

Benchmarking lattice determinations of x-dependent hadron structure

Chris Monahan (he/him/his)
Colorado College

A report on (and context for) PRL 135 (2025) 191901



The 43rd International Symposium on Lattice Field Theory
August 1, 2026



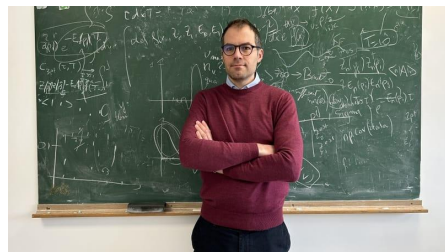
Co-authors



Kostas Orginos, W&M



Joe Karpie, JLab



Savvas Zafeiropoulos
CNRS Marseille

“Window observables for benchmarking parton distribution functions”
PRL 135 (2025) 191901

Chosen as a Jefferson Lab Significant Accomplishment
Read more at “Widening windows to peer into the proton”



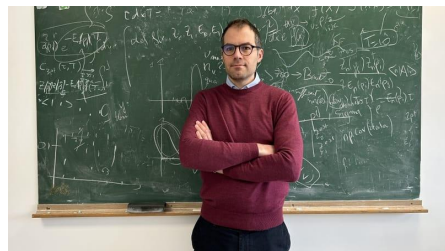
Co-authors and collaborators



Kostas Orginos, W&M



Joe Karpie, JLab



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Andrea Shindler
Peter Stoffer
Oliver Witzel

Claim 1:

We do not understand protons and neutrons.

Claim 2:

But! We can, with dedicated community effort.

Claim 1:

We do not understand protons and neutrons.

We have not yet calculated nucleon properties with precision.

Claim 2:

But! We can, with dedicated community effort.

In particular, through quantified, complete error budgets and appropriate benchmarks.

We propose:

Window observables serve as one set of appropriate benchmarks

Can be defined in regions of mutual validity, without extrapolation, for both lattice and experiment

Provide precise benchmarks by taking advantage of correlations in momentum fraction

Outline

Part 1: claim 1 in context

Collinear x-dependent hadron structure on the lattice
The need for benchmarking

Part 2: our proposal

Window observables
Some preliminary studies

Plenaries
G. Bali, Fri. 11:00
A. Shindler, Sat. 09:45

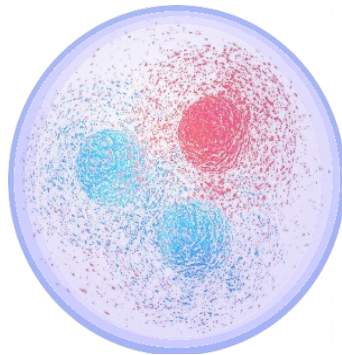
Hadron structure at a glance



How is the spin of the hadron split amongst its constituents?

How does the proton interact with the other fundamental forces?

How is the momentum of a fast moving hadron spread amongst its constituent quarks and gluons?



What are the internal correlations of the proton?

How is the momentum of the constituents correlated with their transverse structure?

Hadron structure at a glance

Global properties → Form factors

$$q^2 = 0$$

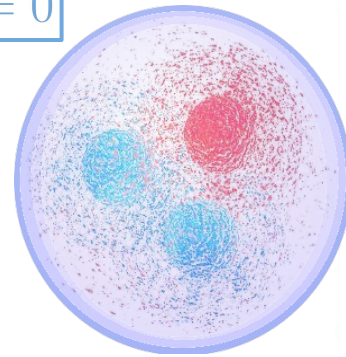
$$q^2 \neq 0$$

How is the spin of the hadron split amongst its constituents?

Extracted from matrix elements of **local** operators

Parton distribution functions (PDFs)

$$q^2 = 0$$



Generalised parton distribution functions (GPDs)

$$q^2 \neq 0$$

Wigner functions

$$???$$

What are the internal correlations of the proton?

How is the momentum of the constituents correlated with their position?

Extracted from matrix elements of **nonlocal** operators

How is the momentum of the constituents correlated with their position?

Hadron structure at a glance

Global
properties

Form factors

Parton distribution
functions (PDFs)

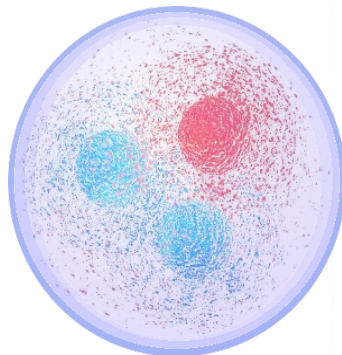
Generalised parton
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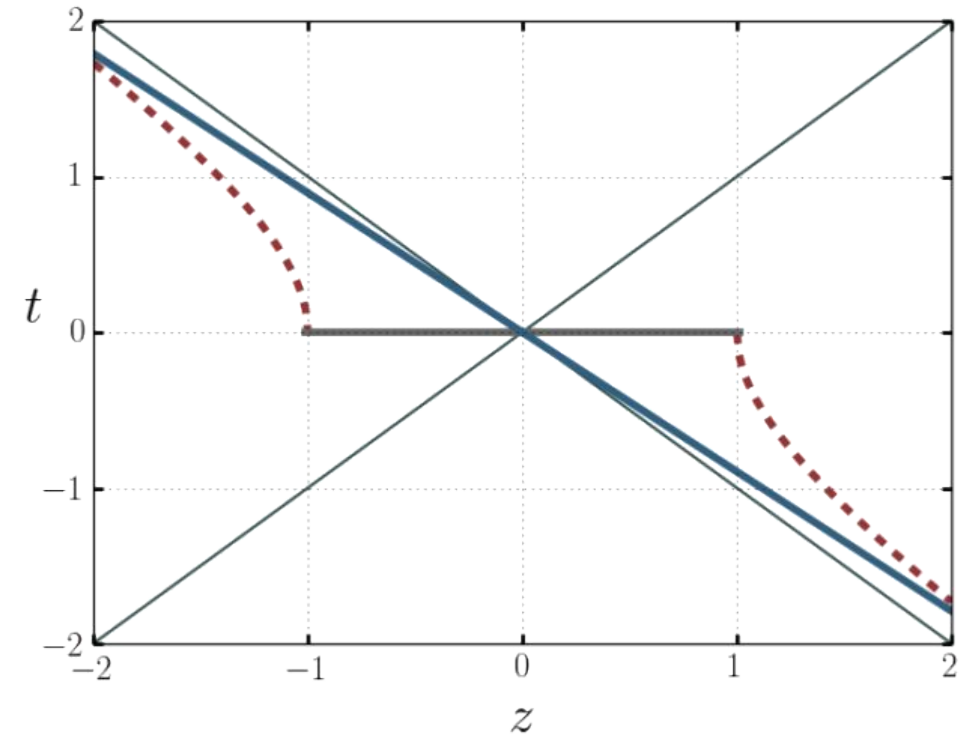
The importance of PDFs

Key theoretical input at LHC

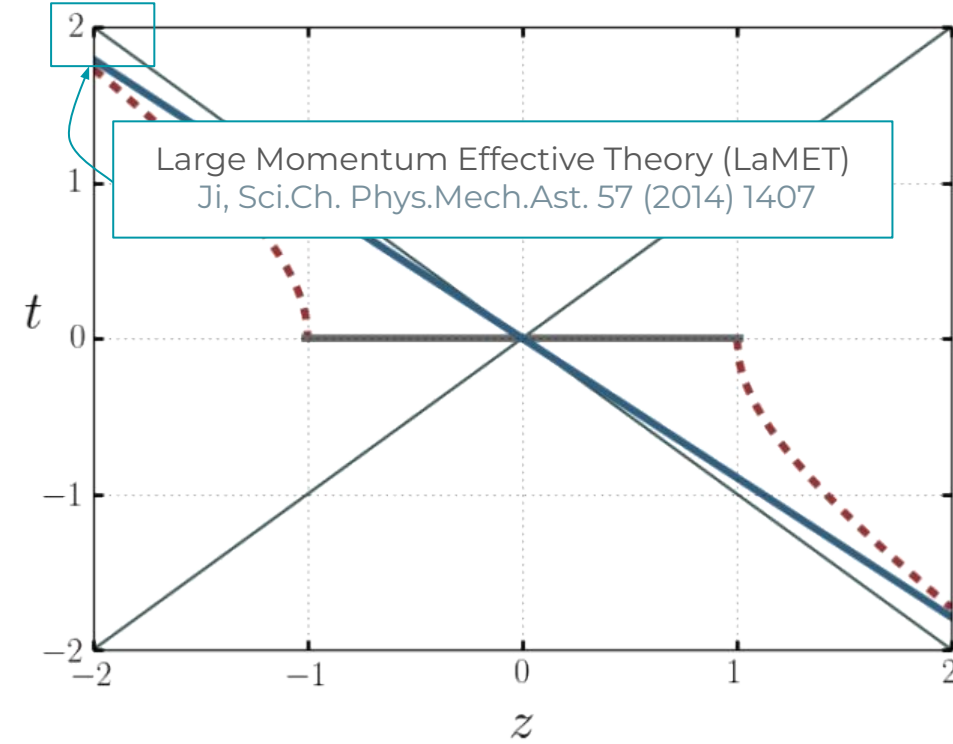
Relevant to high-energy neutrino-nucleus scattering

Central to the “quantitative QCD” era

Towards light-cone physics



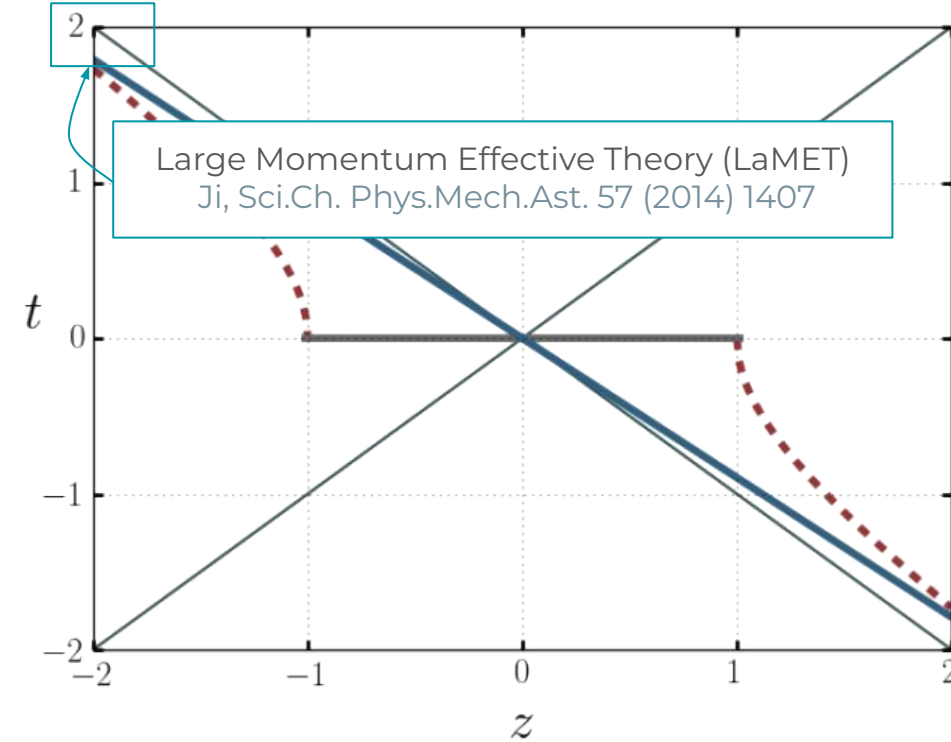
Towards light-cone physics



Towards light-cone physics

Ji, PRL 110 (2013) 262002
Radyushkin, PRD 96 (2017) 034025

Note history of different approaches:
Davoudi & Savage, PRD 86 (2012) 054505
Musch et al., PRD 83 (2011) 094507
Braun & Müller, EPJC 55 (2008) 349
Detmold & Lin, PRD 73 (2006) 014501
Liu & Dong, PRL 72 (1994) 1790



The distribution zoo

$$\langle P|G^{\mu\nu}(z)W(z,0)G^{\rho\sigma}(0)|P\rangle \longrightarrow M_{pp}(\nu = z\cdot P, z^2)$$

The distribution zoo

$$f_g(x) \sim \int_{-\infty}^{\infty} dz^- e^{-iz^- x P^+} M_{pp}(\nu, 0)$$

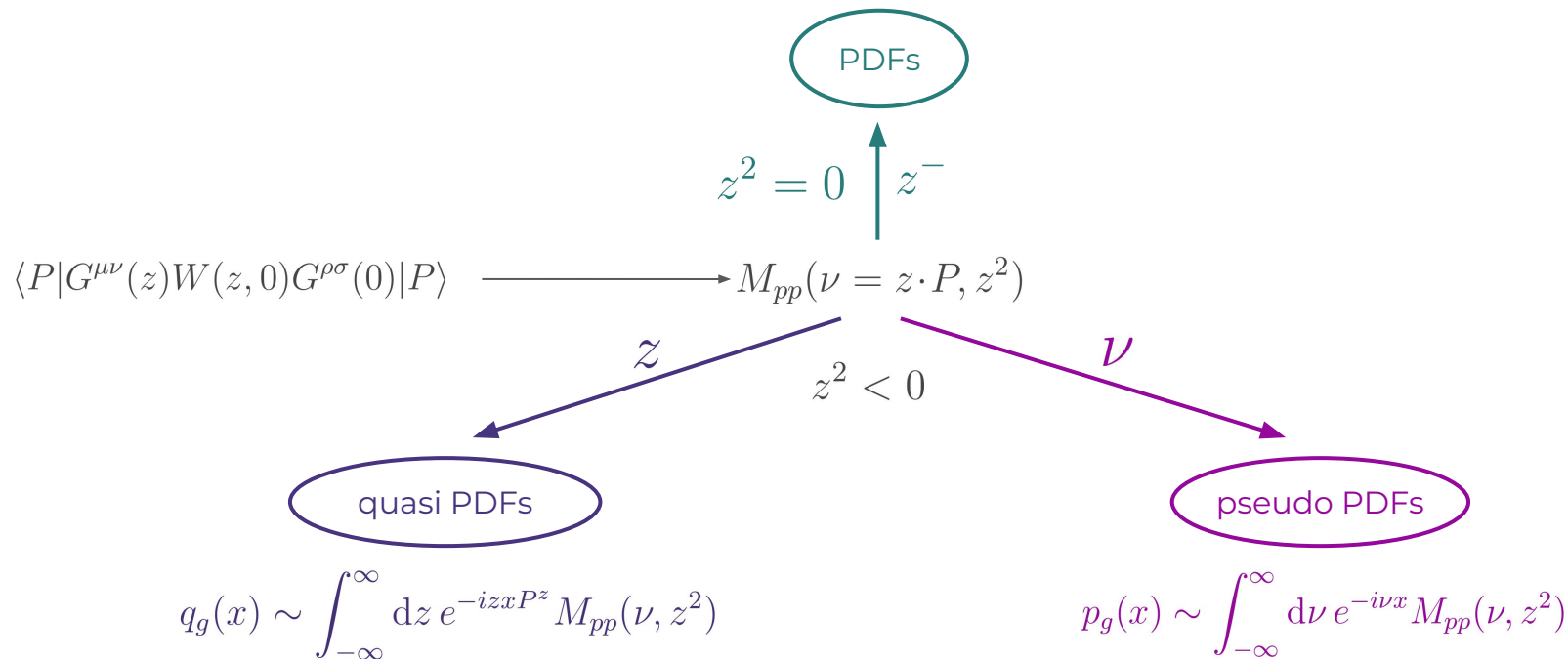
PDFs

$$z^2 = 0 \quad \uparrow \quad z^-$$

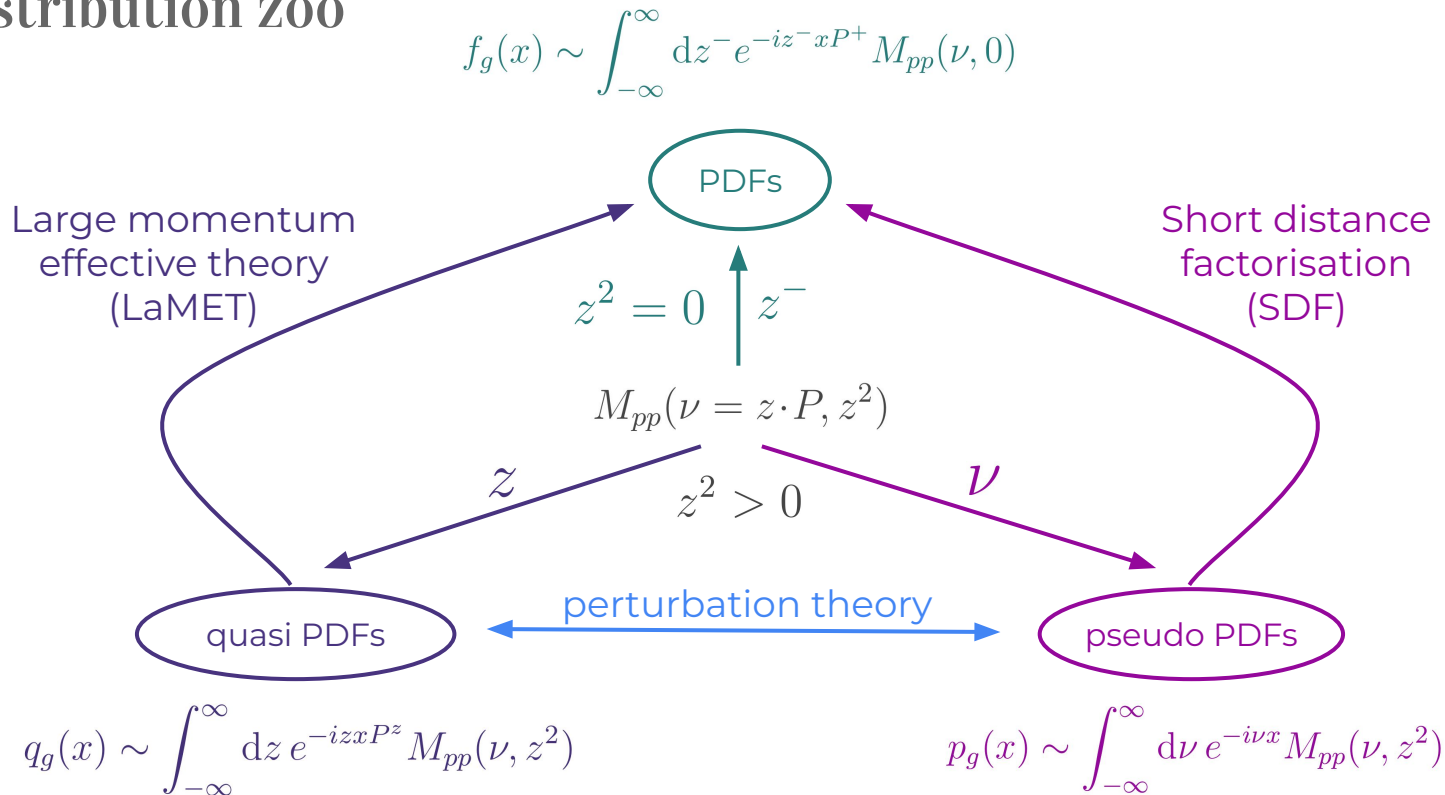
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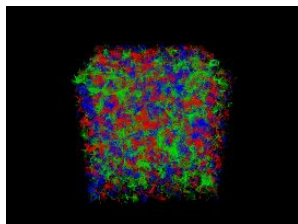
Complementary windows into hadron structure

“Validation from variety”

Wilson Award

N. Klco, Thurs. 11:30

Plenary
G. Bali, Fri. 11:00



CSSM, Adelaide



LHC

loffe-time space

$$I_{q/H}(\nu, \mu^2)$$

$$a_n(\mu^2)$$

Mellin space

$$f_{q/H}(x, \mu^2)$$

momentum-fraction space

Plenary
A. Shindler, Sat. 09:45

Plenary
C. Aidala, Fri. 10:00



Examining uncertainties

$$\langle P | G^{\mu\nu}(z) W(z, 0) G^{\rho\sigma}(0) | P \rangle \longrightarrow M_{pp}(\nu = z \cdot P, z^2)$$

Examining uncertainties

$$\langle P | G^{\mu\nu}(z) W(z, 0) G^{\rho\sigma}(0) | P \rangle \longrightarrow M_{pp}(\nu = z \cdot P, z^2)$$



Typical lattice sources of uncertainty:

statistics

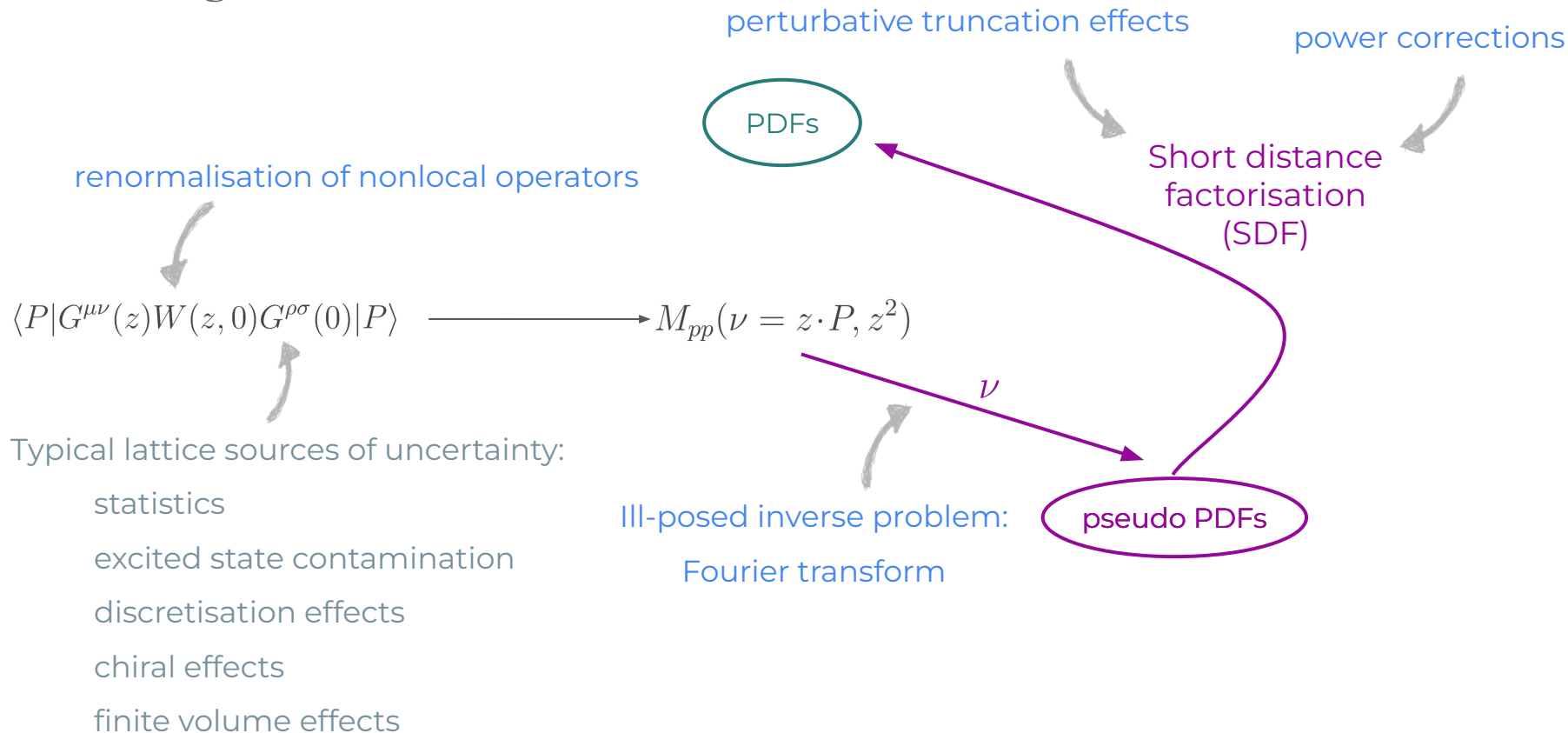
excited state contamination

discretisation effects

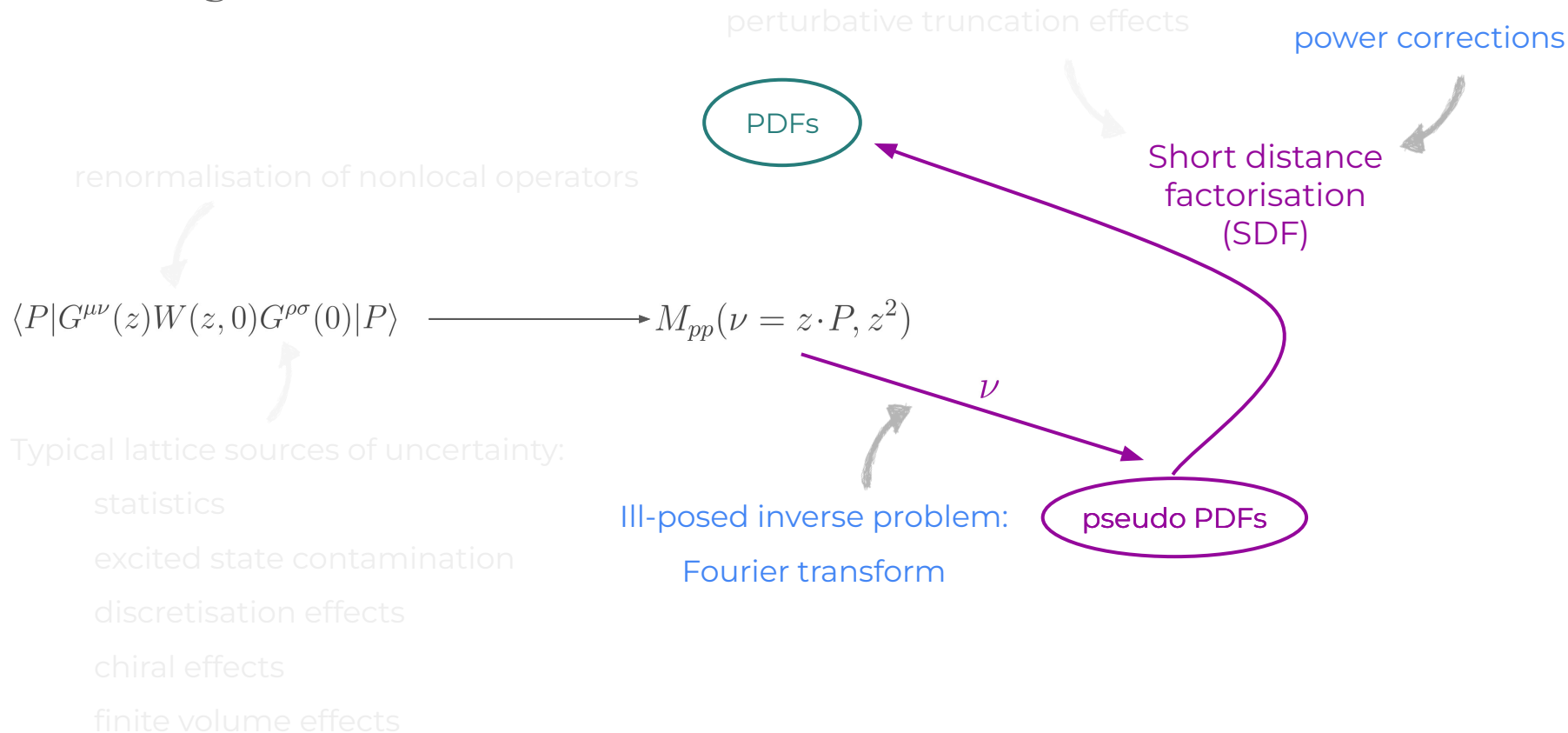
chiral effects

finite volume effects

Examining uncertainties



Examining uncertainties



Ill-posed inverse problems

Determination of PDFs via Fourier transform is an ill-posed inverse problem

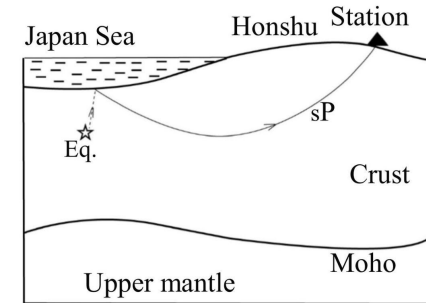
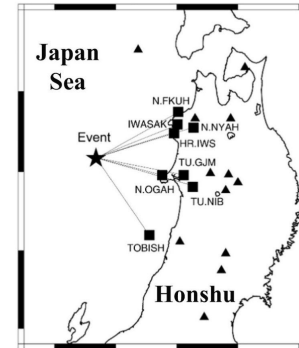
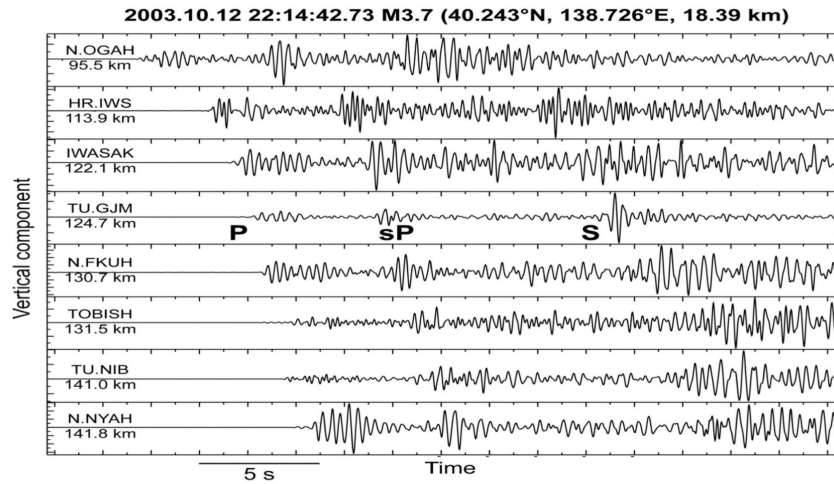
“Starting with the effects to discover the causes”

Plenary
S. Hashimoto, Mon. 11:00

Ill-posed inverse problems

Determination of PDFs via Fourier transform is an ill-posed inverse problem

“Starting with the effects to discover the causes”



Zhao, Phys. EPI 296 (2019) 106314

Ill-posed inverse problems

Determination of PDFs via Fourier transform is an ill-posed inverse problem

“Starting with the effects to discover the causes”

Given map $M(p)$, we want model parameters $\{p\}$ that match observed data

$$d_{\text{obs.}} = M(p) \quad \Rightarrow \quad p = M^{-1}(d_{\text{obs.}})$$

Well-posed (according to Hadamard) if solution:

1. Exists
2. Is unique
3. And is stable

Fourier transforms of discrete, noisy data = ill-posed

Determination of PDFs via Fourier transform is an **ill-posed inverse problem**

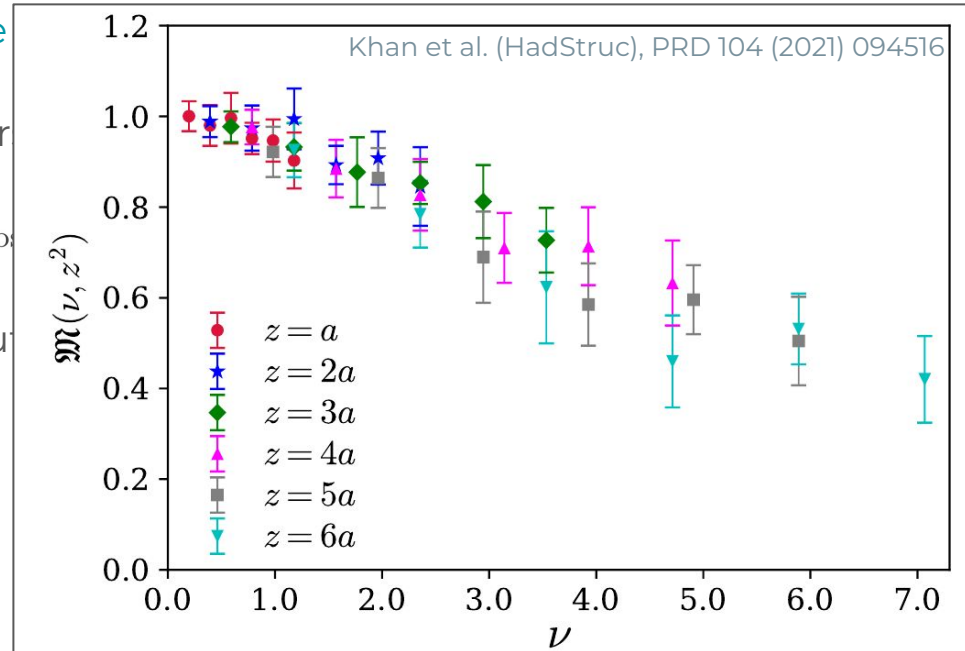
“Starting with the effects to discover the

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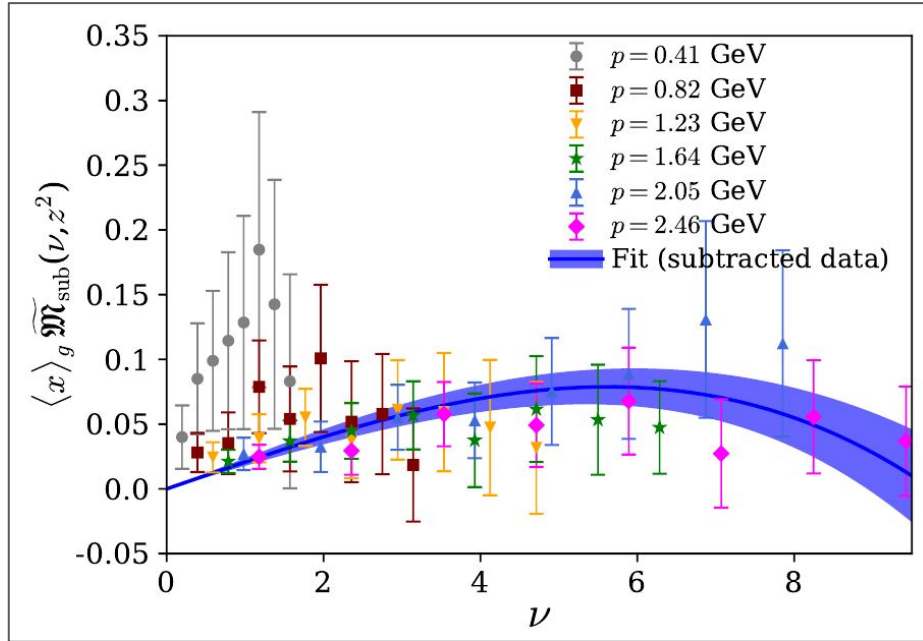
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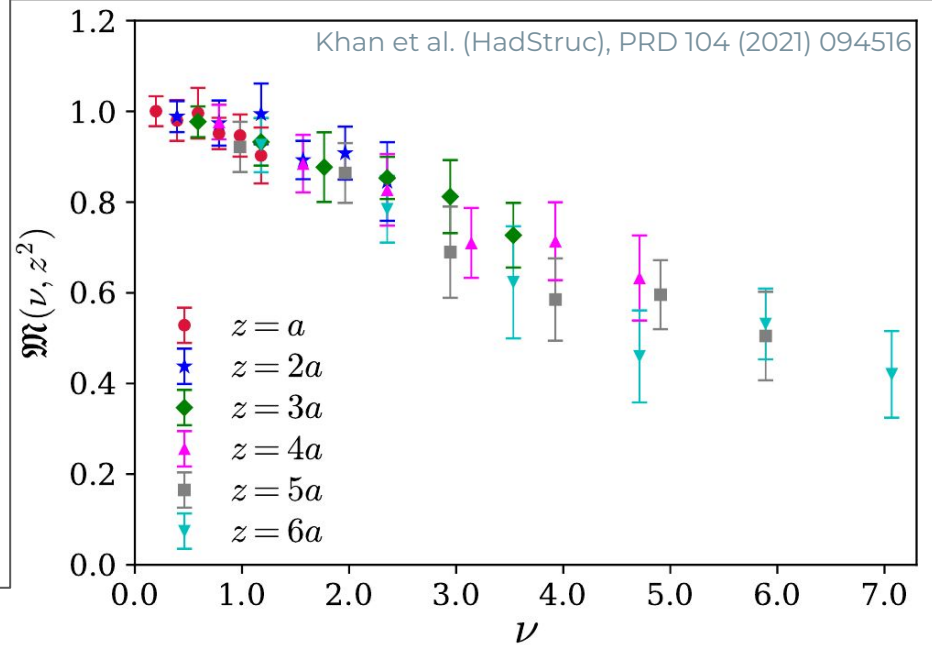


Fourier transforms of discrete, noisy data = ill-posed

Determination of PDFs via Fourier transform is an ill-posed inverse problem



Egerer et al. (HadStruc), PRD 106 (2022) 094511



Power corrections

Perturbative matching of light-cone and quasi PDFs via LaMET valid up to power corrections

$$f_g(x, \mu) \sim C(x, \mu) \otimes q_g(x, \mu) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2}\right)$$

Power corrections

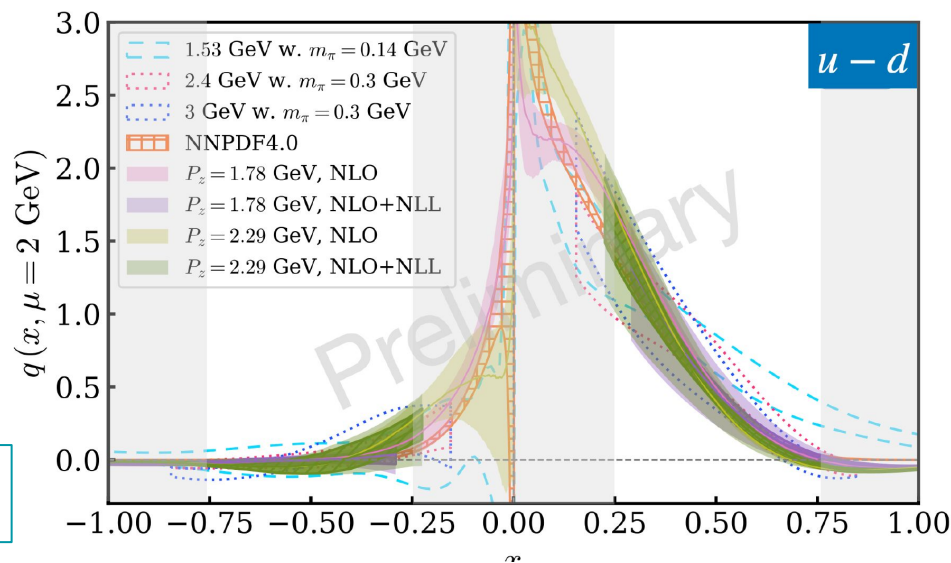
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large for $x \sim 0$

large for $x \sim 1$

Thanks to Q. Shi
Wed. 09:20



Outline

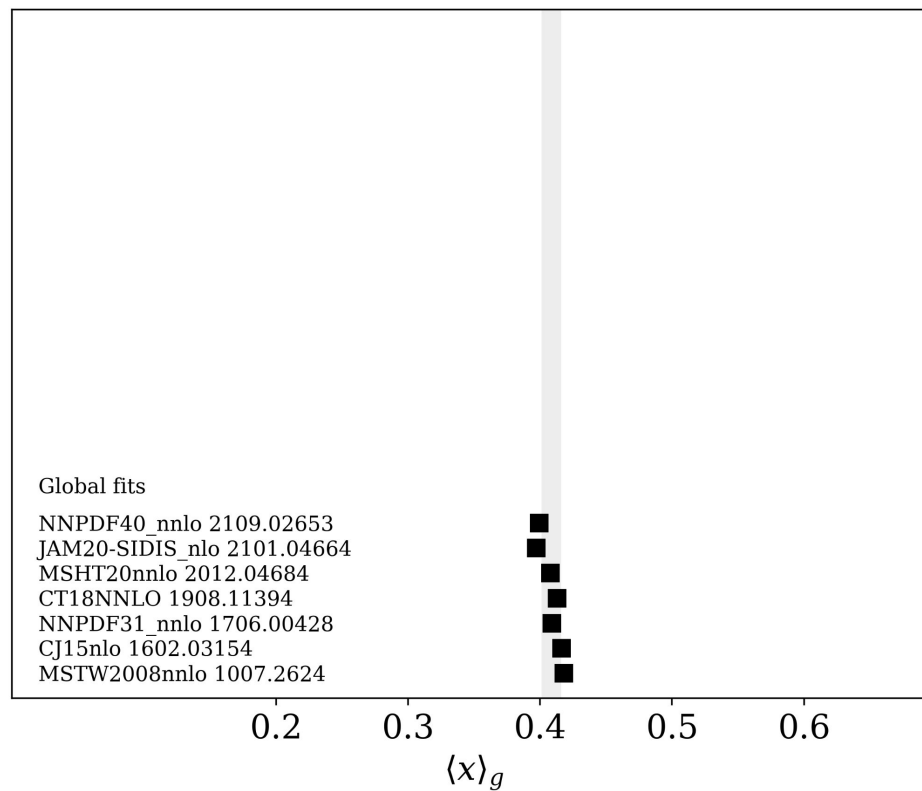
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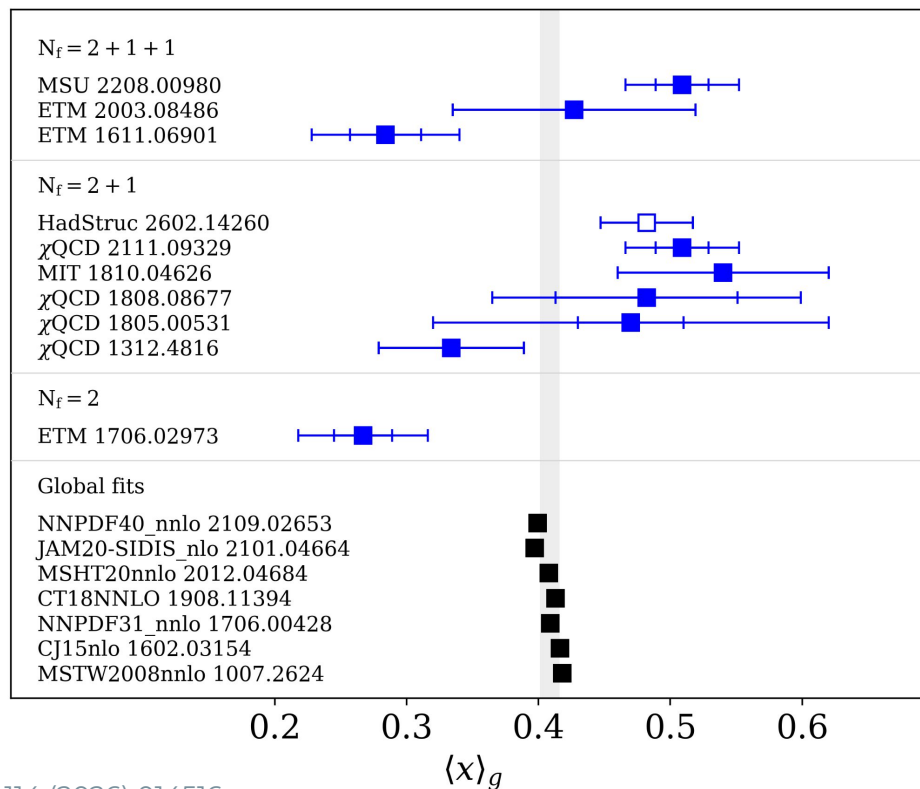
Window observables
Some preliminary studies

Gluon momentum fraction of the nucleon

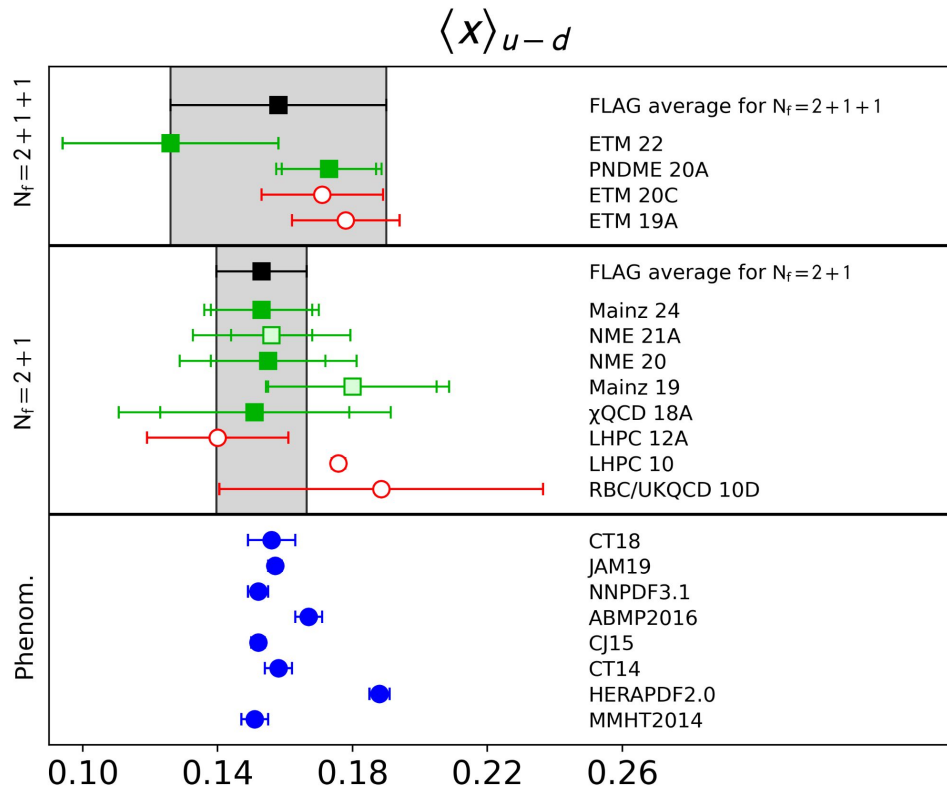


Gluon momentum fraction of the nucleon

A. Sturzu, Tu. 16:10

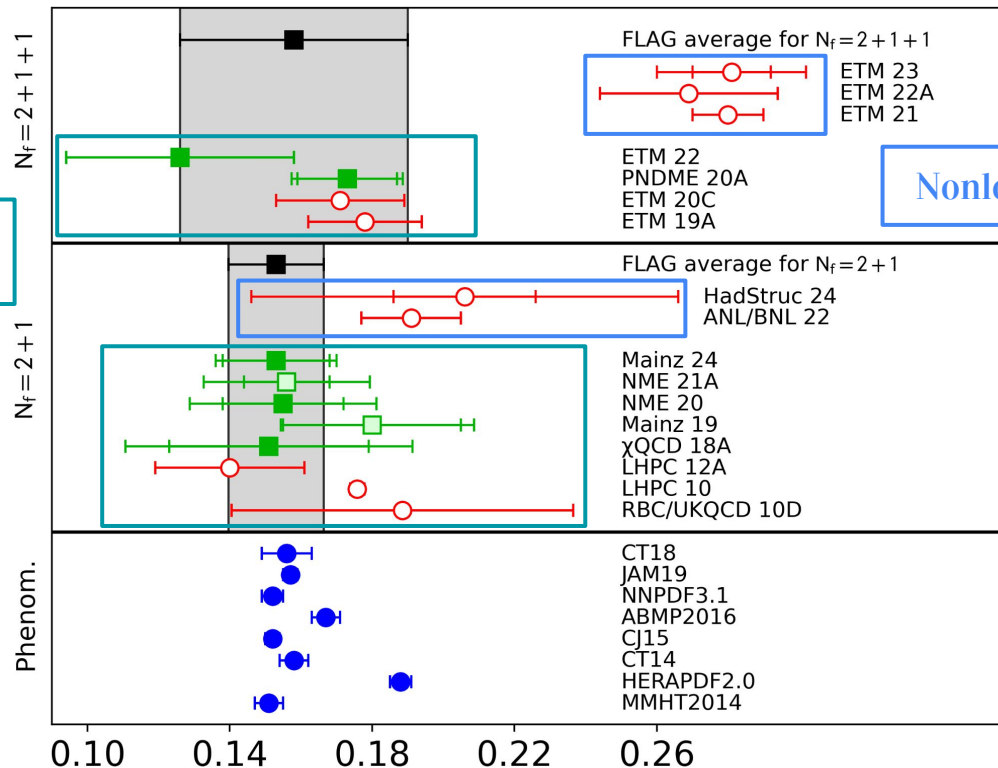


Local operators



Local and nonlocal operators

$$\langle X \rangle_{u-d}$$



Part 1: claim 1 in context

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Our proposal: window observables

Karpie et al., PRL 135 (2025) 191901

Propose **window observables** for benchmarking lattice calculations and global fits

$$W^{(f)}(x_{\pm}) = \int_{-1}^1 dx w(x; x_{\pm}) f(x)$$

For example, window moments:

$$a_n(x_-, x_+) = \int_{x_-}^{x_+} dx x^n f(x)$$

Motivated by analogy with (and enormous success of) windows for muon g-2

Blum et al., RBC/UKQCD PRL 121 (2018) 022003

Plenary
S. Hashimoto, Mon. 11:00

In the spirit of Shoji Hashimoto's take-home message
"Don't ask if the ~~spectral functions~~ PDFs can be fully reconstructed;
Ask if phenomenologically useful observables can be constructed."

Our proposal: window observables

Karpie et al., PRL 135 (2025) 191901

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Goal is “validation from variety” and precision

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Two key advantages as benchmarks that help achieve this goal:

1. Can be defined in regions of mutual validity for both lattice and experiment
2. Provide precise benchmarks by taking advantage of correlations in x

Our proposal: window observables

Karpie et al., PRL 135 (2025) 191901

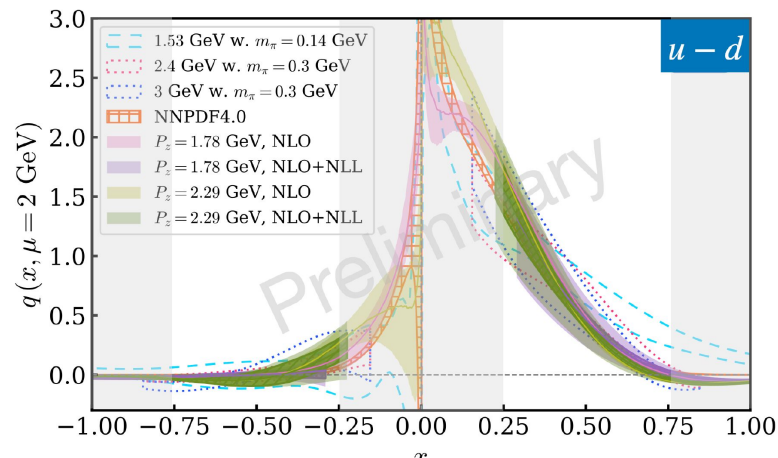
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Proof-of-principle results

Egerer et al., PRD 105 (2022) 034507

Extracted window moments from lattice data

- 2+1 clover-improved Wilson fermions with tree-level tadpole-improved Symanzik glue
- configurations generated by the JLab-LANL-MIT-WM collaboration

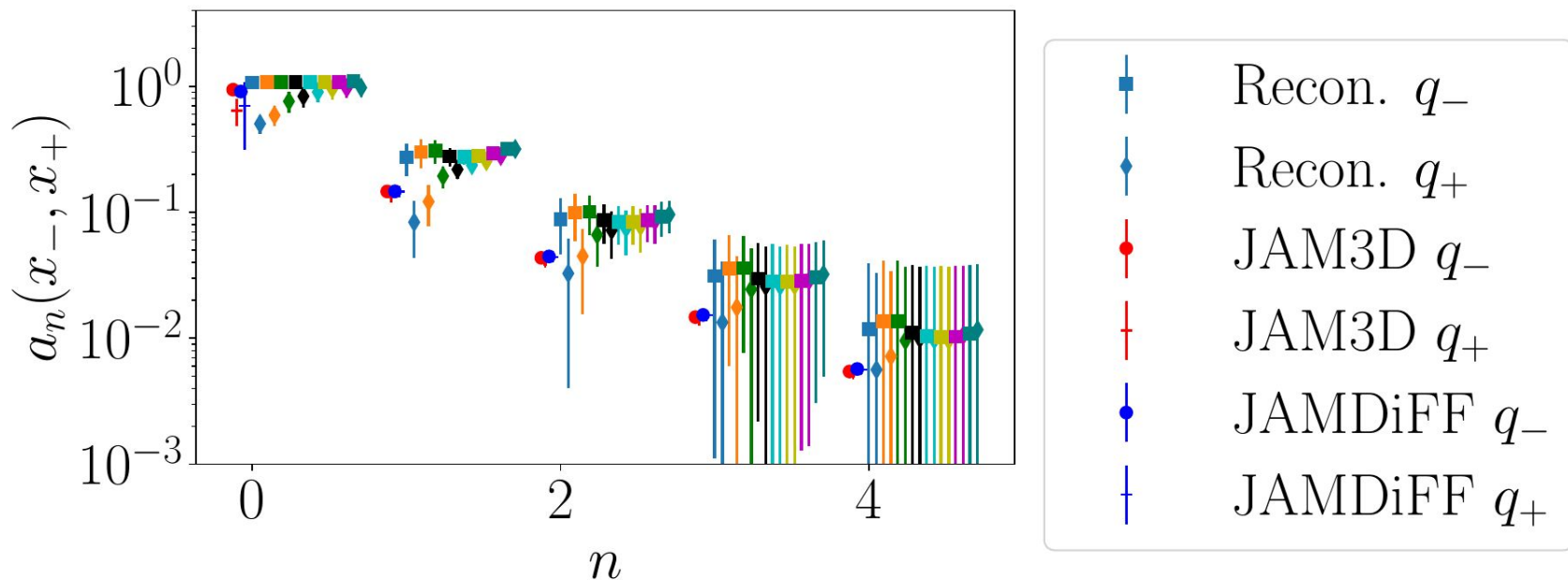
ID	a (fm)	m_π (MeV)	$L^3 \times T$	N_{cfg}	N_{srce}	N_D
<i>a094m358</i>	0.094(1)	358(3)	$32^3 \times 64$	1121	64	64

Compared with two datasets from the JAM collaboration

- JAM3D
- JAMDiFF

Gamberg et al., PRD 106 (2022) 034014
Cocuzza et al., PRL 132 (2024) 091901
Gamberg et al., PRD 109 (2024) 034024

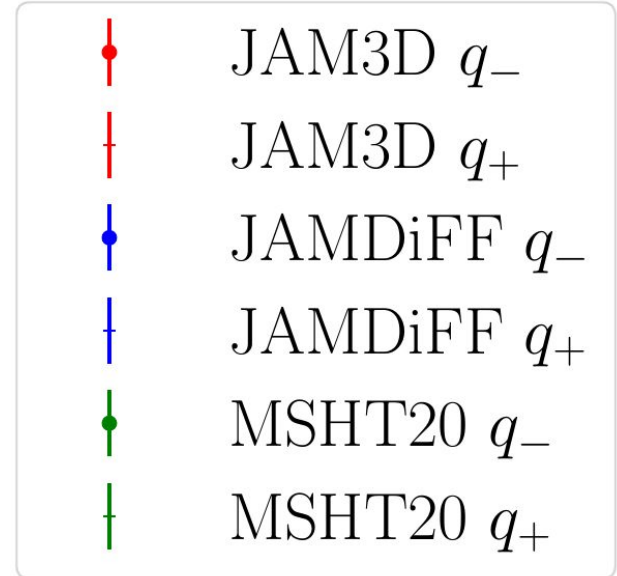
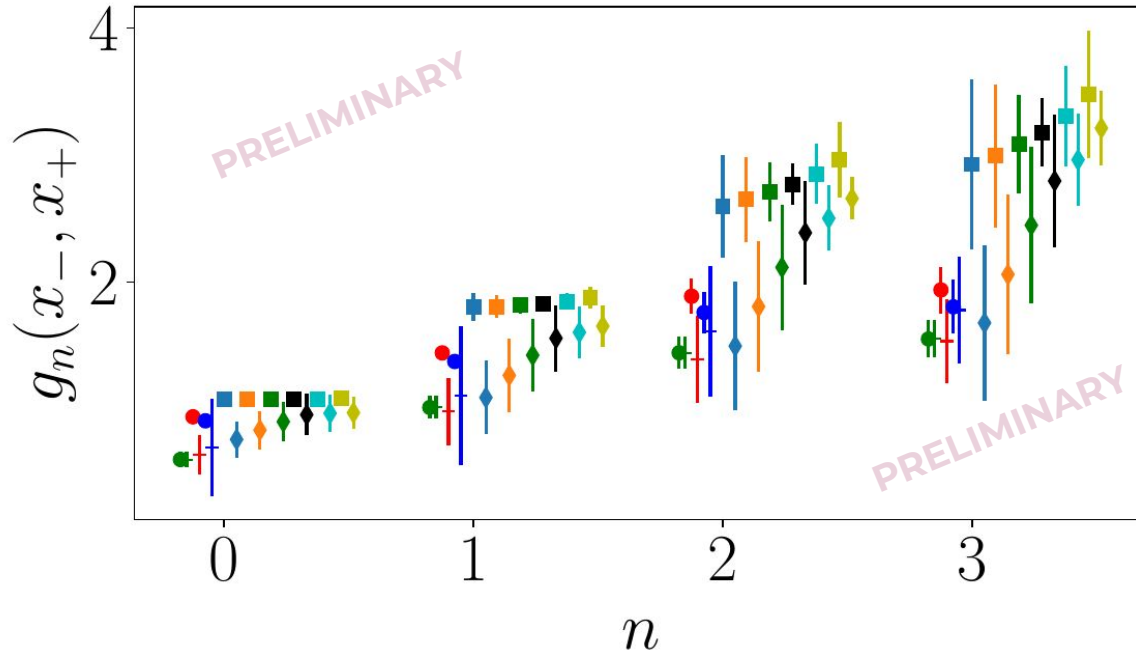
Proof-of-principle comparison



Egerer et al., PRD 105 (2022) 034507

Gamberg et al., PRD 106 (2022) 034014
Cocuzza et al., PRL 132 (2024) 091901
Gamberg et al., PRD 109 (2024) 034024

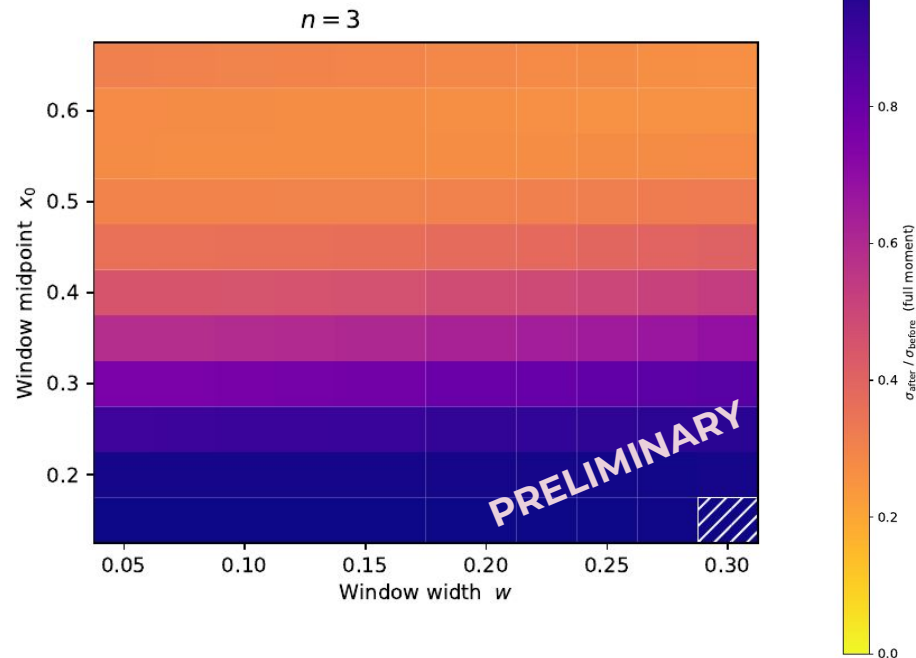
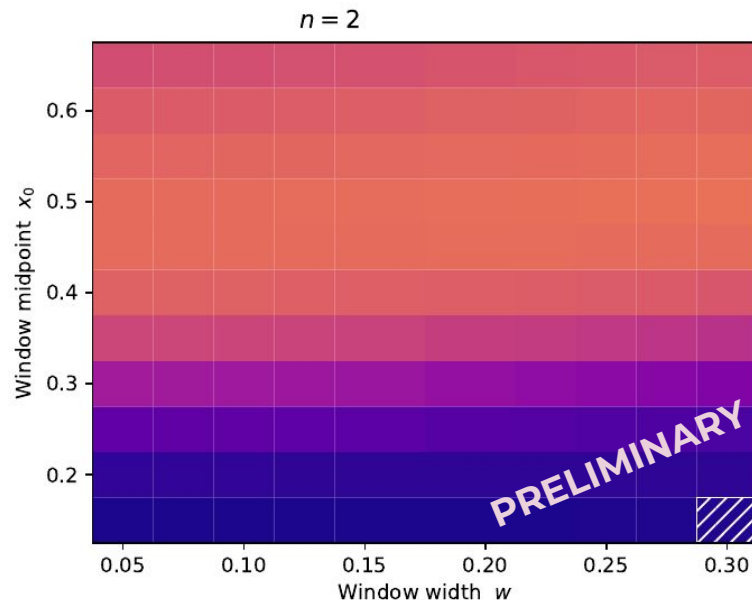
Proof-of-principle comparisons



Egerer et al., PRD 105 (2022) 034507

Gamberg et al., PRD 106 (2022) 034014
 Cocuzza et al., PRL 132 (2024) 091901
 Gamberg et al., PRD 109 (2024) 034024
 Bailey et al., EPJC 81 (2021) 341
 C. Flore, arXiv:2409.18751

Future impact?



Tests via the epump framework

Schmidt et al., PRD 98 (2018) 094005

(Reformulated) Claim 1:

We are entering the “quantitative QCD” era for protons and neutrons on the lattice.

We have not (yet) calculated nucleon properties with precision.

(Reformulated) Claim 2:

Our community history shows that dedicated community effort can achieve great things.

In particular, through quantified, complete error budgets and appropriate benchmarks.

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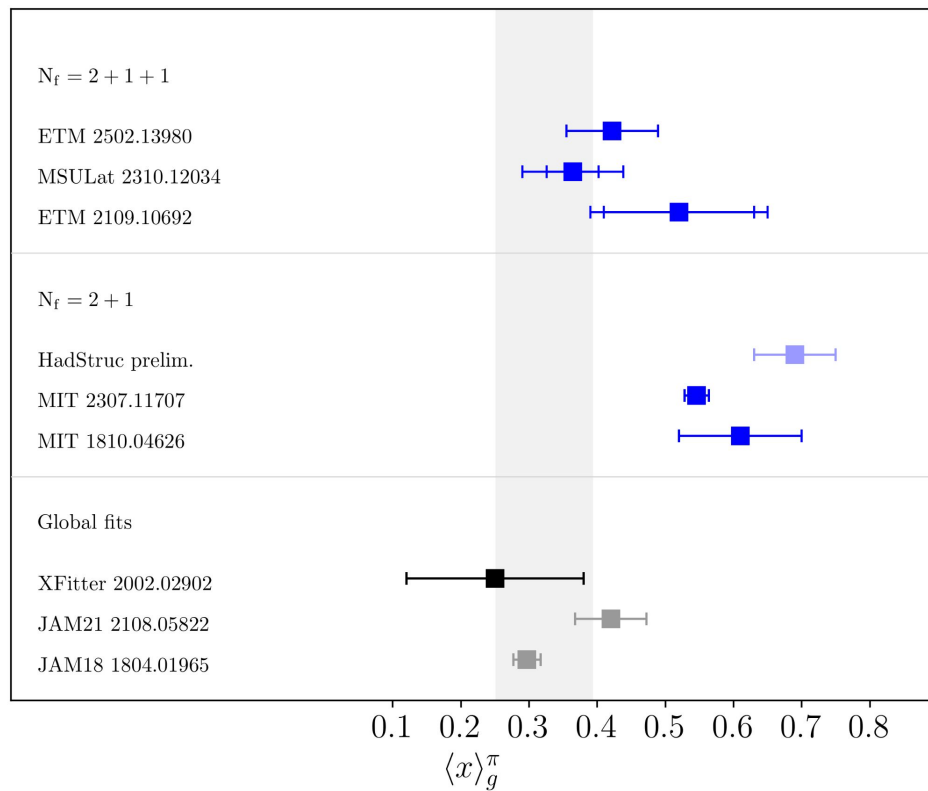
cjmonahan@coloradocollege.edu



The 43rd International Symposium on Lattice Field Theory
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Gluon momentum fraction of the pion



PDFs from and for experiment

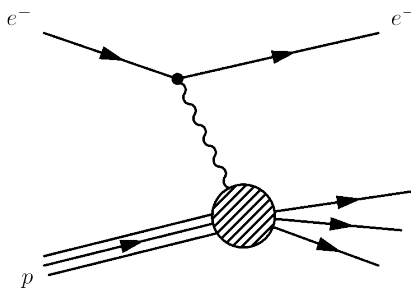
Extracted from and input into inclusive high-energy lepton-hadron scattering processes

$$e^- (\gamma^*) H \rightarrow e^- X$$

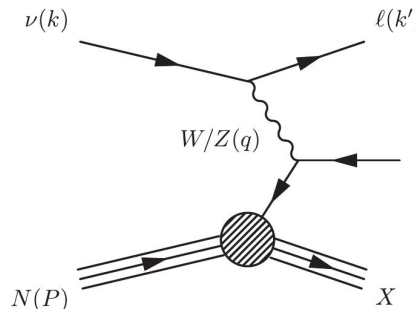
$$\nu_\ell (W^{\pm*}) H \rightarrow \ell^\pm X$$

$$\nu_\ell (Z^*) H \rightarrow \nu_\ell X$$

Cross-sections factorise: perturbative scattering coefficient x nonperturbative PDFs



Universität Zürich



Xie et al. (CTEQ-TEA), PRD 109 (2024) 113001

Additional details for inverse problems

Overcoming noise

$$d_{\text{obs.}} = M(p) \quad \Rightarrow \quad p = M^{-1}(d_{\text{obs.}})$$

Three obvious strategies to overcome noise in inverse problems

1. Acquire more accurate data to reduce $|p_1 - p_2|$ arising from noise
2. Change the map $M(p)$ and acquire different data
3. Restrict the space in which parameters are sought

In other words, introduce a **prior model**:

1. **Penalisation procedure** (e.g. a regularisation)
2. **Bayesian framework**

Note that some authors consider the first option a subset of the latter, with Bayesian methods having the advantage of making the impact of prior information explicit.

Noise and ill-posed inverse problems

In the presence of noisy data, we can capture stability through [stability estimates](#)

$$|p_1 - p_2| \leq f(|M(p_1) - M(p_2)|)$$

Provides a numerical, but subjective, approach to ill-posedness:

when noise is significantly amplified, the problem is ill-posed

noise can also lead to data falling outside the range of $M(p)$

Natural to introduce noise through a modified forward map

statistical properties of noise become important

Monte Carlo statistical uncertainty well-modelled by Gaussian noise

less clear how to analytically model lattice systematic uncertainties

$$d_{\text{obs.}} = M(p) + \eta$$

Some historical strategies

Three obvious strategies to overcome noise in inverse problems

1. Acquire more accurate data to reduce $|p_1 - p_2|$ arising from $|\eta|$
2. Change the map $M(p)$ and acquire different data
3. Restrict the space in which parameters are sought

Lin et al. (LP³), PRD 91 (2014) 054510

Lin et al. (LP³), PRD 98 (2018) 054504

Ishikawa et al., SCPMA 62 (2019) 991021

Karpie et al., JHEP 04 (2019) 057

Dutrieux et al., PRD 111 (2025) 034515

Del Debbio et al., EPJC 85 (2025) 2

Izubuchi et al., PRD 100 (2019) 034516

Cichy, Del Debbio & Giani, JHEP 2019 (2019) 137

Direct Fourier transform

Low-pass filtering

“Derivative method”

Gaussian weighting

Backus-Gilbert

Maximum entropy method

Bayesian reconstruction

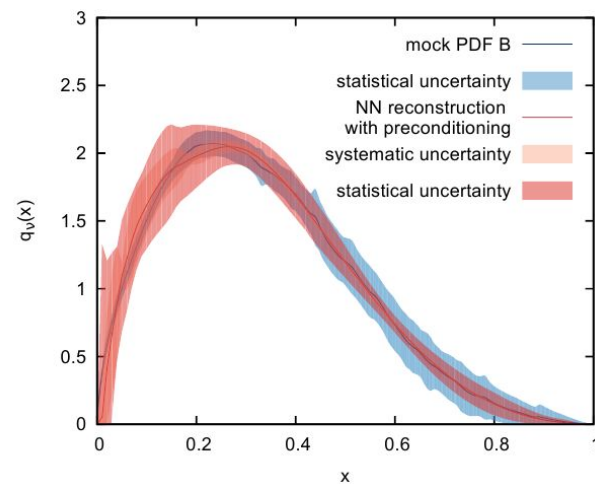
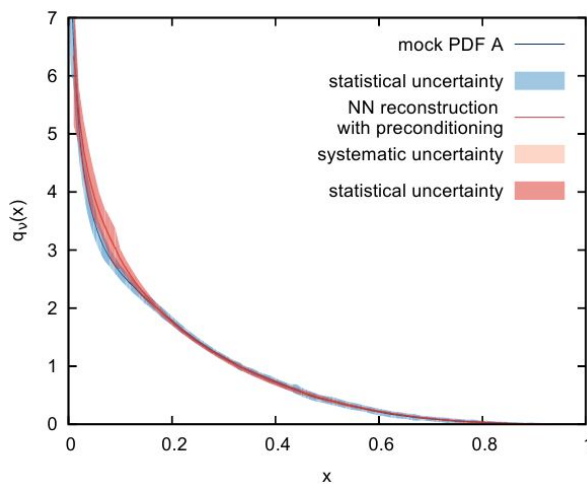
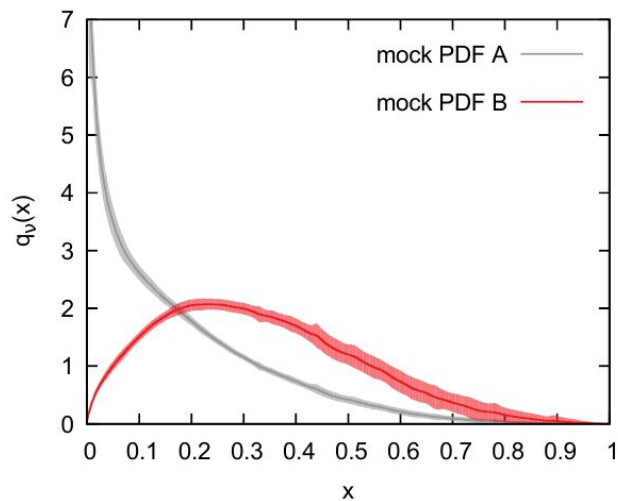
Neural nets

Gaussian processes

Global fits

Model data and closure tests

Reconstruction methods first systematically studied in Karpie et al., JHEP 04 (2019) 057
Generated two PDF models and analysed advanced reconstruction methods



Gaussian process regression

Applied to x-dependent distributions in
Alexandrou et al., PRD 102 (2020) 094508
Dutrieux et al., PRD 111 (2024) 034515
Cahuana Medrano et al., JHEP 04 (2026) 182

Non-parametric approach to generating flexible models that implement desired constraints

- can be interpreted as a neural network with a single wide hidden layer

Gaussian process: set of random variables that follows a joint multivariate Gaussian distribution

Goal of regression is to find mean and covariance of the distribution such that the mean is an estimate of data and covariance describes deviations

Typically implemented via Bayesian inference

- requires choice of prior covariance function and corresponding hyperparameters

Gaussian process regression

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Non-parametric approach to generating flexible models that implement desired constraints

Can be interpreted as an implementation of neural networks with a single wide hidden layer

Gaussian process:

- collection of random variables $f(z)$, with continuous real index z , that follows a joint multivariate Gaussian distribution with mean function $\mu(z)$ and covariance function $k(z, z_o)$
- for points $z_1 \dots z_n$, the distribution of $f(z)$ is given by

$$p(f_1, \dots, f_n) = \det(2\pi K)^{-1/2} e^{-1/2 \sum_{ij} (f_i - \mu_i) K_{ij}^{-1} (f_j - \mu_j)}$$

where

$$f_i = f(z_i) \quad \mu_i = \mu(z_i) \quad K_{ij} = k(z_i, z_j)$$

Goal: find $\mu(z)$ and $k(z, z_o)$ such that $\mu(z)$ is an estimate of matrix elements $h(z)$ and $k(z, z_o)$ describes deviations of $\mu(z)$ from $h(z)$

Bayesian inference

Applied to x-dependent distributions in
Alexandrou et al., PRD 102 (2020) 094508
Dutrieux et al., PRD 111 (2024) 034515
Cahuana Medrano et al., JHEP 04 (2026) 182

Regression (determination of $\mu(z)$) typically undertaken with Bayesian inference

- requires specification of a prior Gaussian process, specified by $\mu_P(z)$ and $k_P(z, z_0)$
- prior Gaussian process adapted to data using Bayes' theorem
- posterior Gaussian process can be calculated analytically

$$\mu(z) = \mu_P(z) + \sum_{ij} k_P(z, z_i) [K_{ij} + (\Delta h_i)^2 \delta_{ij}]^{-1} (h_j - \mu_P(z_j))$$

Choice of prior covariance and hyperparameters is critical to avoid over/underfitting

GPR as Bayesian prior

data likelihood given by correlated likelihood with data

$$P(M|f_i^q) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp \left[-\frac{1}{2}(M_k - B \cdot f^q)^T \Sigma^{-1} (M_k - B \cdot f^q) \right]$$

In the case relevant to GPDs (or PDFs), data take the form

$$\mathcal{M}(\nu, \bar{\nu}, t, z^2) = B(\nu, \bar{\nu}) \otimes f^q(t, z^2)$$

where $B(\nu, \bar{\nu})$ is a linear operator and we wish to infer $f^q(t, z^2)$

vector index i labels vertices of fine mesh for reconstruction

Assuming a prior given by a multivariate normal distribution

$$I = \mathcal{N}(g_i, K)$$

Bayes' theorem gives the posterior probability distribution

Prior distribution density

$$P(f_i^q|I) = \frac{1}{\sqrt{\det(2\pi K)}} \exp \left[-\frac{1}{2}(f_i - g_i)^T K^{-1} (f_i - g_i) \right]$$

$$P(f_i^q|M, I) = \frac{P(M|f_i^q)P(f_i^q|I)}{P(M|I)}$$

evidence term

given the data (also a multivariate normal distribution)

$$M = \mathcal{N}(M_k = \mathcal{M}(\nu_k, \bar{\nu}_k, t_k, z_k^2), \Sigma)$$

covariance of lattice data

discrete set of lattice data at specific kinematic points (labelled by k)