(t)$$

$$m_{\overline{\mathrm{MS}}}(\mu) = \lim_{t\to 0}\zeta^{-1}_{AP}(\mu, t)\, m_{\mathrm{GF}}(t)$$

- Flowed sink, local source: the ratio cancels Z_χ and source renormalization
- Continuum limit at fixed t, then SFTX matching to $\overline{\mathrm{MS}}$
- RG-improved matching stabilizes the $t \to 0$ extrapolation
- Test case: RBC/UKQCD $N_f = 2+1$ DWF ensembles, η_s, η_c, D_s
- $m_s^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 89(3)\,\mathrm{MeV}$,
  $m_c^{\overline{\mathrm{MS}}}(3\,\mathrm{GeV}) = 972(16)\,\mathrm{MeV}$, $m_c/m_s = 12.1(4)$





Black, Harlander, Hasenfratz, Rago and Witzel,
Renormalized quark masses using gradient flow, arXiv:2506.16327

GF Schemes for Fermion Operators

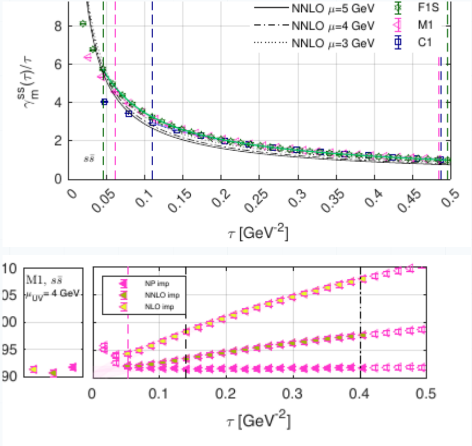
$$O^X_{\mathrm{GF}}(t,x) = \widetilde{Z}^X_\chi(t)\,\bar{\chi}(t,x)\Gamma\chi(t,x), \qquad X = A, V$$

$$\widetilde{Z}^A_\chi(t) = Z_A\,R_A(t), \qquad \widetilde{Z}^V_\chi(t) = Z_V\,R_V(t)$$

- Fix Z_χ from axial/vector Ward-normalized two-point functions
- Avoid backward flow for local ringed definitions
- Reuse ringed SFTX coefficients as $t \to 0$

Concerns

- Final formulae seem largely independent of the unflowed correlators used in the scheme definitions
- Since C_m is determined in infinite volume, pushing t down at fixed a may reintroduce ordinary cutoff effects
- Finite-t renormalization of bare $t = 0$ operators is delicate: $Z_O(t, a)$ has no continuum limit by itself, and continuum matching does not encode lattice mixing or power divergences



Black, Hasenfratz and Witzel, Gradient Flow Renormalization Schemes for Composite Fermion Operators, arXiv:2607.00493

Perturbative Matching with Flowed Fields

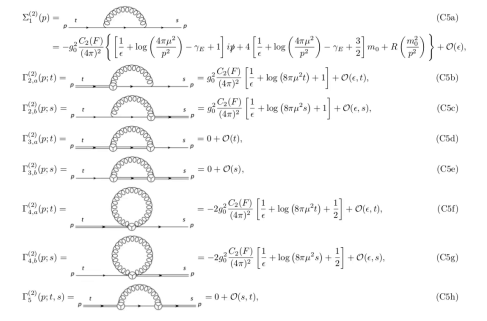
$$\chi(t,x) = \int d^D y\, K_t(x-y)\psi(y)$$
$$+ \int_0^t ds \int d^D y\, K_{t-s}(x-y)\Delta'\chi(s,y)$$

$$\langle \chi(t,x)\bar{\chi}(s,y)\rangle = \int \frac{d^D p}{(2\pi)^D}\, e^{ip(x-y)}\, \frac{e^{-(t+s)p^2}}{M_0 + i\, \not{p}}$$
$$+ \mathcal{O}(g_0^2)$$

$$\text{flowed loops} \Rightarrow \frac{1}{\epsilon} + \log(8\pi\mu^2 t) + \text{finite} \Rightarrow Z_\chi$$

$$\chi_{\mathrm{R}} = Z_\chi^{1/2}\chi, \qquad \bar{\chi}_{\mathrm{R}} = \bar{\chi} Z_\chi^{1/2}$$

$$Z_\chi = 1 + \frac{3C_F}{16\pi^2\epsilon}\, g^2 + \mathcal{O}(g^4)$$



M. Lüscher, JHEP 04 (2013) 123

C. Rizik, C. Monahan, A.Sh.,
Phys. Rev. D 102 (2020) 034509