Reducing $SU(3)$ LGT circuit complexity

using permutation-symmetric $SU(N)$ Clebsch-Gordan coefficients

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Now is the time to think about $SU(3)$

- Simpler gauge groups have been more a suitable target for algorithm development in the NISQ era.
- But fault tolerance is on the horizon.
- Recent hardware and QEC advancements make it plausible to soon see:
  - dozens to hundreds of logical qubits;
  - $O(10^5)$ logical operations.

But $SU(3)$ LGT is still a hard problem

- Quantum simulations of $SU(3)$ involve complications simply not present in simpler gauge theories like $SU(2)$.
- These complications drive complexity in circuits.
- Even FTQCs will be subject to resource constraints.

However: the problem has substructure.

Performing a simulation of $SU(3)$

Constructing a basis

Performing time evolution

State preparation (i.e. GS, wave packets)

Choosing a fermion discretization

Extracting interesting observables

Performing a simulation of $SU(3)$

Goal for talk

Constructing a basis

Classical pre-computation of CGCs

Performing time evolution

State preparation (i.e. GS, wave packets)

Choosing a fermion discretization

Extracting interesting observables

Outline

1. Hamiltonian simulation framework
2. Permutation-symmetric CGCs
3. Resulting circuit simplifications

1. Hamiltonian simulations in the reduced electric basis

The irrep (electric) basis

- Build the single-link Hilbert space from eigenstates of the electric Casimir in the Kogut-Susskind Hamiltonian.

- Each end of the link transforms in its own copy of *the same* irrep of $SU(3)$.

- Link quantum numbers are given by the shared irrep, and independent left & right quantum numbers for specific states in the multiplet.

- Lattice basis states are tensor products of links.

- Proposed by T. Byrnes, Y. Yamamoto, Phys. Rev. A, 2006

$$H_{KS} = \frac{g^2}{2a}\sum_{\vec{s},i} E^2(\vec{s},\hat{e}_i) + \frac{1}{g^2 a}\sum_{\vec{s},i<j}\left(2N - \square(\vec{s},\hat{e}_i,\hat{e}_j) - \square^\dagger(\vec{s},\hat{e}_i,\hat{e}_j)\right)$$

$$\mathscr{H}_{\text{Link}} = \left\{ \left| R, \vec{r}_L, \vec{r}_R \right\rangle \right\}$$

$$\left|\Lambda\right\rangle = \bigotimes_{\ell} \left| R_\ell, \vec{r}_L, \vec{r}_R \right\rangle$$

The *reduced* electric basis

- The BY basis has many unphysical (gauge-variant) states.
- Throw them out using the Gauss law. This gives the *reduced electric basis.*
  - A. Ciavarella, et al., Phys. Rev. D, 2021; P. Balaji, et al., Phys. Rev. D, 2025; P. Balaji, et al., Phys. Rev. D, 2026
- In practice, this means that all links meeting at a site must form an $SU(3)$ singlet.

$$|\Lambda\rangle = \bigotimes_{\vec{s}} \sum_{r} \underbrace{\left(\prod_{\ell_L(s)} \phi(r_L) \right)}_{\text{Phase convention}} \underbrace{\left\langle \mathbf{1}, \overbrace{\Gamma_s}^{\text{Singlet multiplicity index for site } s} \middle| \otimes_{\ell_L(s)} (\bar{R}_\ell, \tilde{r}_L) \otimes_{\ell_R(s)} (R_\ell, r_R) \right\rangle}_{\text{CGC}} \overbrace{\left| \otimes_{\ell_{L,R}(s)} (R_\ell, r_{L,R}) \right\rangle}^{\text{“Half-links” meeting at } s}$$

Reduced electric basis matrix elements

- Electric Hamiltonian is diagonal in this basis.
- Magnetic Hamiltonian matrix elements decompose into "site factors"
- To circuitize magnetic Hamiltonian time evolution, break up into a sum of Givens rotation generators, then Trotterize.

$$
\boxed{\langle \Lambda_n | \square | \Lambda_m \rangle} = \left(\prod_{k=1}^{4} \sqrt{\frac{\dim(R^m_{\ell_k})}{\dim(R^n_{\ell_k})}} \right) \left\{\!\!\left\{ \begin{matrix} \bar{R}^m_{\ell_4} & \Gamma^m_{s_1} & \bar{R}^m_{\ell_1} \\ f & \vec{C}_{s_1} & \bar{f} \\ \bar{R}^n_{\ell_4} & \Gamma^n_{s_1} & \bar{R}^n_{\ell_1} \end{matrix} \right\}\!\!\right\} \left\{\!\!\left\{ \begin{matrix} R^m_{\ell_1} & \Gamma^m_{s_2} & \bar{R}^m_{\ell_2} \\ f & \vec{C}_{s_2} & \bar{f} \\ R^n_{\ell_1} & \Gamma^n_{s_2} & \bar{R}^n_{\ell_2} \end{matrix} \right\}\!\!\right\}
$$
$$
\times \left\{\!\!\left\{ \begin{matrix} R^m_{\ell_2} & \Gamma^m_{s_3} & R^m_{\ell_3} \\ f & \vec{C}_{s_3} & \bar{f} \\ R^n_{\ell_2} & \Gamma^n_{s_3} & R^n_{\ell_3} \end{matrix} \right\}\!\!\right\} \left\{\!\!\left\{ \begin{matrix} \bar{R}^m_{\ell_3} & \Gamma^m_{s_4} & R^m_{\ell_4} \\ f & \vec{C}_{s_4} & \bar{f} \\ \bar{R}^n_{\ell_3} & \Gamma^n_{s_4} & R^n_{\ell_4} \end{matrix} \right\}\!\!\right\}
$$

$$
\boxed{\left\{\!\!\left\{ \begin{matrix} R & G & S \\ f & \vec{C} & \bar{f} \\ R' & G' & S' \end{matrix} \right\}\!\!\right\}} = \sum_{\sigma=1}^{\dim(f)} \sum_{r=1}^{\dim(R)} \sum_{r'=1}^{\dim(R')} \sum_{s=1}^{\dim(S)} \sum_{s'=1}^{\dim(S')} \sum_{\vec{c} \in \vec{C}} \phi(\sigma)
$$
$$
\times \langle R', r' | (R, r) \otimes (f, \sigma) \rangle \; \langle S', s' | (S, s) \otimes (\bar{f}, \tilde{\sigma}) \rangle
$$
$$
\times \langle \mathbf{1}, G | (R, r) \otimes (S, s) \otimes (\vec{C}, \vec{c}) \rangle_F
$$
$$
\times \langle \mathbf{1}, G' | (R', r') \otimes (S', s') \otimes (\vec{C}, \vec{c}) \rangle_F
$$

Try it out yourself with ymcirc.

Observations

- Circuit depth of a magnetic Trotter step is controlled by the number of non-vanishing matrix elements of $\left\langle \Lambda_n \mid \Box \mid \Lambda_m \right\rangle$.
- Site factors are sums of products of $SU(3)$ CGCs.
- There is some freedom to choose conventions for CGCs.
- Conjecture: choosing a smart convention for computing CGCs could cause more matrix elements to vanish.

2. Permutation-symmetric CGCs

$SU(3)$ Clebsch-Gordan Coefficients and S_n

- Reminder: CGCs are just matrix elements connecting tensor product and direct sum bases.
- If there are n repeated irreps $\left|R\right\rangle$ in a tensor product state, there is a nontrivial group action by the permutation group S_n on the state.
- $SU(3)$ group action commutes with S_n group action.

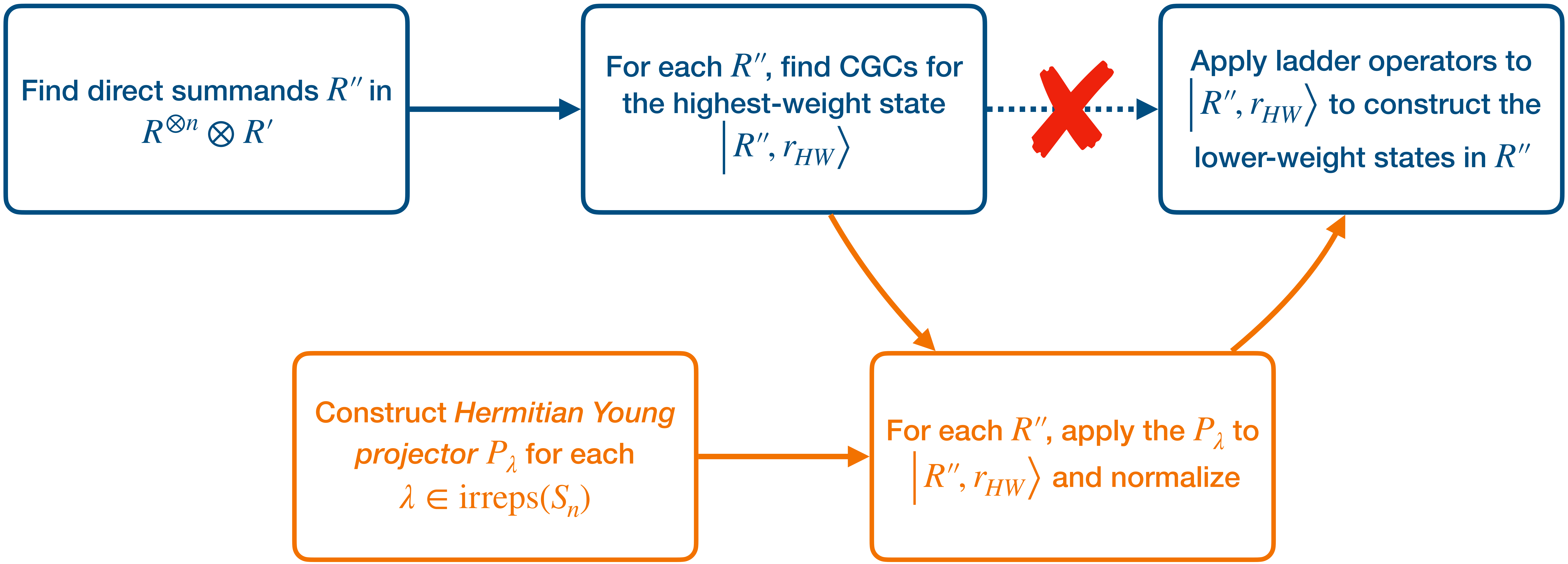
$$U\sigma(\vec{R})\left|\vec{R},R'\right\rangle = \left(\prod_{i=1}^{n}\left|D^R(U)\sigma(\vec{R})_i\right\rangle\right)\left|D^{R'}(U)R'\right\rangle = \sigma(\vec{R})U\left|\vec{R},R'\right\rangle$$

Idea: pick $SU(3)$ CGCs so that the direct summands are *also* S_n irreps

“Diagonalizing” both $SU(3)$ and S_n
Ingredients

- Don’t need to re-invent the wheel.
- Compute $SU(3)$ CGCs numerically via $SU(N)$ algorithm due to AKHD.
  - A. Alex, et al., Journal of Mathematical Physics, 2011.
- For n identical tensor factors R, compute *Hermitian Young projection operators* P_λ which have an $R^{\otimes n}$ tensor product as their carrier space.
  - $\lambda \in \mathrm{irreps}(S_n)$ is a Young diagram with n boxes (i.e. a partition of n).
  - Algorithm due to AZ&W.
    - J. Alcock-Zeilinger, H. Weigert, Journal of Mathematical Physics, 2017.

$SU(N)$ CGC algorithm from AKHD



"Diagonalize S_n" using AZ&W

“Diagonalizing” both $SU(3)$ and S_n

Why does this work?

Lower weight states obtained using lowering operator

Group actions commute. $\Longrightarrow$ $E_\alpha^- P_\lambda \left| R'', r_{HW} \right\rangle$

will still be in an irrep $\lambda \in \mathrm{irreps}(S_n)$.

“Diagonalizing” both $SU(3)$ and S_n

Example: direct-sum decomposition of three adjoints

$$8 \otimes 8 \otimes 8 = 64 \oplus 35'_2 \oplus 35_2 \oplus 27_6 \oplus 10'_4 \oplus 10_4 \oplus 8_8 \oplus 1_2$$

AKHD’s algorithm

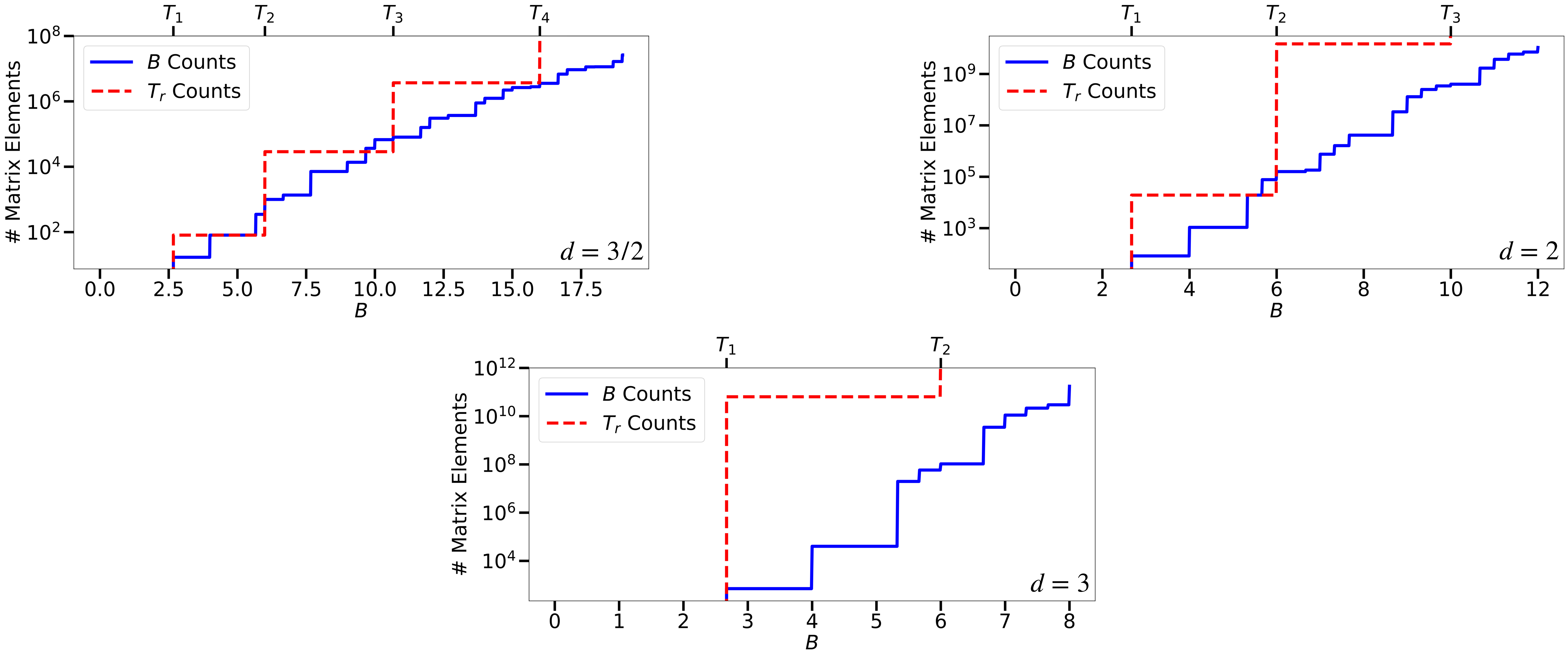
Decompositions computed with pyclebsch.

$$\begin{aligned} 8 \otimes 8 \otimes 8 = 64^{(3)} &\oplus 35'^{(2,1)} \oplus 35^{(2,1)} \oplus 27^{(1,1,1)} \oplus 27_2^{(2,1)} \oplus 27^{(3)} \\ &\oplus 10'^{(1,1,1)} \oplus 10'^{(2,1)} \oplus 10'^{(3)} \oplus 10^{(1,1,1)} \oplus 10^{(2,1)} \oplus 10^{(3)} \\ &\oplus 8^{(1,1,1)} \oplus 8_3^{(2,1)} \oplus 8^{(3)} \oplus 1^{(1,1,1)} \oplus 1^{(3)} \end{aligned}$$

AKHD supplemented with AZ&W

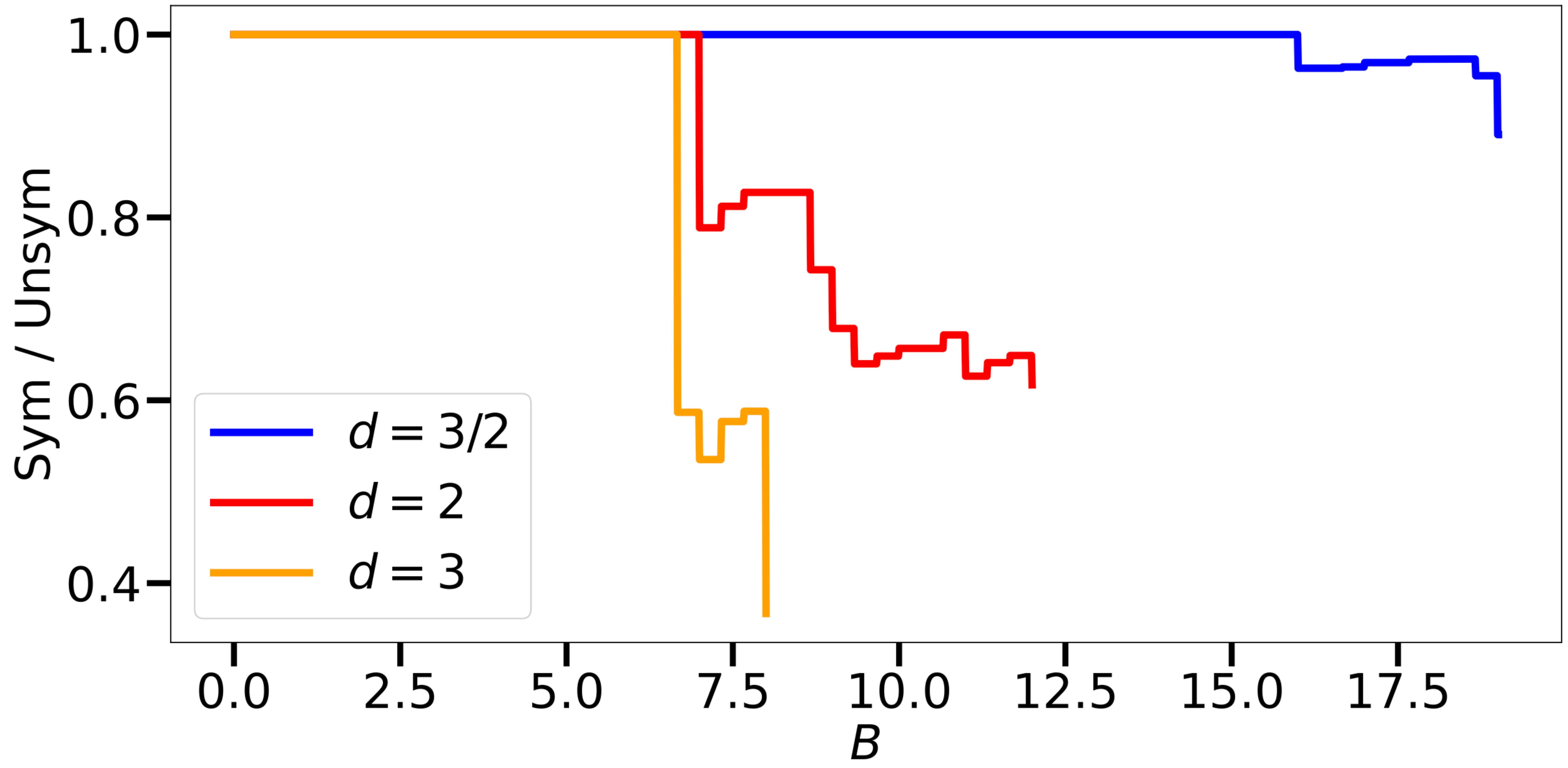
3. Resulting circuit simplifications

☐ matrix element counts *before* symmetrization



P. Balaji, et al., Phys. Rev. D, 2026

☐ matrix element counts *after* symmetrization



These ratios *directly* translate into circuit depth reductions.

P. Balaji, et al., Phys. Rev. D, 2026

Fine-tuning irrep truncations

Singlet probabilities

$$\left|\vec{R}\right\rangle = \sum_r \left\langle \mathbf{1}, \Gamma_s \mid \otimes_{\ell_L(s)} (\bar{R}_\ell, \tilde{r}_L) \otimes_{\ell_R(s)} (R_\ell, r_R) \right\rangle \Big| \otimes_{\ell_{L,R}(s)} (R_\ell, r_{L,R}) \Big\rangle$$

- Using classically pre-computed CGCs, we can evaluate the ground state probability of encountering various "singlet-forming links meeting at site" using MC to evaluate the Euclidean path integral.

- Instead of truncating on link irreps or local energy density, opens the possibility of truncating on "most likely singlets in the ground state."

$$\frac{\left\langle \hat{P}_{\vec{R}} \right\rangle}{\left\langle \hat{P}_{\vec{1}} \right\rangle}$$

Ratio of g.s. expectation of singlet projectors

P. Draper and D. Gupta, to appear

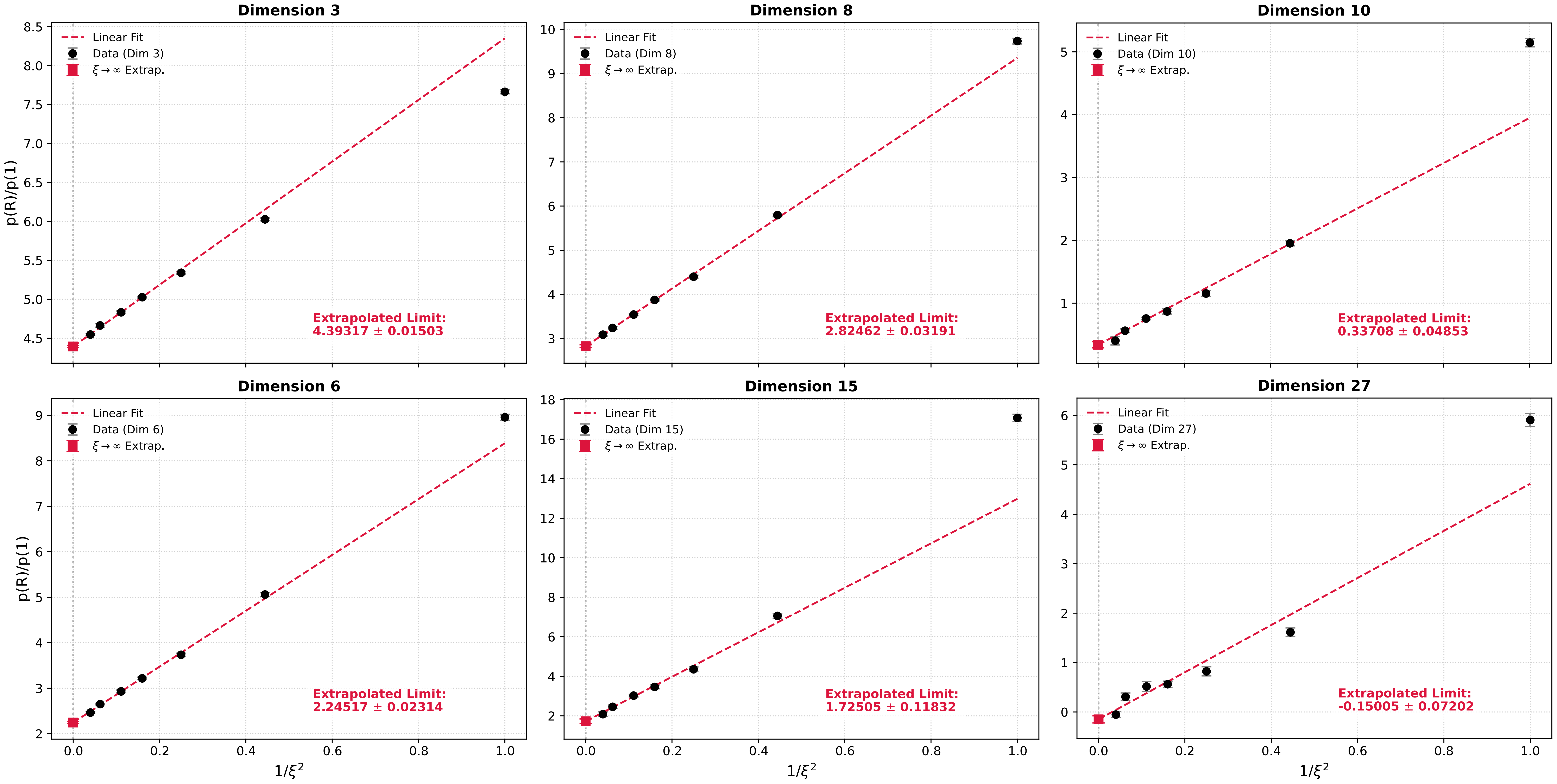
vertex content	N_1	$R(\vec{R})$
$3\,3\,3\,\bar{3}\,\bar{3}\,\bar{3}$	6	2.80(31)
$3\,3\,\bar{3}\,\bar{3}\,8$	4	1.51(26)
$8\,8\,3\,\bar{3}$	3	0.92(20)
$3\,3\,\bar{3}\,\bar{3}$	2	0.90(7)
$3\,\bar{3}\,8$	1	0.42(4)
$3\,\bar{3}$ (pair)	1	0.50(2)

$\beta = 6,\ \xi \to \infty$

Fine-tuning irrep truncations

Link probabilities

Preliminary (courtesy of D. Gupta)



- Results of analogous MC analysis for individual links yields relative probabilities of link irreps.
- CGCs still needed to construct lattice basis states, but not projectors.

Summary

- Implementing quantum simulations of $SU(3)$ LGT is a major challenge, but it breaks up into smaller, manageable problems.

- Improving pre-computation of CGCs via S_n symmetrization exemplifies this structure of the problem space because:

    - It was a technical but well-scoped sub-problem.

    - Solving this narrow technical challenge cascaded to other parts of the problem space (i.e. circuit depth reductions, current investigations of improved irrep basis truncation schemes).

Thank you!

Backup slides

Hermitian Young Symmetrizer action

on a highest-weight state

- Input:
    - $SU(N)$ irrep tensor product with (say) an irrep R' repeated n times.
    - Target irrep R to compute CGCs for.

Hermitian Young Symmetrizer action

on a highest-weight state

- Procedure:
  - S_n-symmetric $SU(N)$ DS decomposition (given via *plethysms* + Littlewood-Richardson rule). Also yields standard Young tableaux $\{\lambda\}$ for all S_n irreps that appear.
  - Use "generalized J_z" operators to identify TPS basis states which have the right quantum numbers. This is the subspace which contains one or more copies of R as direct summands.
  - Group action of S_n elements is a product of a linear combination of permutations on the R' factors. Write this as a set of standard Young tableax.
  - Each Young tableau defines an (anti)symmetrization of the TP basis states. AZ&W proves that the "naive" symmetrizer constructed from sets of them them fails to be Hermitian for $n > 5$, and provides a recipe for restoring Hermiticity which involves applying additional (anti)symmetrizations that map the naive operator to a "lexically ordered one" where the Young tableau is row- or column-ordered.

MC details for “fine-tuning” irrep truncations

- Reported values are fits to the $\xi \to \infty$ (Hamiltonian) limit.
- Obtained at $\beta = 6$, no improvement terms, Spatial Wilson action, temporal Kogut-Susskind action.
- $4 \times 4 \times 4 \times L_t$ lattice, with $L_t = 24\xi$.
- 6 chains were run with cold or hot starts.