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27 JULY 2026

26MM.XXXXXX



A (LATTICE!) VECTOR-VECTOR-AXIAL ANOMALY IN 4D



Aleksey
Cherman

UMN

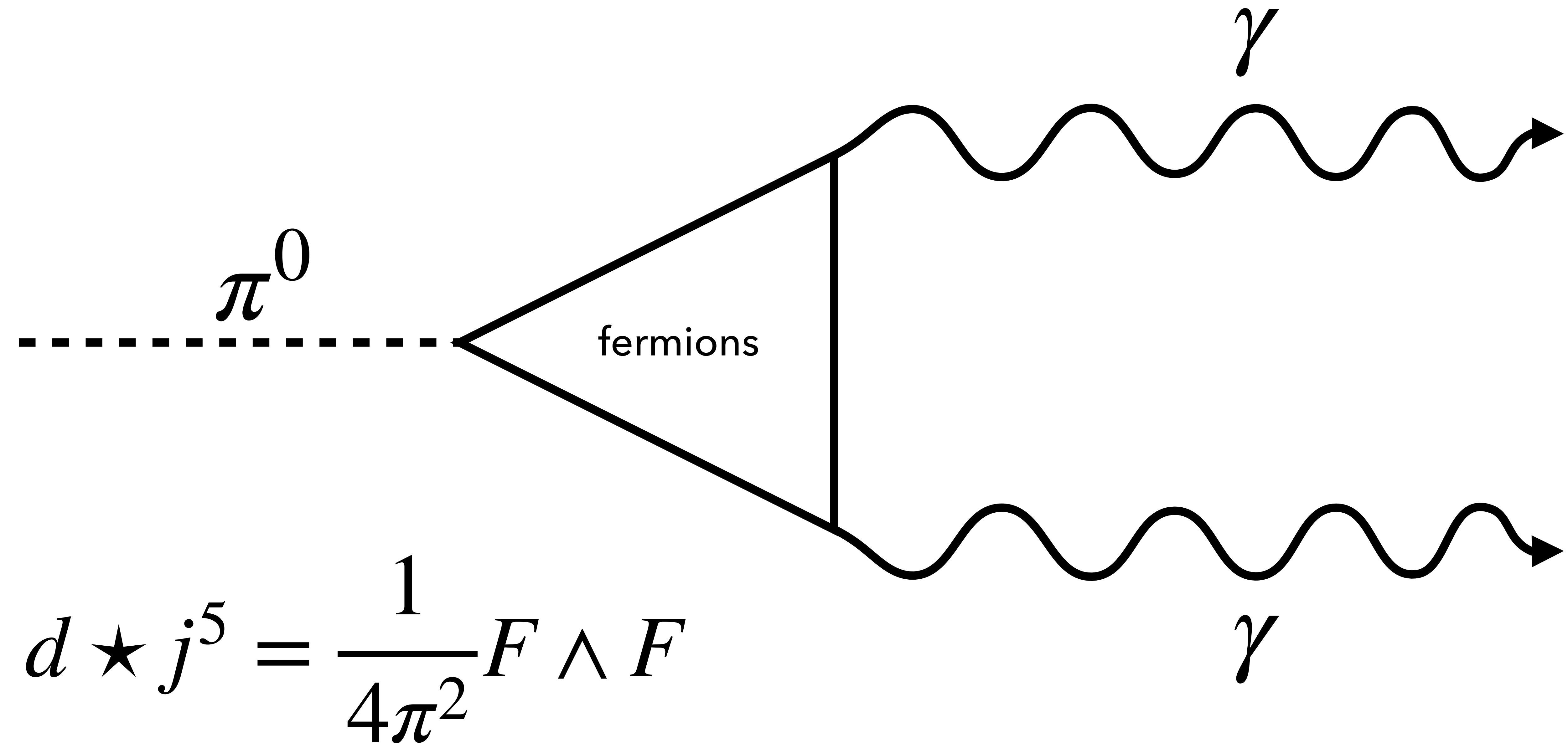


Shi
Chen

pd at UMN

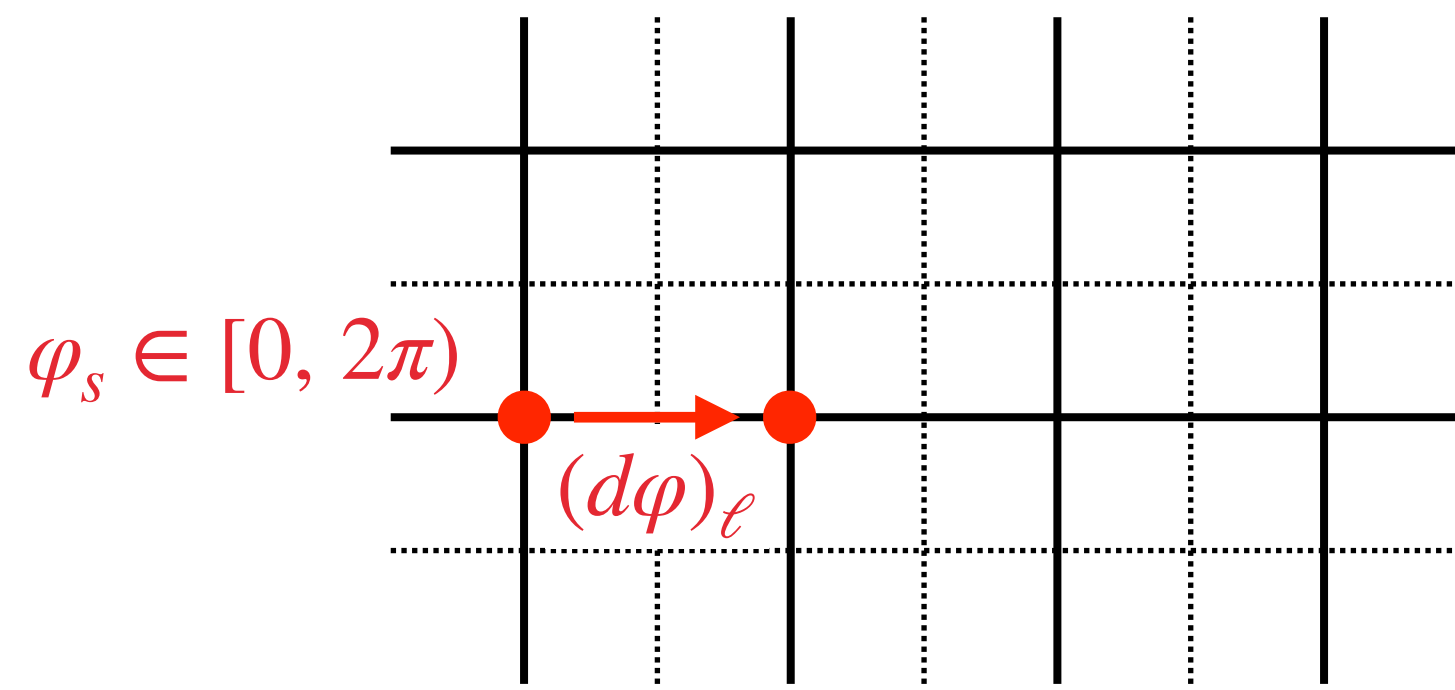
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MOTIVATION: THE CHIRAL / ABJ ANOMALY



COMPACT BOSON + XY

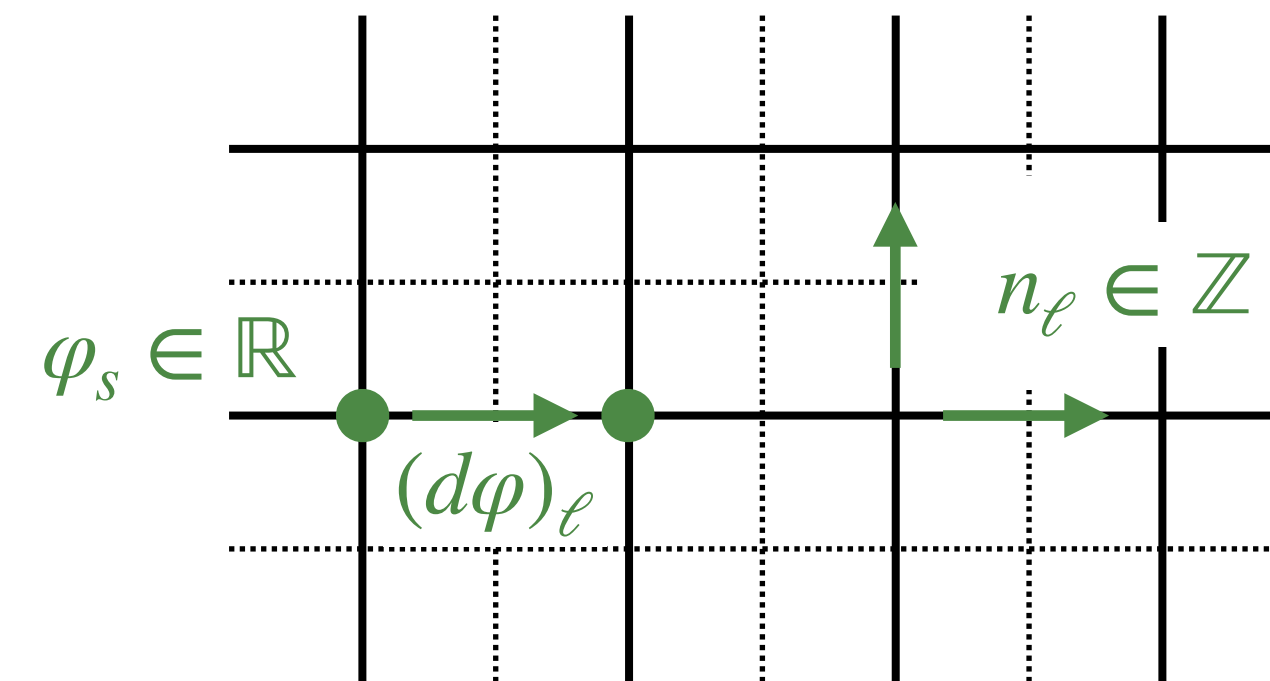
$$S = \frac{\kappa}{2} \sum_{\ell} [1 - \cos(d\varphi)_{\ell}]$$



$$\varphi \rightarrow \varphi + c \quad \text{global shift}$$

$$\varphi_s \rightarrow \varphi_s + 2\pi \quad \text{local shift}$$

$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi n_{\ell}]^2 \quad (\text{quadratic!})$$



$$\varphi \rightarrow \varphi + c \quad \text{global shift}$$

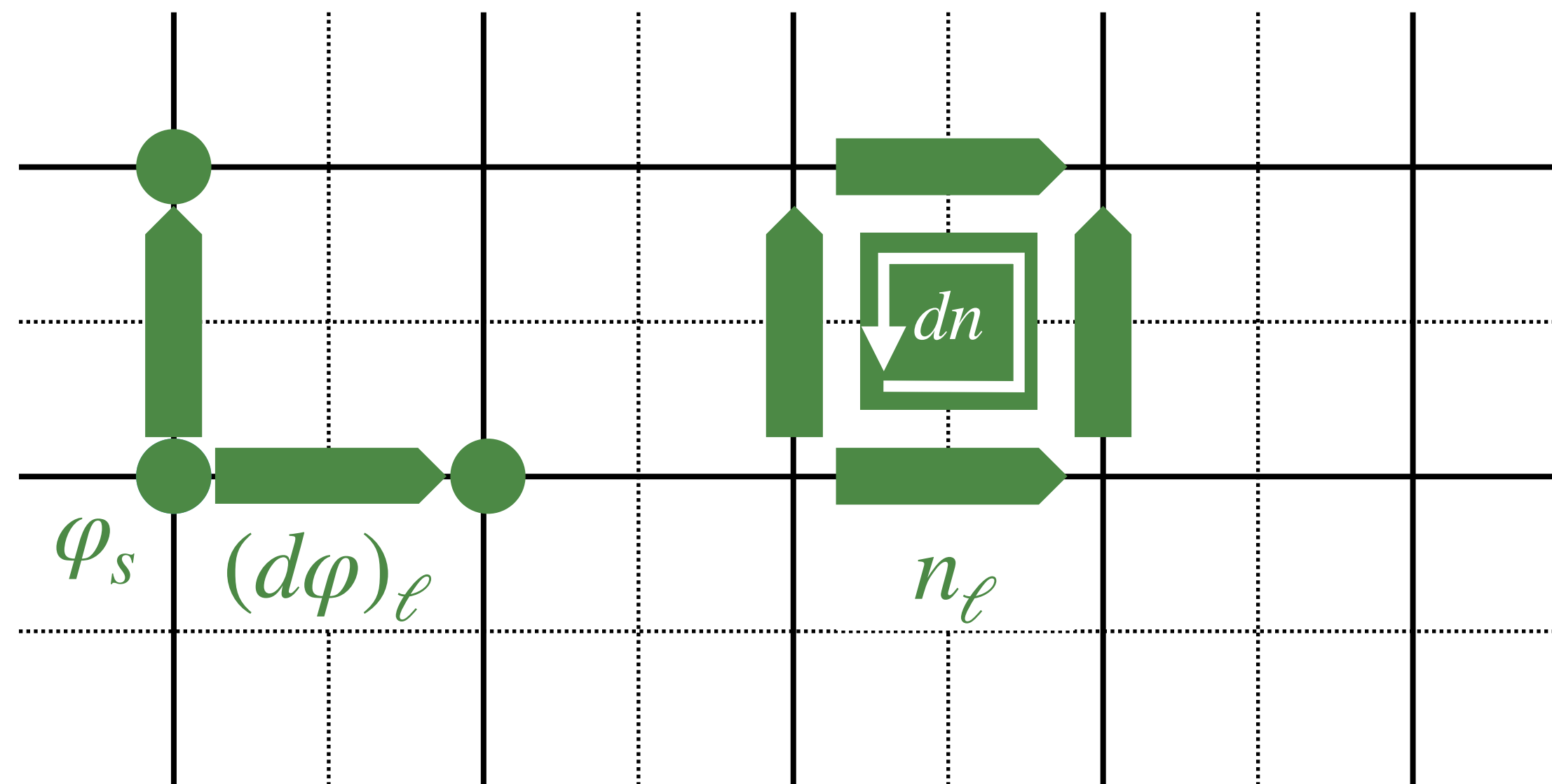
$$\begin{aligned} \varphi_s &\rightarrow \varphi_s + 2\pi q_s \\ n_{\ell} &\rightarrow n_{\ell} + (dq)_{\ell} \end{aligned} \quad \begin{array}{l} \text{discrete gauge} \\ \text{redundancy} \\ (q_{\ell} \in \mathbb{Z}) \end{array}$$

$$U(1) \cong \mathbb{R}/2\pi\mathbb{Z}$$

$$w_A = \frac{1}{2\pi} \oint_{\partial A} \omega = \frac{1}{2\pi} \int_A d\omega$$

$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_{\ell} - 2\pi n_{\ell} \right]^2$$

$$\frac{1}{2\pi} \sum_{\ell \in \partial A} (d\varphi - 2\pi n)_{\ell} = \frac{1}{2\pi} \sum_{p \in A} \left[\overbrace{d(d\varphi - 2\pi n)}^{d^2 = 0} \right]_p = - \sum_{p \in A} (dn)_p \in \mathbb{Z}$$



$$dn \neq 0$$

Dynamical vortices,
2D BKT,
gapped phase for small κ .

LATTICE EXTERIOR ALGEBRA

$$\begin{aligned}
 d^2 &= 0 \\
 \langle da, b \rangle &= \langle a, \delta b \rangle \quad \checkmark \\
 \delta^2 &= 0 \\
 \Delta &= d\delta + \delta d \\
 \langle \Delta f, f \rangle &= \langle df, df \rangle + \langle \delta f, \delta f \rangle \\
 \sum a \wedge \star b &= \langle a, b \rangle \\
 (a + b) \wedge c &= a \wedge c + b \wedge c \\
 (a \wedge b) \wedge c &= a \wedge (b \wedge c)
 \end{aligned}$$

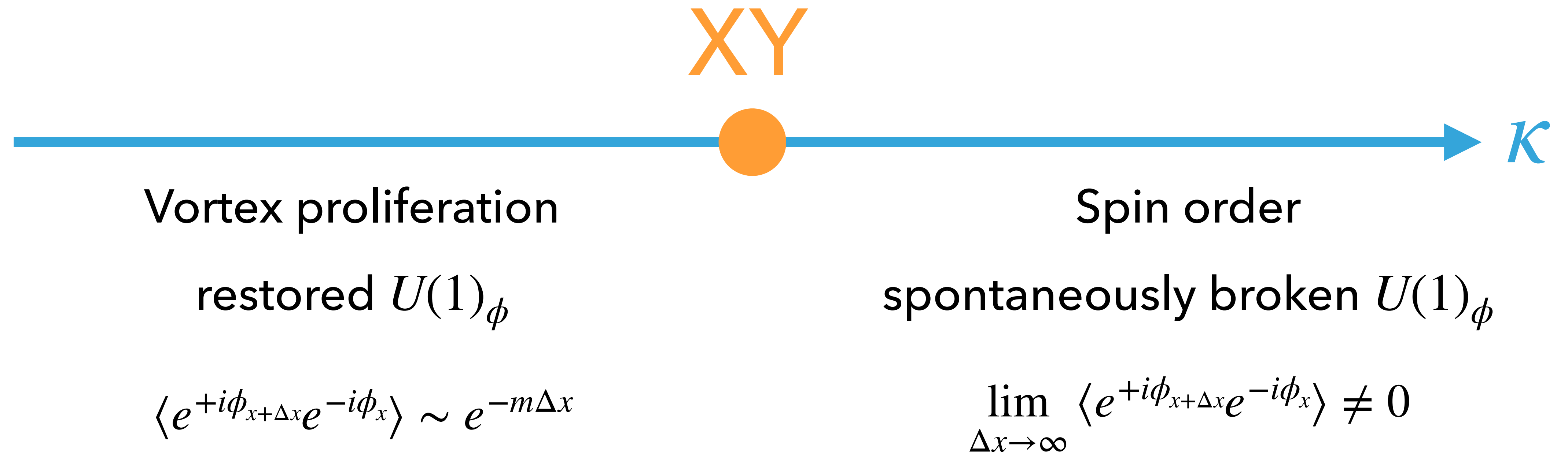
$$d(a \wedge b) = da \wedge b + (-1)^{\deg a} a \wedge db$$

$$\begin{aligned}
 &\times \quad a \wedge b = (-1)^{nm} b \wedge a \\
 &\quad \star \star a = (-1)^{p(D-p)} \\
 &\quad \delta = (-1)^{D(k+1)+1} \star d \star \\
 &\quad a \wedge b \neq (-1)^{nm} b \wedge a \\
 &\times \quad \star \star a = (-1)^{p(D-p)} T \circ \\
 &\quad \delta = (-1)^{D(k+1)+1} T \circ \star d \star \\
 &\quad \text{with } T \text{ a lattice translation}
 \end{aligned}$$

For experts:

$\wedge$ really means $\smile$ throughout!

$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_{\ell} - 2\pi n_{\ell} \right]^2$$



$$S = \frac{\kappa}{2} \sum_{\ell} (d\phi - 2\pi n)_{\ell}^2 + i \sum_h \theta_h (dn \wedge dn)_h$$

Villainized XY

Lagrange multiplier $\theta_h \in \mathbb{R}$

Eliminate the sign problem
by satisfying the constraint

Vortex worldsheets are
not allowed to intersect

$$\mathcal{Z} = \oint D\phi Dn D\theta e^{-S} = \oint D\phi Dn e^{-S} \prod_h [(dn \wedge dn)_h = 0]$$


$$S = \frac{\kappa}{2} \sum_{\ell} (d\phi - 2\pi n)_{\ell}^2 + i \sum_h \theta_h (dn \wedge dn)_h$$

Villainized XY

Lagrange multiplier

Global symmetries

mixed 't Hooft anomaly
ON THE LATTICE

$$U(1)_V: \quad \phi \rightarrow \phi + \alpha$$

$$U(1)_A: \quad \theta \rightarrow \theta + \beta$$

$$d \star j_A = \frac{1}{4\pi^2} F_V \wedge F_V!$$

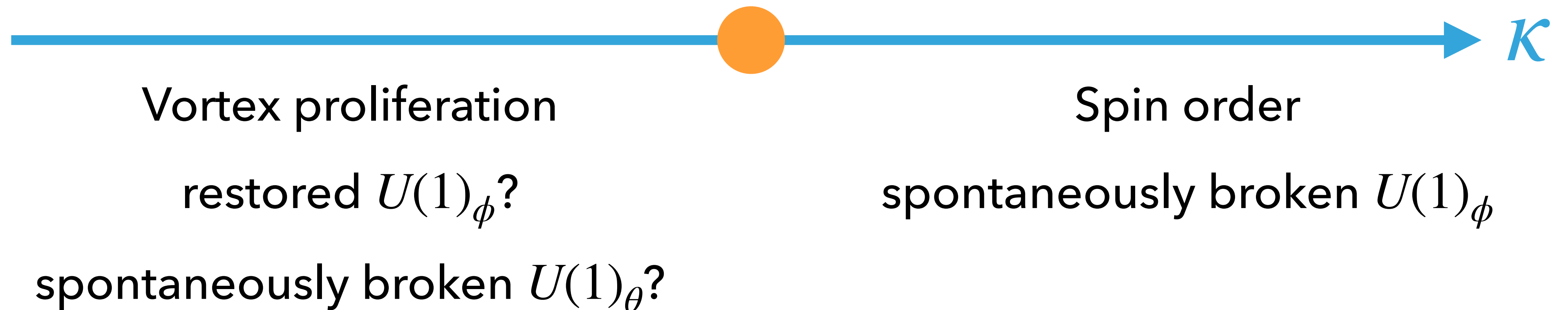
$$S = \frac{\kappa}{2} \sum_{\ell} (d\phi - 2\pi n)_{\ell}^2 + i \sum_h \theta_h (dn \wedge dn)_h$$



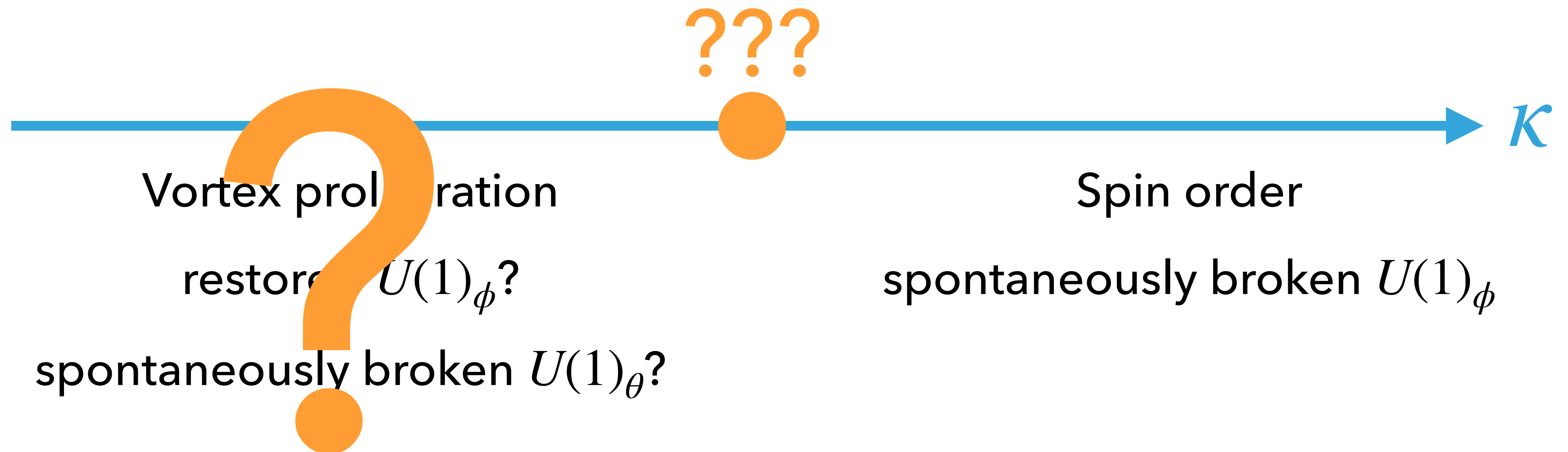
Spin order

spontaneously broken $U(1)_{\phi}$

$$S = \frac{\kappa}{2} \sum_{\ell} (d\phi - 2\pi n)_{\ell}^2 + i \sum_h \theta_h (dn \wedge dn)_h$$



$$S = \frac{\kappa}{2} \sum_{\ell} (d\phi - 2\pi n)_{\ell}^2 + i \sum_h \theta_h (dn \wedge dn)_h$$



CODE AVAILABILITY



<https://github.com/evanberkowitz/supervillain/>

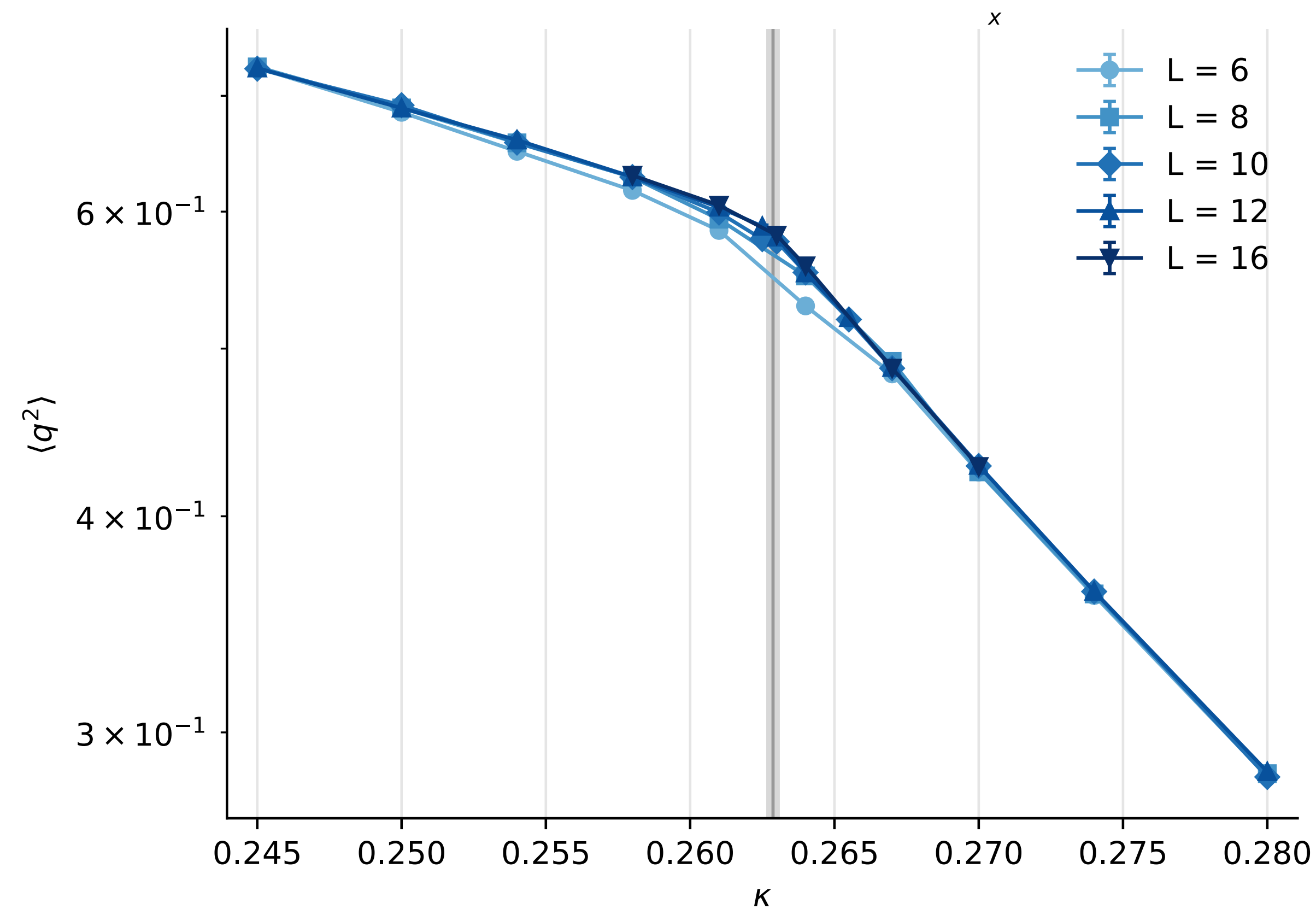
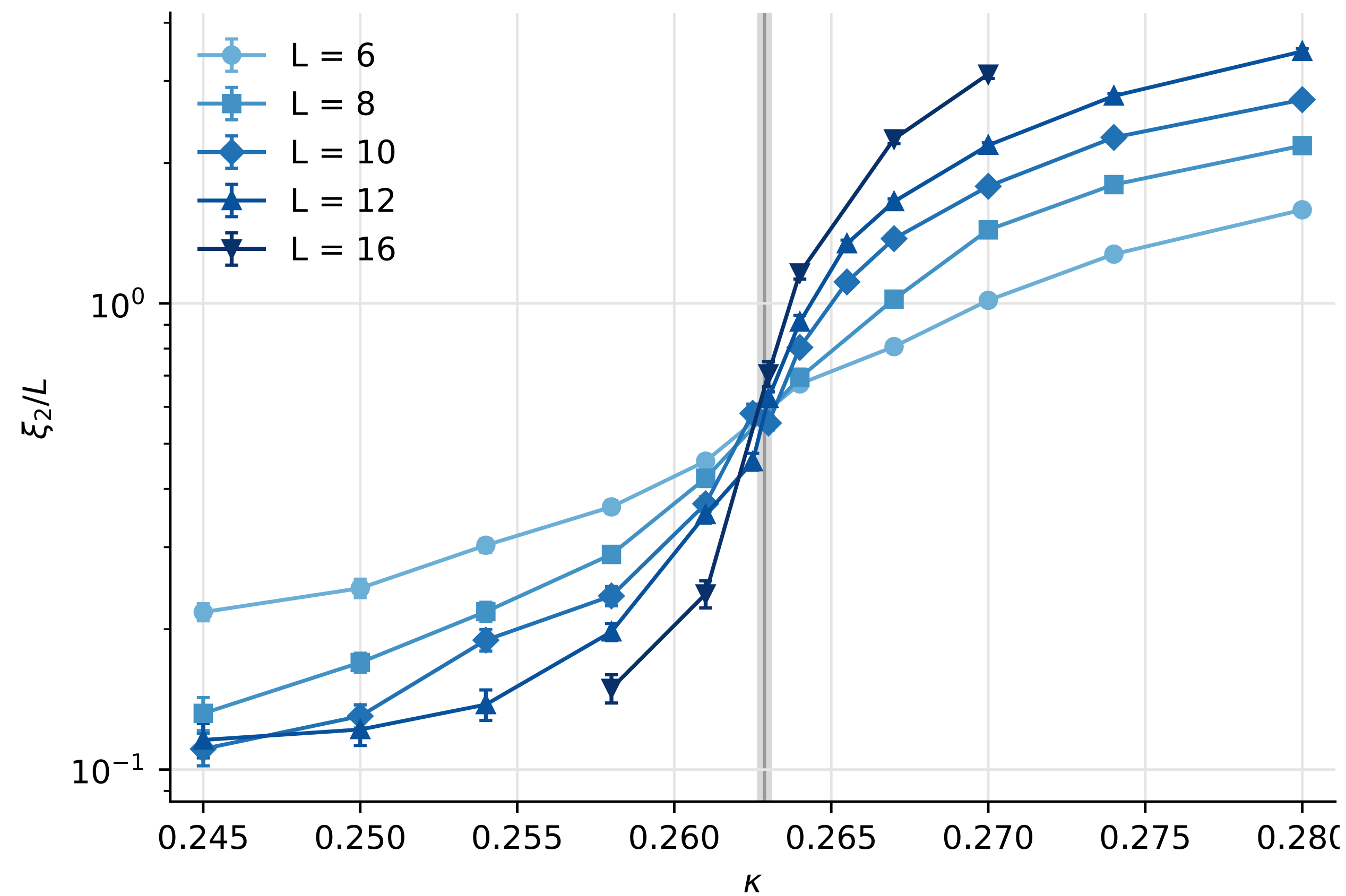
<https://supervillain.readthedocs.io/>

The WIP [model development branch](#) has additional docs



XY

KILLING THE INTERSECTIONS SHOULD REALLY MATTER!



At the critical κ
 $\langle (dn \wedge dn)^2 \rangle \approx 0.6$

*How shall I hold my soul, that it may not
be touching yours...*

Ranier Maria Rilke

NO INTERSECTIONS

A FORK IN THE ROAD

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}n \mathcal{D}\theta e^{-S} = \int \mathcal{D}\phi \mathcal{D}n e^{-S} \prod_h [(dn \wedge dn)_h = 0]$$

Take this model seriously

Proposals must maintain the constraint

Z is Gaussian, take advantage of heatbath etc.

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}n e^{-S} \zeta^{\sum_h |dn \wedge dn|}$$

Relax the constraint but punish violations.

$\zeta = 1$ is the unconstrained Villain model

$\zeta \rightarrow 0$ recovers the constrained model

A FORK IN THE ROAD

$$\mathcal{Z} = \int D\phi Dn D\theta e^{-S} = \int D\phi Dn e^{-S} \prod_h [(dn \wedge dn)_h = 0]$$

Take this model seriously

Proposals must maintain the constraint

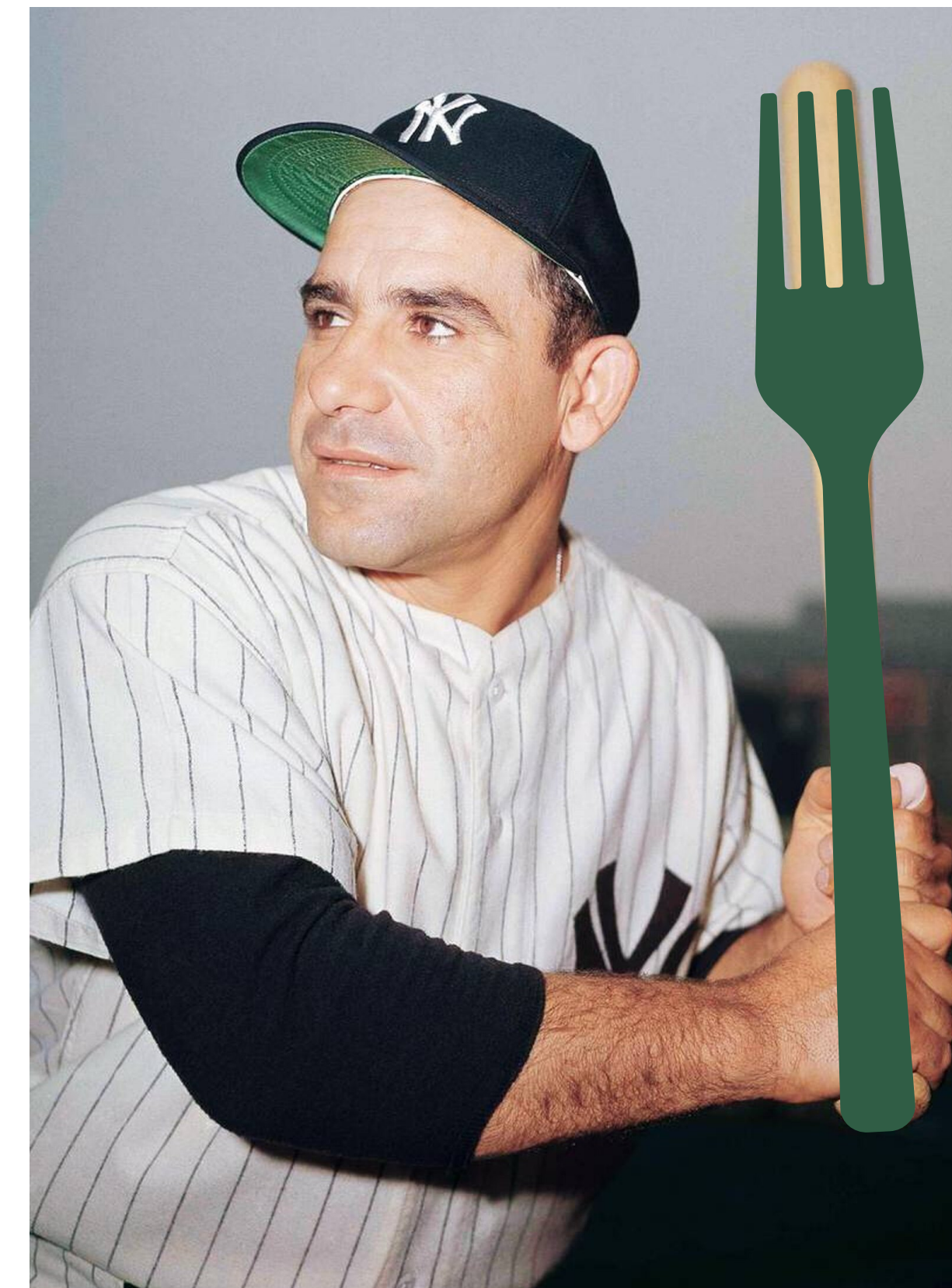
Z is Gaussian, take advantage of heatbath etc.

$$\mathcal{Z} = \int D\phi Dn e^{-S} \zeta \sum_h |dn \wedge dn|$$

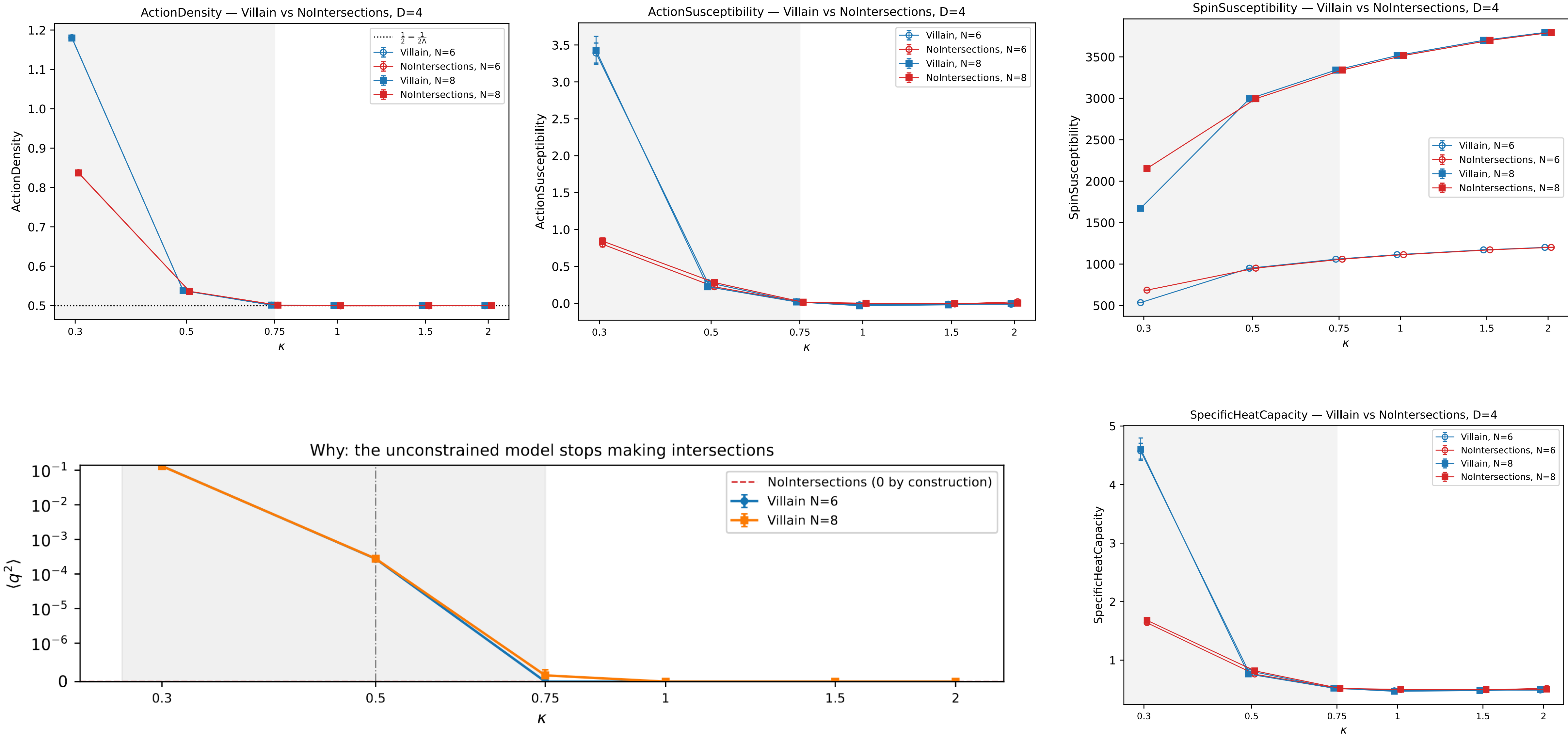
Relax the constraint but punish violations.

$\zeta = 1$ is the unconstrained Villain model

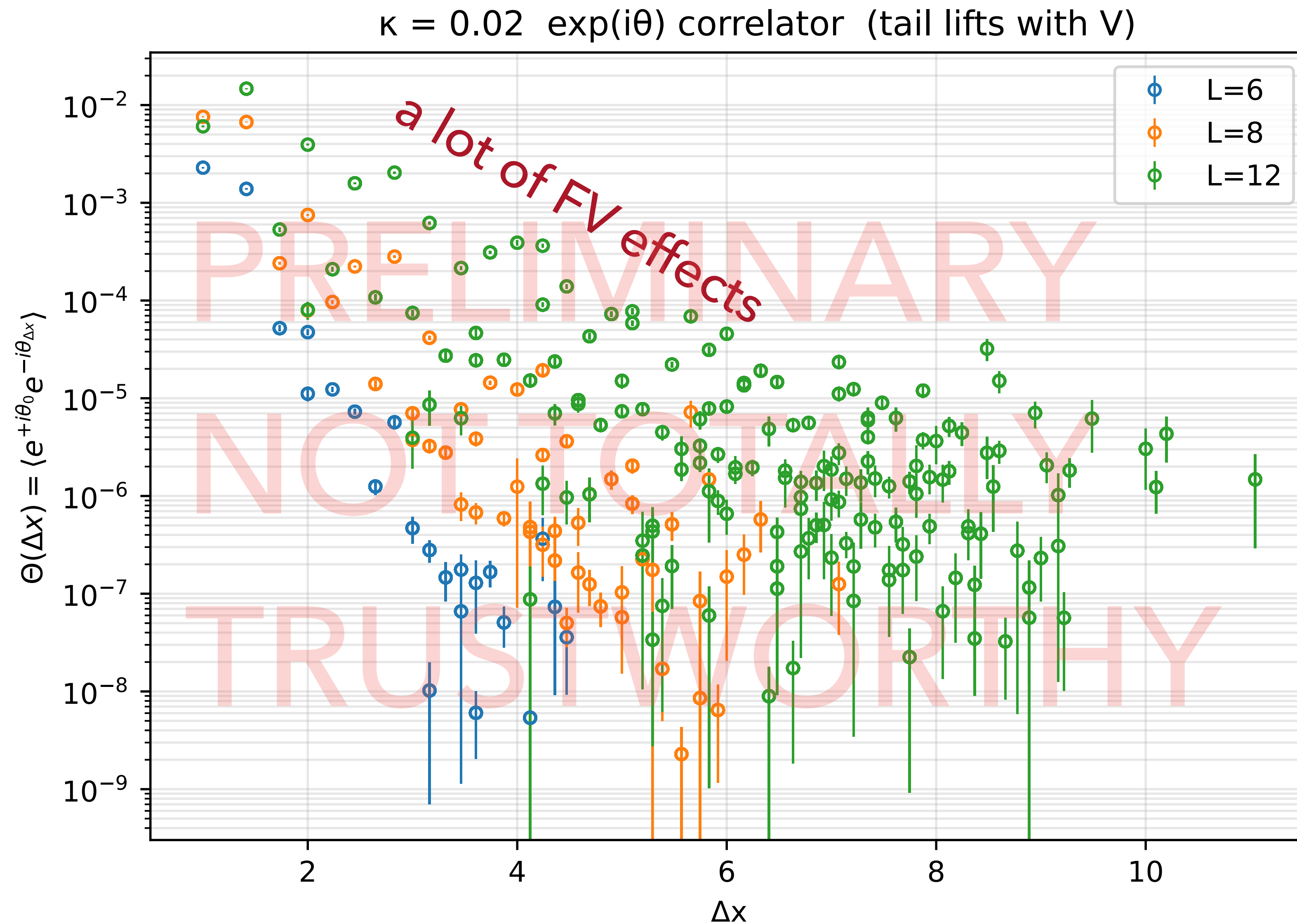
$\zeta \rightarrow 0$ recovers the constrained model



RESULTS AT HIGH KAPPA MATCH UNCONSTRAINED VILLAIN



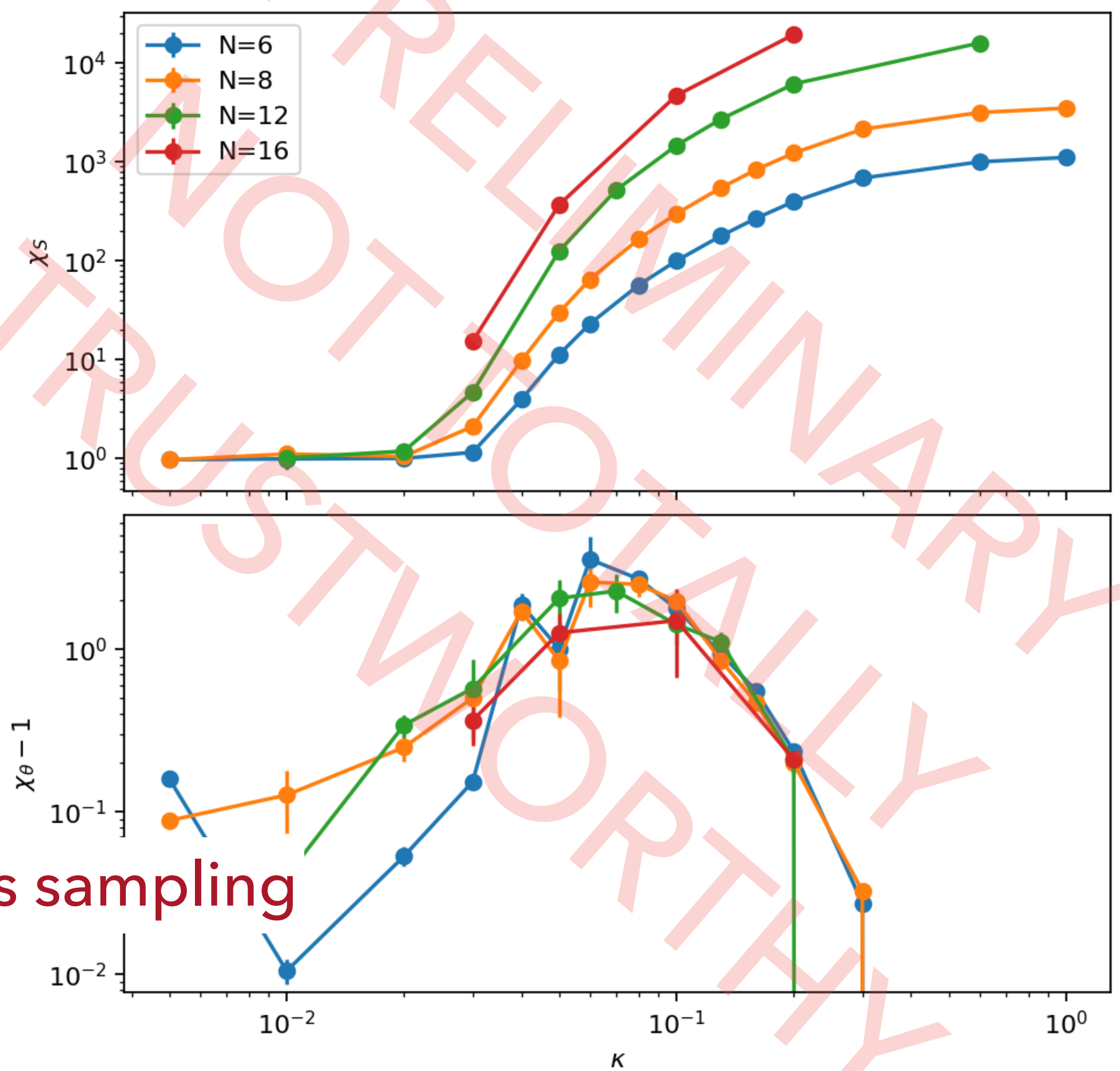
RESULTS $\langle e^{+i\theta_0} e^{-i\theta_{\Delta x}} \rangle$



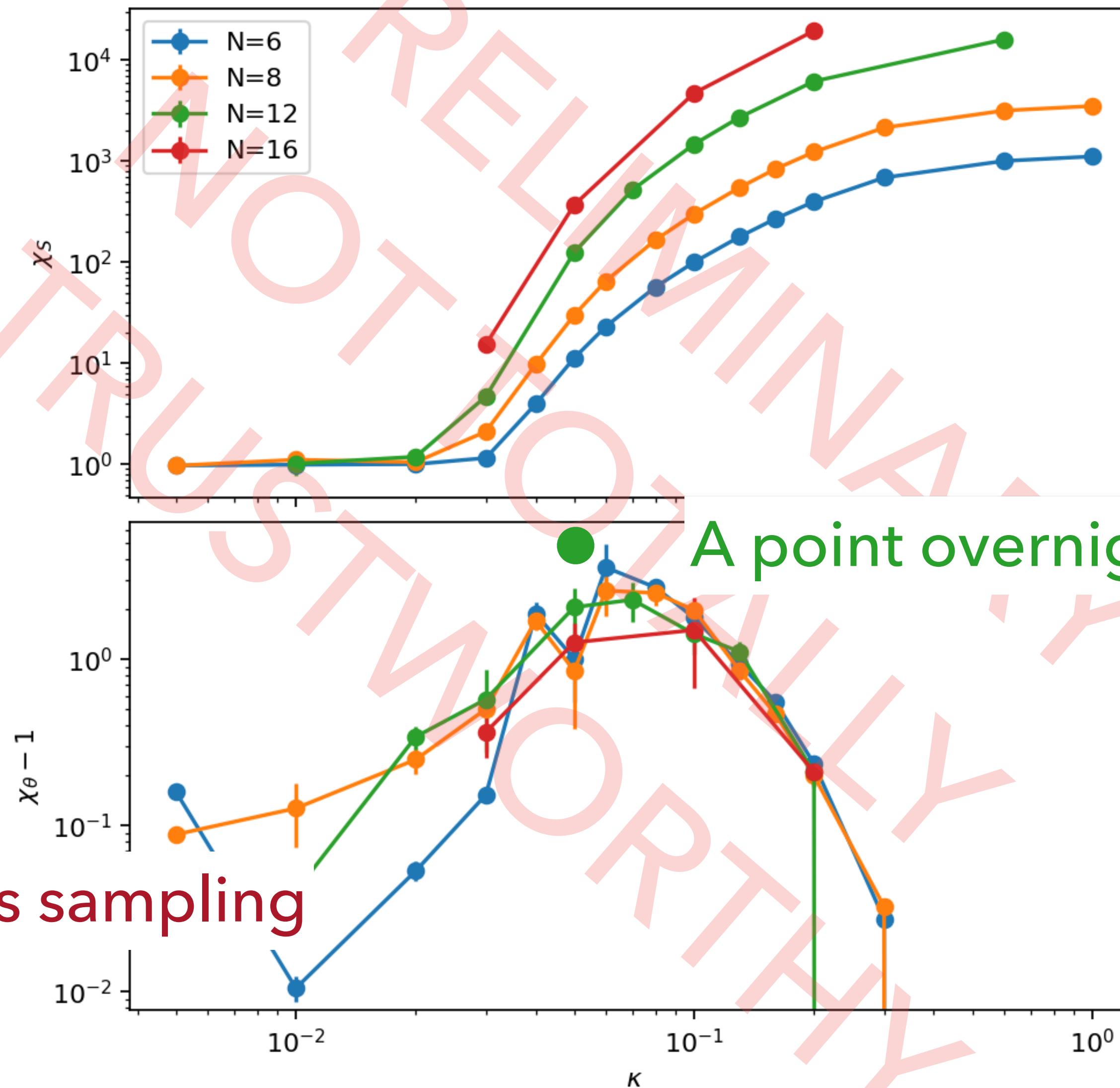
*Not really seeing
a plateau / vev*

RESULTS:

No-Intersection model: the two $U(1)$ s against κ



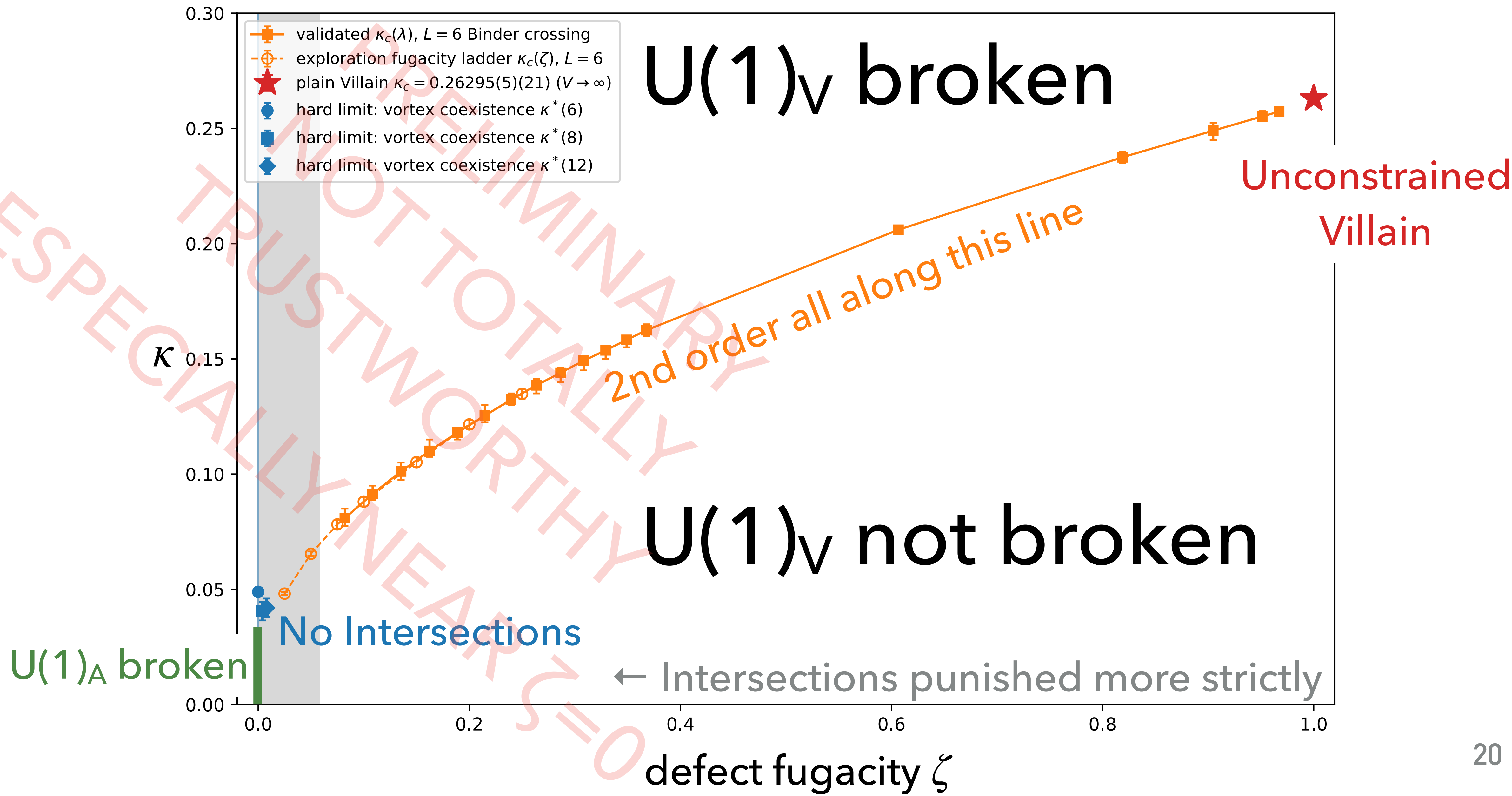
Problems sampling

No-Intersection model: the two $U(1)$ s against κ 

A point overnight using a surface worm

Problems sampling

CURRENT UNDERSTANDING OF THE PHASE



VILLAINY CAN PAY

- 't Hooft anomaly on the lattice guarantees order
- Theo Jacobson's no-intersection model seems to have *some* sort of interesting behavior as $\kappa \rightarrow 0$
- Calculation so far don't provide any reason to think there's a 1st order transition.
- Could there be an exotic 4D CFT with the right anomaly emerges at some critical point? What could it be?

Code



Docs



**YOU WILL NEVER FIND A MORE
WRETCHED HIVE OF SCUM AND VILLAINY.
WE MUST BE CAUTIOUS.**



**WITH OUR COMBINED STRENGTH,
WE CAN END THIS DESTRUCTIVE
CONFLICT AND BRING
ORDER
TO THE GALAXY.**



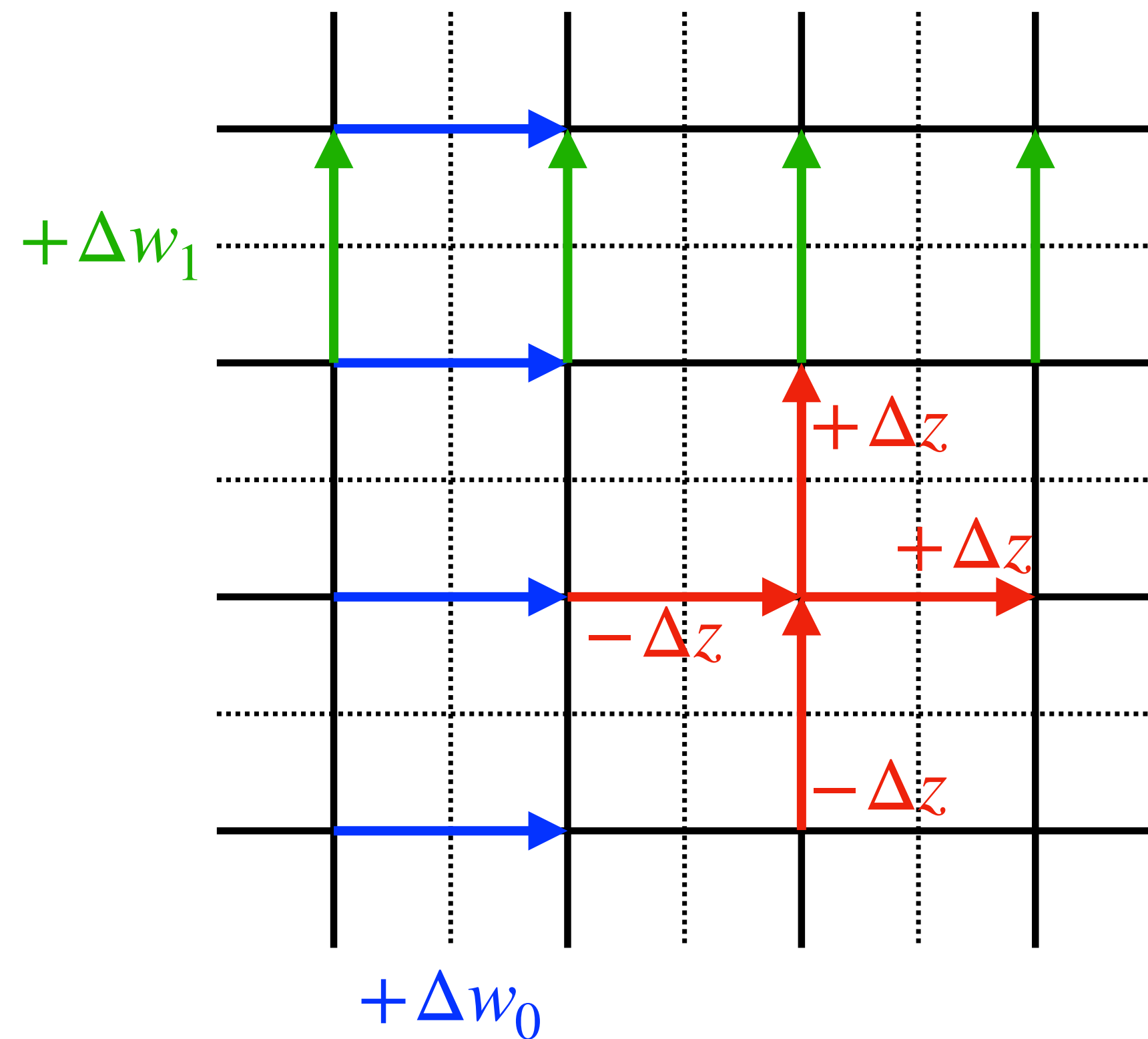
Star Wars IV + V, Lucasfilm

BACKUP SLIDES

ALGORITHMS

Dual 3450 has terms like this

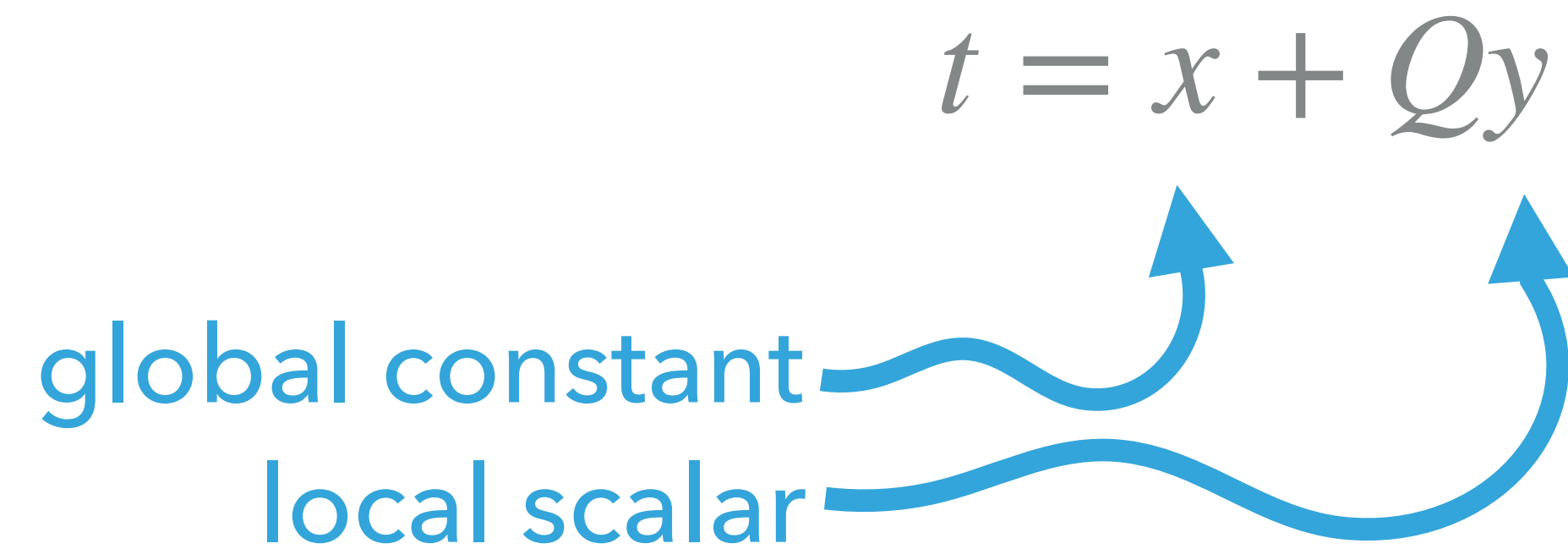
$$(dn)_p = 0$$



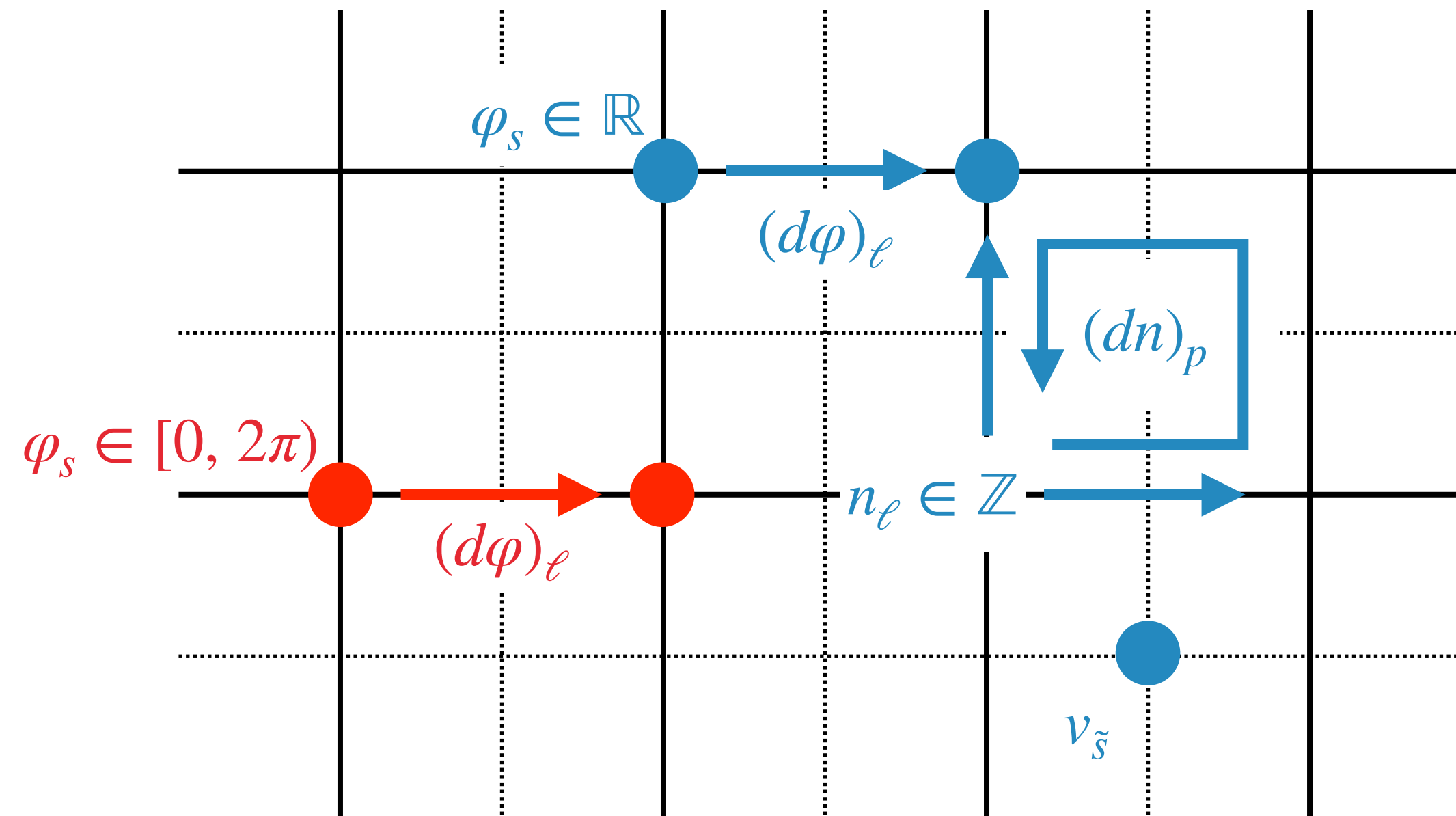
Worms also work!

All the dual theories have terms like this

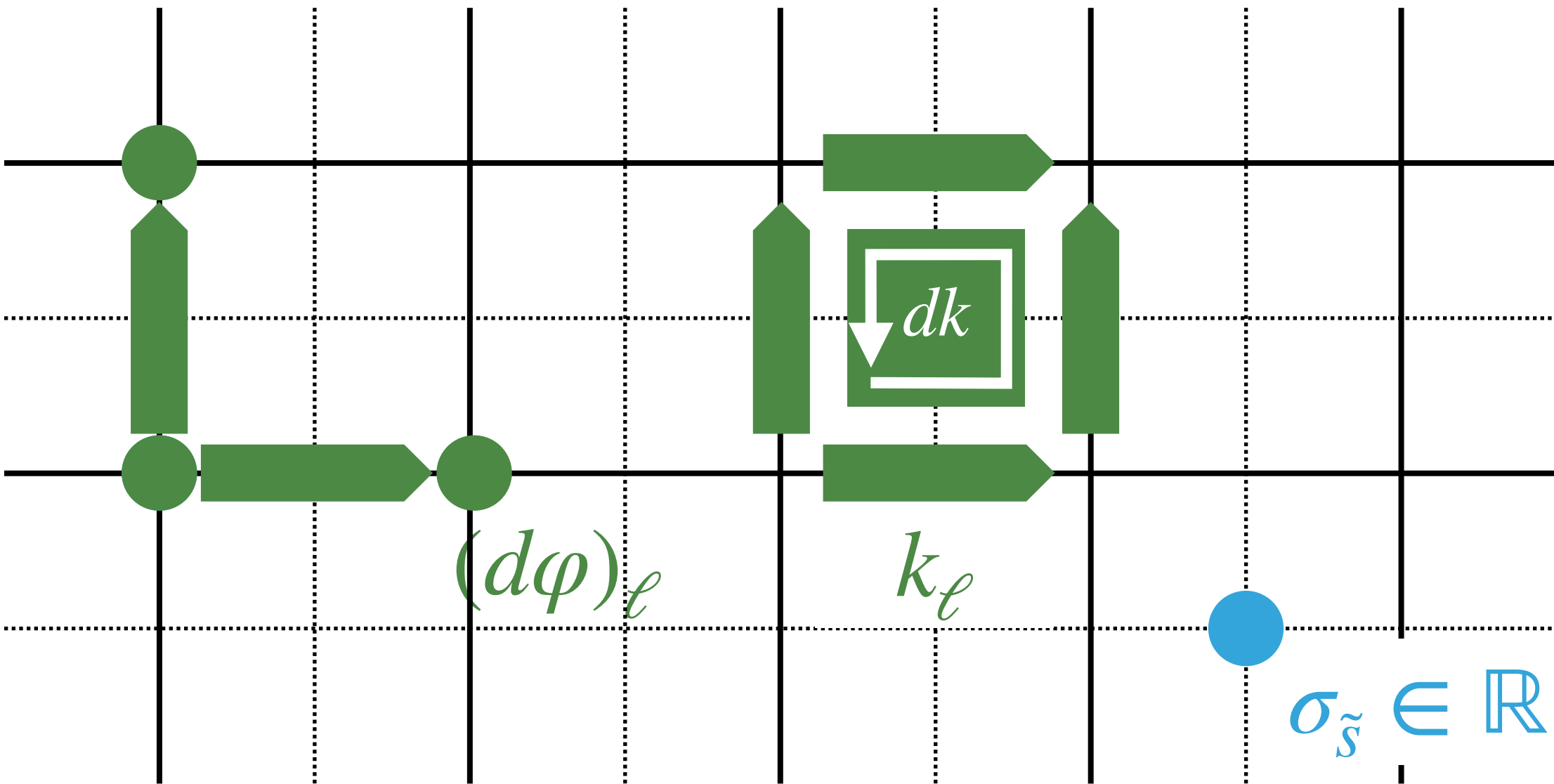
$$dt \equiv 0 \pmod{Q}$$



Can do standard Metropolis for both x, y



$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi k_{\ell}]^2 + i \sum_p \sigma_{\star p} (dk)_p$$



$$j_s = \kappa(d\varphi - 2\pi k)$$

$$j_w = \star \frac{1}{2\pi}(d\varphi - 2\pi k)$$

$$d \star j_s = \kappa d \star (d\varphi - 2\pi n) = 0 \quad (\varphi \text{ EOM})$$

$$d \star j_w = \frac{1}{2\pi}(d^2\varphi - 2\pi dk) = -dk = 0 \quad (\sigma \text{ EOM})$$

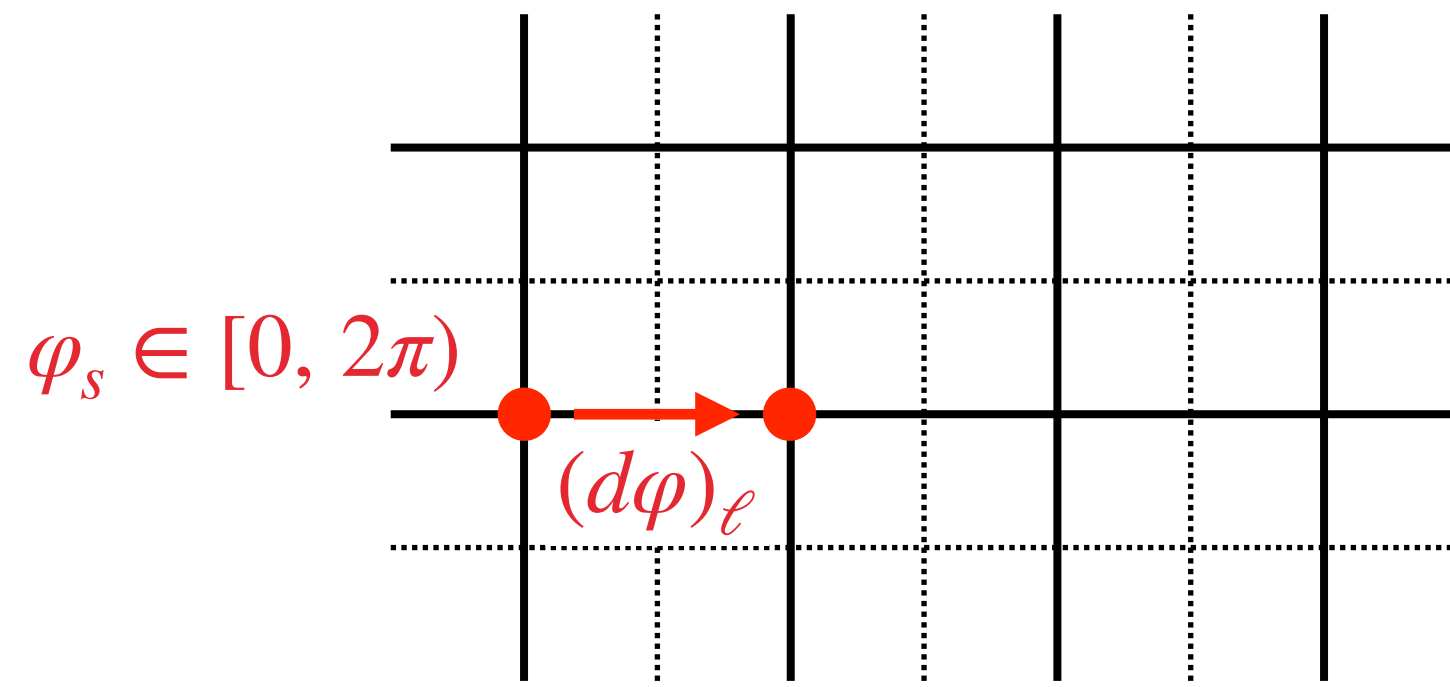
$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi k_{\ell} - a_{\ell}]^2 + i \sum_p \sigma_{\star p} (dk + r)_p$$

$$\begin{aligned} a &\rightarrow a + d\alpha + 2\pi m \\ \varphi &\rightarrow \varphi + \alpha + 2\pi n \\ k &\rightarrow k + dn - m \\ r &\rightarrow r + dm \end{aligned}$$

$$j_w = \star \frac{1}{2\pi}(d\varphi - 2\pi k - a)$$

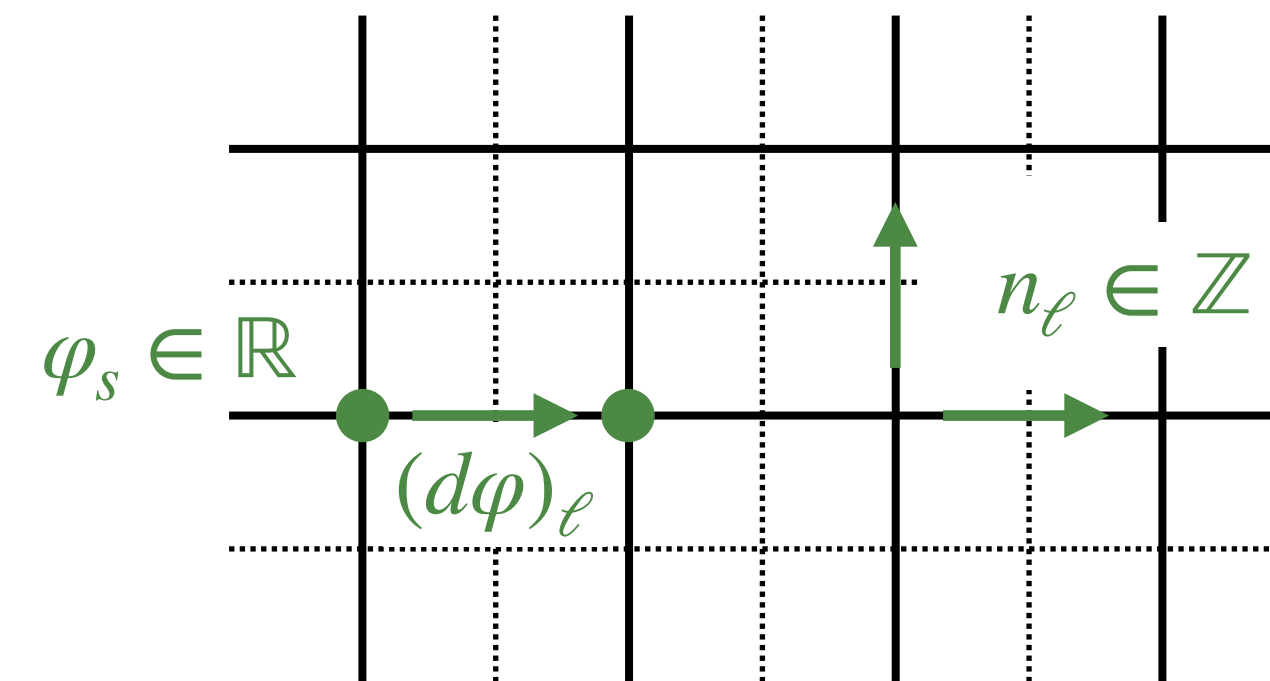
(NOT 0 by σ EOM)

$$S = \frac{\kappa}{2} \sum_{\ell} [1 - \cos(d\varphi)_{\ell}]$$



plaquette vorticity $\in \{-1, 0, +1\}$

$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi n_{\ell}]^2 \quad (\text{quadratic!})$$



plaquette vorticity $\in \mathbb{Z}$

Very easy to keep only vorticity 0 (mod W)

$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_{\ell} - 2\pi n_{\ell} \right]^2 + i \sum_p \frac{2\pi v_{\star p}}{W} (dn)_p$$

$$\sum_{c^r} (dA)_{c^r} B_{\star c^r} = \sum_{c^{r-1}} (-1)^r A_{c^{r-1}} (dB)_{\star c^{r-1}}$$

Lattice summation by parts

$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_{\ell} - 2\pi n_{\ell} \right]^2 + i \sum_{\ell} 2\pi(\delta v)_{\ell} n_{\ell} / W$$

 Poisson resum link-by-link

$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_{\ell} - 2\pi k_{\ell} \right]^2 + i \sum_p \frac{2\pi v_{\star p}}{W} \sum_{(n)} \exp \left\{ -\frac{\kappa}{2} (\theta - 2\pi n)^2 + i n \tilde{\theta} \right\} = \frac{1}{\sqrt{2\pi\kappa}} \sum_m \exp \left\{ -\frac{1}{2\kappa} \left(m - \frac{\tilde{\theta}}{2\pi} \right)^2 - i \left(m - \frac{\tilde{\theta}}{2\pi} \right) \theta \right\}$$

trade path integration variables (and strong/weak)
 $\theta \rightarrow d\varphi, \tilde{\theta} \rightarrow d\sigma$

$$\tilde{S} = \frac{\tilde{\kappa}}{2} \sum_{\tilde{\ell}} \left[(d\sigma)_{\tilde{\ell}} - 2\pi \tilde{k}_{\tilde{\ell}} \right]^2 + i \sum_{\tilde{p}} \varphi_{\star \tilde{p}} (d\tilde{k})_{\tilde{p}} \quad \text{Exact T-duality on the lattice!}$$

$$\tilde{\kappa} = \frac{1}{(2\pi)^2 \kappa}$$

DUALITY: A WORLDLINE FORMULATION

Gorantla, Lam, Seiberg, Shao 2021
[PRB 103 205116](#), [J Math Phys 62 102301](#)

$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_{\ell} - 2\pi n_{\ell} \right]^2 + i \sum_p \frac{2\pi v_{\star p}}{W} (dn)_p$$

Lattice summation by parts

$$S = \frac{\kappa}{2} \sum_{\ell} \left[(d\varphi)_{\ell} - 2\pi n_{\ell} \right]^2 + i \sum_{\ell} 2\pi(\delta v)_{\ell} n_{\ell} / W$$

 Poisson resum n link-by-link (exact lattice duality)
trade path integration variables (and strong/weak)

$$S = \frac{1}{2\kappa} \sum_{\ell} \left[m - \delta v / W \right]_{\ell}^2 - i \sum_s (\delta m)_s \varphi_s \quad (m_{\ell}, v_p \in \mathbb{Z}) \quad \text{Worldline: } m \text{ counts bosons}$$

DUALITY: A WORLDLINE FORMULATION

Gorantla, Lam, Seiberg, Shao 2021
[PRB 103 205116](#), [J Math Phys 62 102301](#)

$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi n_{\ell}]^2 + i \sum_p \frac{2\pi v_{\star p}}{W} (dn)_p$$

Path integrate v to get **constraint** $dn \equiv 0 \pmod{W}$

Lattice summation by parts

$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi n_{\ell}]^2 + i \sum_{\ell} 2\pi(\delta v)_{\ell} n_{\ell} / W$$

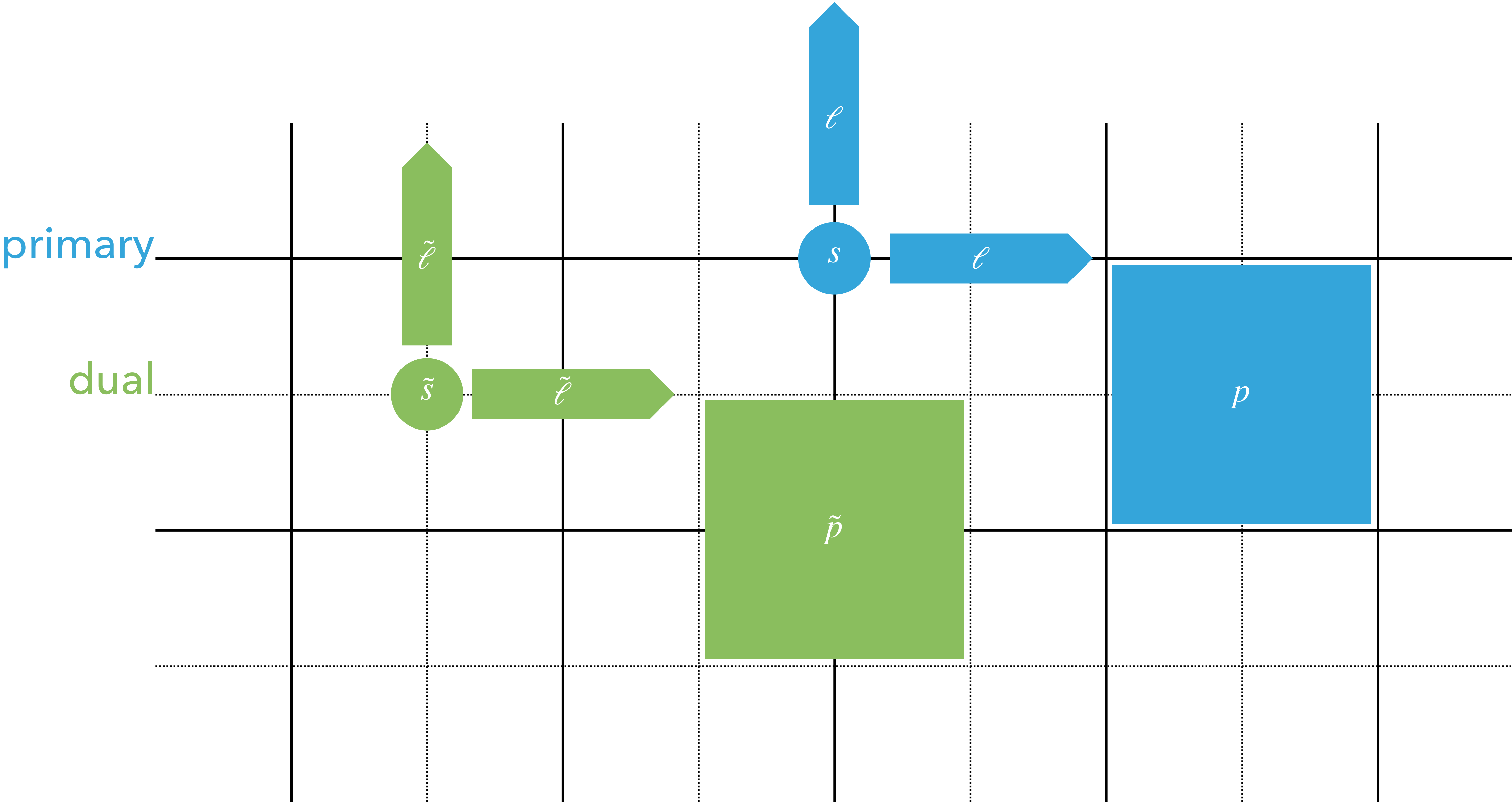
Poisson resum n link-by-link (exact lattice duality)
 trade path integration variables (and strong/weak)

$$S = \frac{1}{2\kappa} \sum_{\ell} [m - \delta v / W]_{\ell}^2 - i \sum_s (\delta m)_s \varphi_s$$

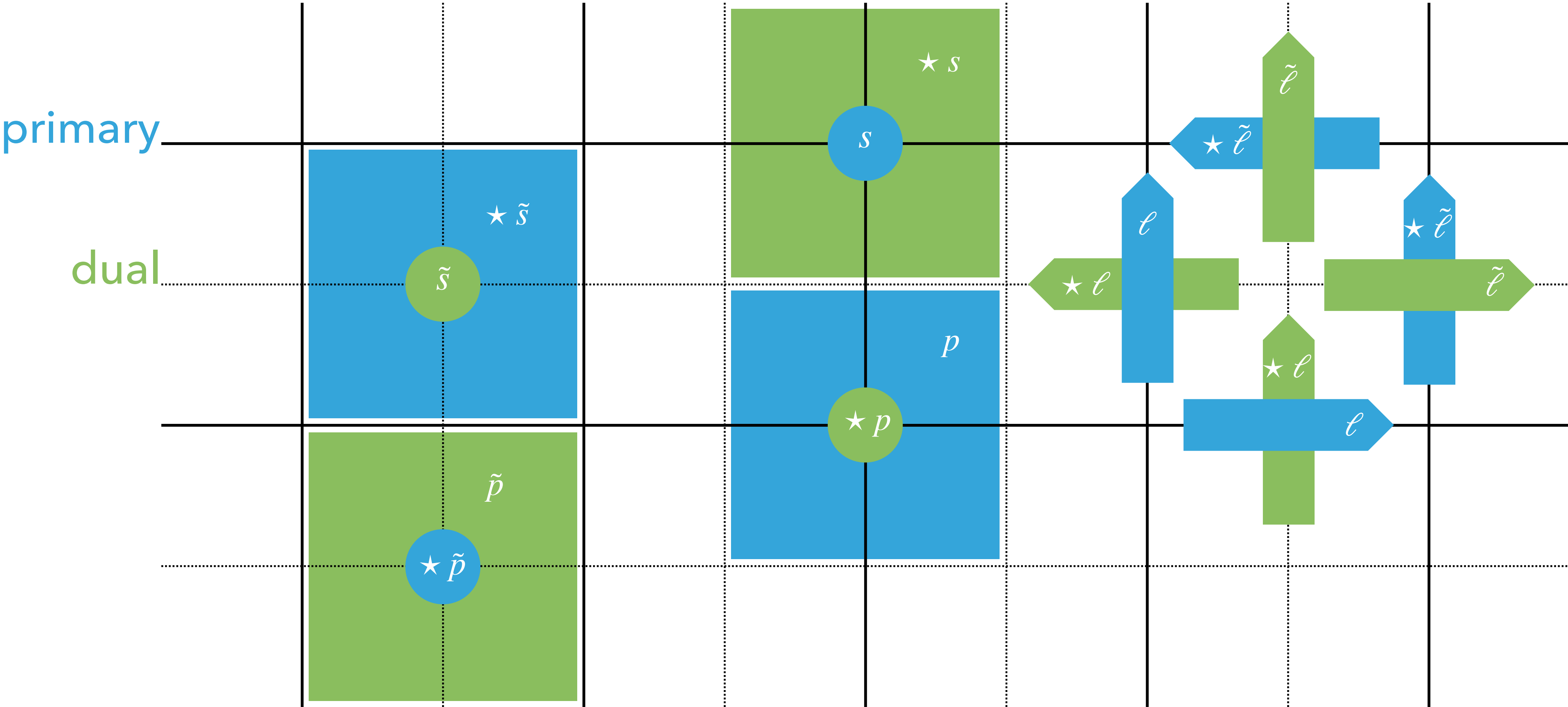
$(m_{\ell}, v_p \in \mathbb{Z})$ Worldline: m counts bosons
 Path integrate φ to get **constraint** $\delta m = 0$

LATTICE EXTERIOR CALCULUS

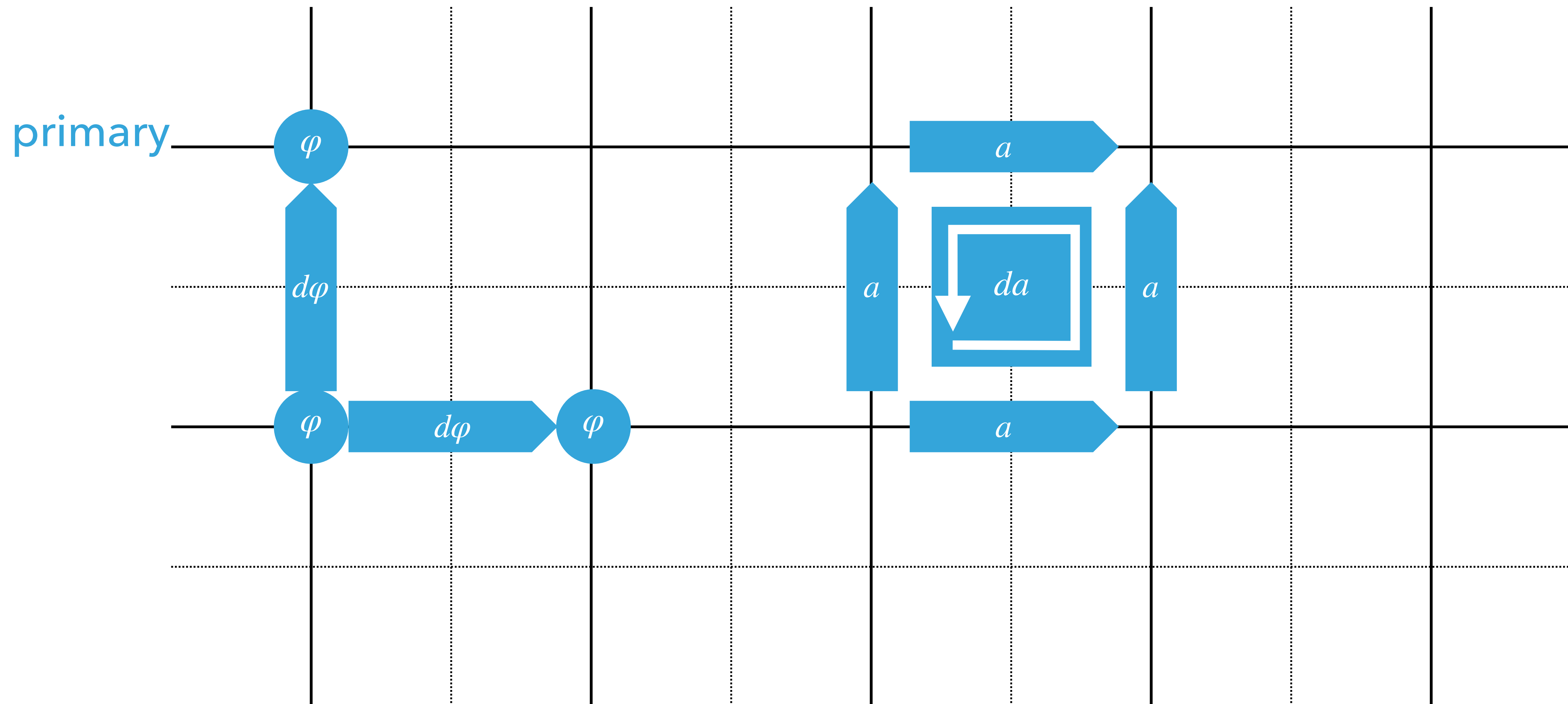
LATTICE



★ maps between primary and dual lattices



d increases the degree of the form by 1

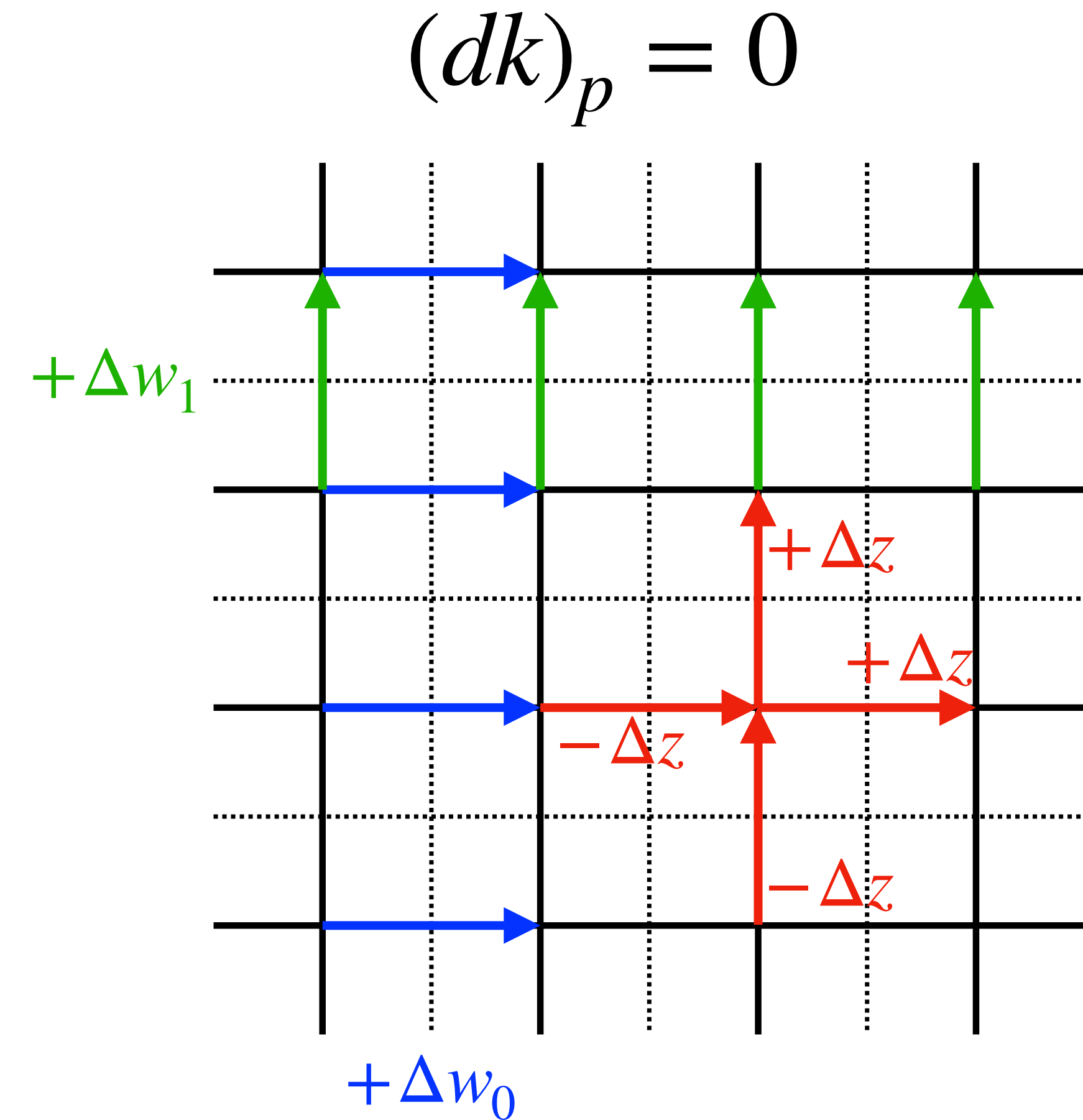
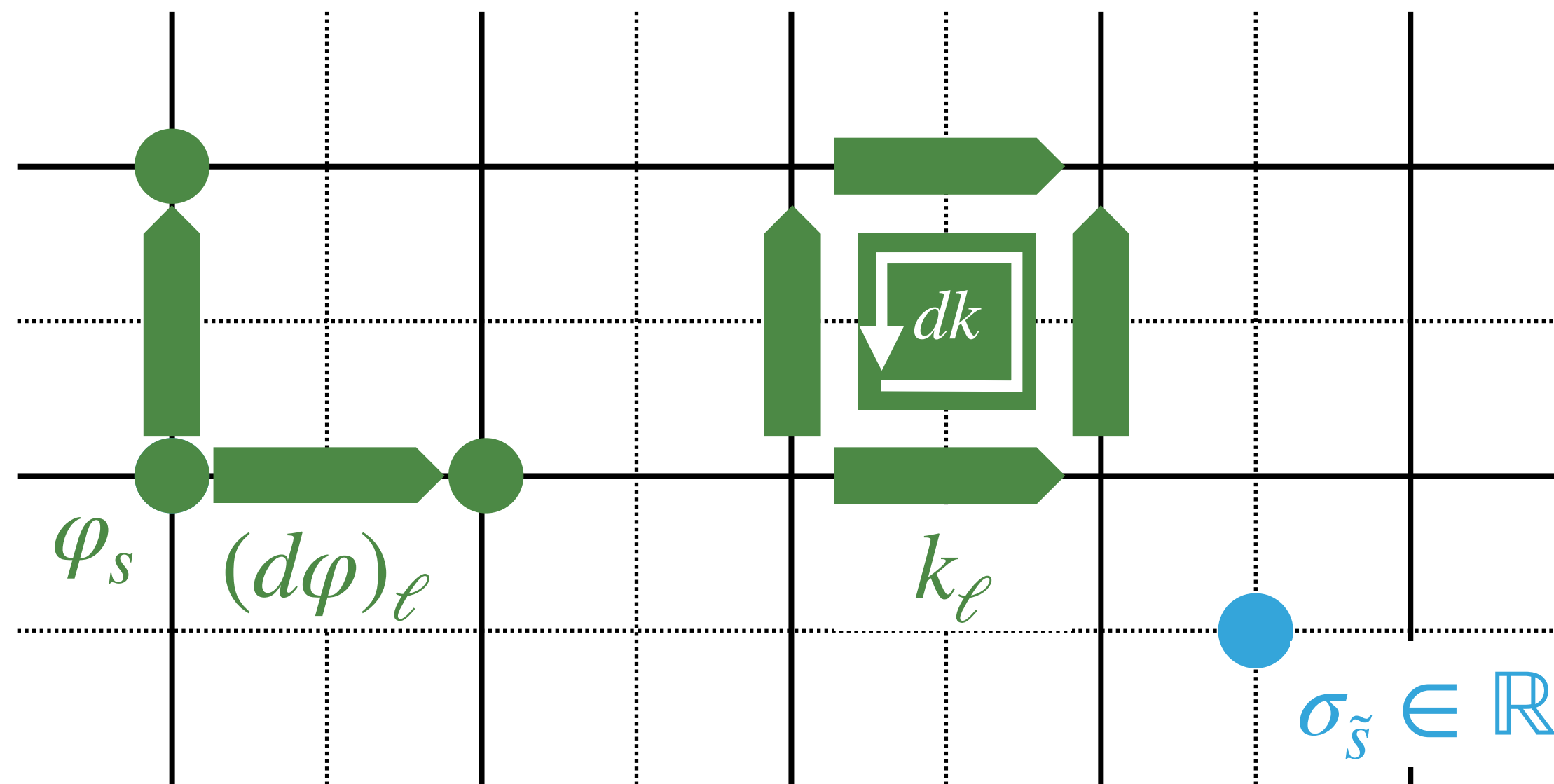




SIGN PROBLEM?



$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi k_{\ell}]^2 + i \sum_p \sigma_{\star p} (dk)_p$$



Worms also work! (forthcoming)

Can solve the **constraint** with smart update algorithms!

THE VILLAIN WORM ($W=\infty$, BUT SIMILAR FOR FINITE W)

$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi k_{\ell}]^2 + i \sum_p \sigma_{\star p} (dk)_p \quad (dk)_p = 0$$

$$V_{\tilde{x}\tilde{y}} = \left\langle e^{i(\sigma_{\tilde{x}} - \sigma_{\tilde{y}})} \right\rangle$$

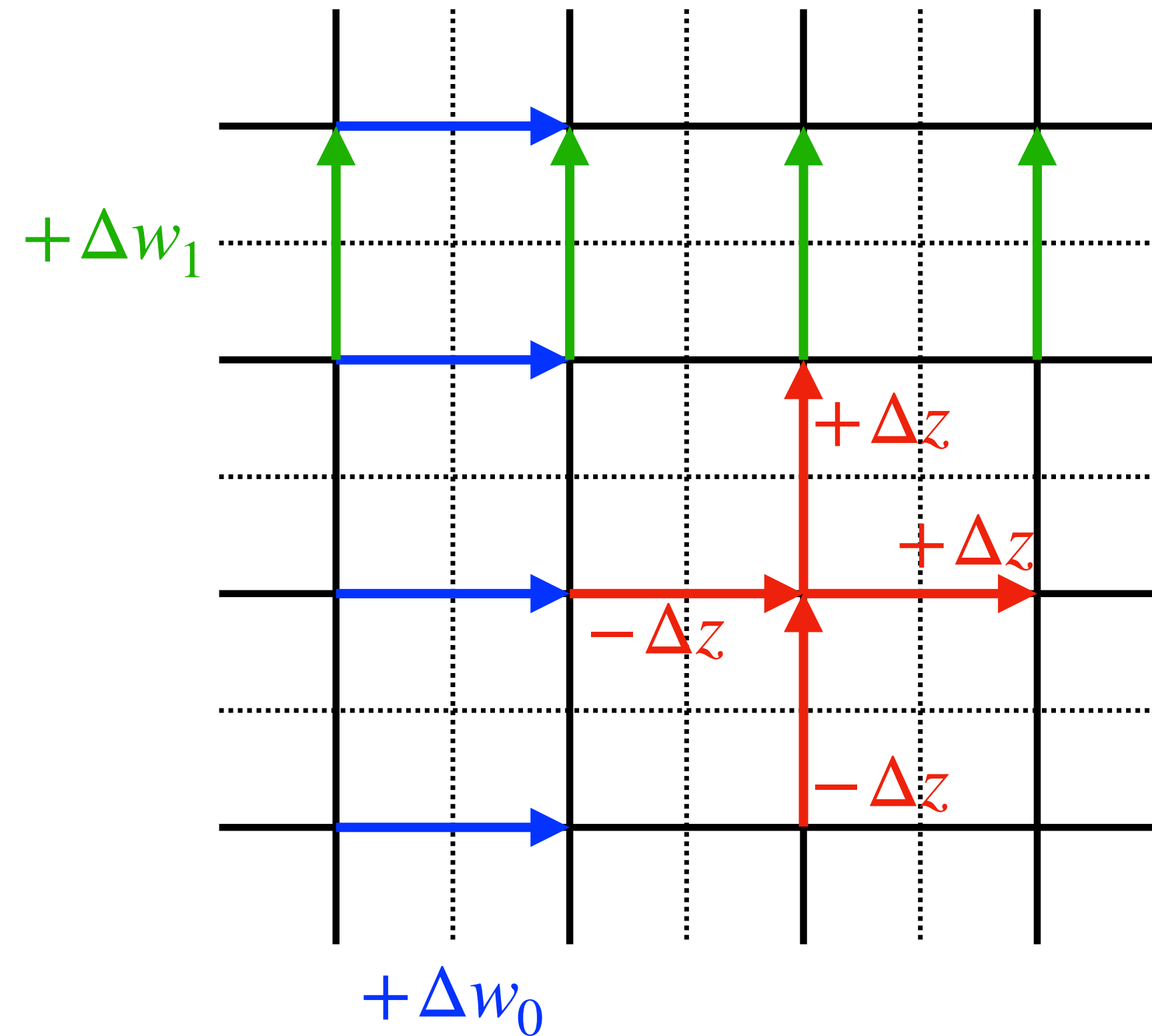
$$= \frac{1}{\mathcal{Z}} \int D\phi Dn D\sigma e^{-S} e^{i(\sigma_{\tilde{x}} - \sigma_{\tilde{y}})}$$

$$= \frac{1}{\mathcal{Z}} \int D\phi Dn e^{-S} [dn_p = \delta_{py} - \delta_{px}]$$

$$\mathcal{G} = \int D\phi Dn d\tilde{h} d\tilde{t} e^{-S} [dn_p = \delta_{p\tilde{t}} - \delta_{p\tilde{h}}]$$

$$V_{\tilde{x}\tilde{y}} = \frac{\left\langle \delta_{\tilde{x}\tilde{h}} \delta_{\tilde{y}\tilde{t}} \right\rangle_{\mathcal{G}}}{\left\langle \delta_{\tilde{h}\tilde{t}} \right\rangle_{\mathcal{G}}}$$

Can measure otherwise-difficult correlators during evolution



THE VILLAIN WORM ($W=\infty$, BUT SIMILAR FOR FINITE W)

$$S = \frac{\kappa}{2} \sum_{\ell} [(d\varphi)_{\ell} - 2\pi k_{\ell}]^2 + i \sum_p \sigma_{\star p} (dk)_p \quad (dk)_p = 0$$

$$V_{\tilde{x}\tilde{y}} = \left\langle e^{i(\sigma_{\tilde{x}} - \sigma_{\tilde{y}})} \right\rangle$$

$$= \frac{1}{\mathcal{Z}} \int D\phi Dn D\sigma e^{-S} e^{i(\sigma_{\tilde{x}} - \sigma_{\tilde{y}})}$$

$$= \frac{1}{\mathcal{Z}} \int D\phi Dn e^{-S} [dn_p = \delta_{py} - \delta_{px}]$$

$$\mathcal{G} = \int D\phi Dn d\tilde{h} d\tilde{t} e^{-S} [dn_p = \delta_{p\tilde{t}} - \delta_{p\tilde{h}}]$$

$$V_{\tilde{x}\tilde{y}} = \frac{\left\langle \delta_{\tilde{x}\tilde{h}} \delta_{\tilde{y}\tilde{t}} \right\rangle_{\mathcal{G}}}{\left\langle \delta_{\tilde{h}\tilde{t}} \right\rangle_{\mathcal{G}}}$$

Can measure otherwise-difficult correlators during evolution

