Bootstrapping Spectral Functions From Euclidean Lattice Data

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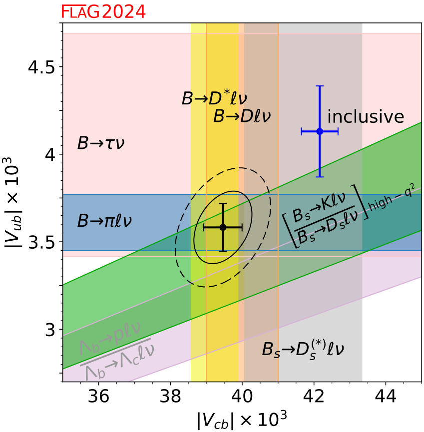
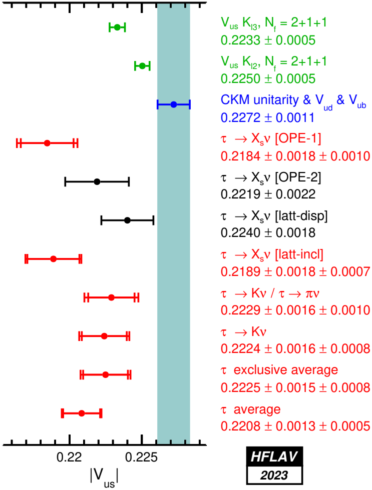
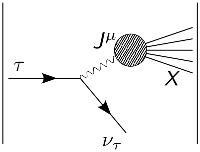
Inclusive-Exclusive/CKM Tensions

- Persistent tensions
- New Physics?
- Systematics?





Inclusive Decays & Spectral Functions

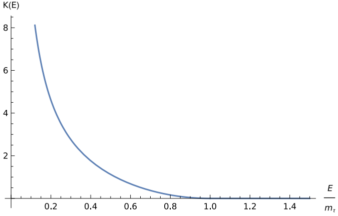


$$\sum_X \left| \cdots \right|^2 = \int_0^\infty dE\, K(E) \rho(E)$$

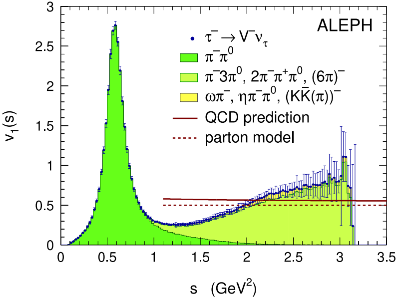
“Kernel”
Kinematics, leptonic

$K(E) = \frac{1}{x}(1 - x^2)^2 \theta(1 - x)$

$x = E/m_\tau$



$\rho(E) \sim \sum_X |\langle 0|J|X\rangle|^2 \delta(E_X - E)$
Spectral function, Hadronic



Spectral functions from Lattice QCD

- Numerically determining ρ requires inverting Laplace transform:
  - Often very ill-conditioned/ill-posed

$$C_E(\tau) = \int_0^\infty dE\, e^{-E\tau} \rho(E)$$

- Problem(s): ρ is a distribution, finite vs infinite volume
  - Key idea: work with *smeared* spectral functions

$$\mathcal{K}[\rho] = \int dE\, K(E) \rho(E)$$

- Standard method(s): Backus-Gilbert/Hansen, Lupo, Tantalo (HLT)
  - See also: Chebyshev

HLT: Exact Data

- Some kernels are easy:

$$K_g(E) = \sum_t g_t e^{-Et}$$

$$\implies \int K_g(E)\rho(E)\, dE = \sum_t g_t C(t)$$

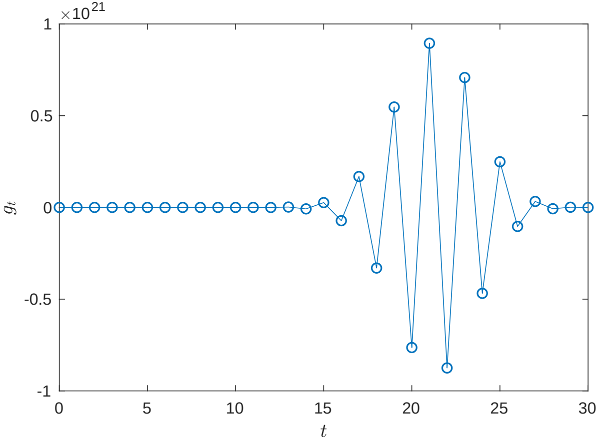
- Key idea: tune g_t so $K_g \approx K$

$$\text{minimize} \quad A[g] = \int dE \left| K_g(E) - K(E) \right|^2$$

- Works very well with exact data...

Backus-Gilbert & HLT: Inexact data

Problem: optimal g large + oscillatory:



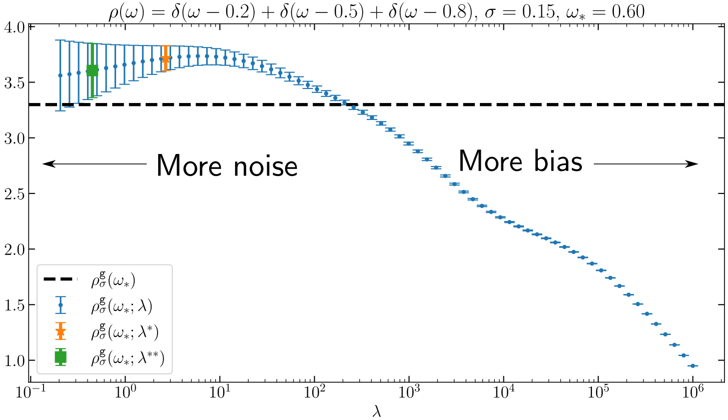
$$\int K\rho \approx \sum_t g_t C_t$$

Figure: [HLT, 1903.06476]

Backus-Gilbert & HLT: Inexact data (cont.)

Introduce covariance Σ, regulator λ, new functional

$$\text{minimize } A[g] + \lambda g^\dagger \Sigma g$$



Question: rigorous systematics?

Rigorous bounds

- Question: how to get rigorous systematics?
- Several methods proposed (see Shoji Hashimoto's plenary)
- Non-exhaustive, biased list (imposing causality):
  - Nevanlinna–Pick Interpolation [Bergamaschi, Jay, Oare 2305.16190]
  - Moment problems [Abbott, Jay, Oare 2508.01377]
  - Linear/convex programming [Lawrence 2408.11766]

Key insight: The above are closely related/equivalent, and generalize HLT

- Also closely related to Conformal bootstrap, S-matrix bootstrap

$$\implies \text{ The Causal Bootstrap}$$

The Causal Bootstrap

- Idea: minimize/maximize target with $\rho(E) \geq 0$ (reflection positivity)
- Formulate as a (convex) optimization problem

$$\begin{aligned} \text{minimize} \quad & \int K(E)\rho(E)\, dE \\ \text{subject to} \quad & \int \rho(E) e^{-Et} = \hat{C}_t \\ \text{and } & \rho(E) \geq 0 \end{aligned}$$

Monte-Carlo data

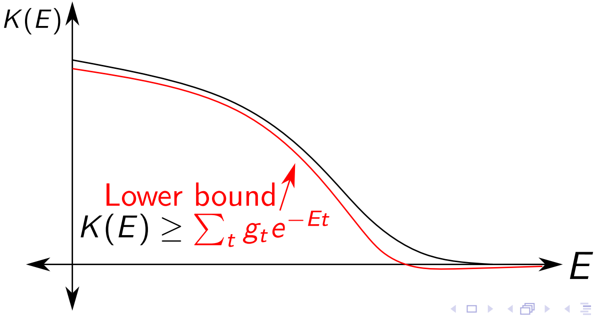
- Referred to as *semidefinite program*
- Note: can also maximize with $K(E) \mapsto -K(E)$

Lagrange Duality

- General duality theory $\Rightarrow$ equivalent dual problem [Lawrence 2408.11766]

$$\text{maximize} \sum_t g_t \hat{C}_t$$
$$\text{subject to } K(E) \geq \sum_t g_t e^{-Et}$$

- Infinite # conditions, but can make finite (conformal bootstrap)



Don't Approximate, Bound

Linear Methods (e.g. HLT)

$$K(E) \approx \sum_t g_t e^{-Et}$$

$$\Big\Downarrow \int \rho$$

$$\int K(E)\rho(E)\, dE \approx \sum_t g_t C_t$$

- Choose g via regulator λ
- Bias-Variance tradeoff

Causal Bootstrap

$$K(E) \geq \sum_t g_t e^{-Et}$$

$$\Big\Downarrow \int \rho \quad [\rho \geq 0]$$

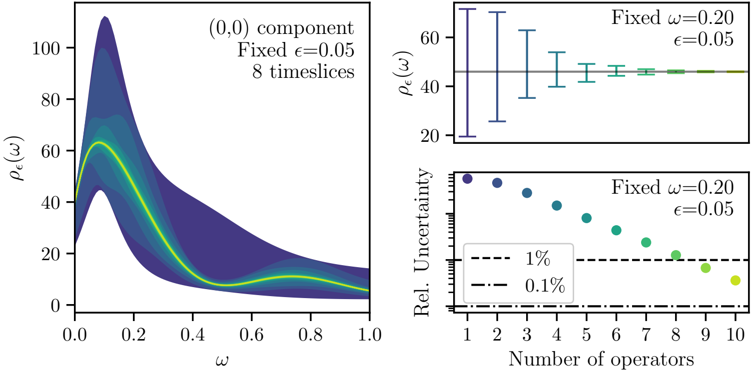
$$\int K(E)\rho(E)\, dE \geq \sum_t g_t C_t$$

- Optimize over g (convex)
- Key fact: can saturate

- More benefits: matrix correlators, noise/inexact data

Matrix Correlators

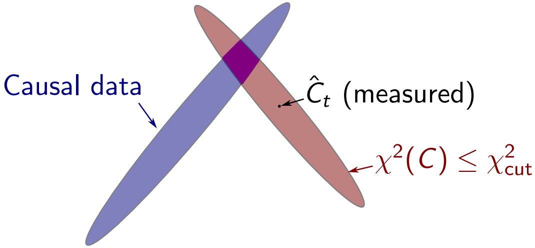
- Can use matrix correlators directly [Abbott, Jay, Oare 2508.01377]



- Exponential (!) improvement w/ more operators
- Key point: even if only want $(0,0)$ component, more ops still help
  - Compare with a_μ^{HVP} long distance window

Inexact Data

- Problem: measured data $\hat{C}_t$ almost never causal
- Solution: Optimize subject to χ^2 cutoff (convex!):

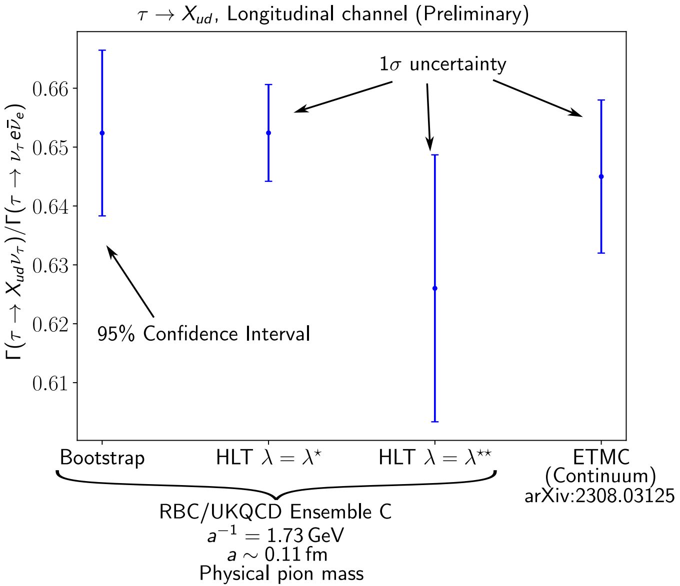


- Dual formulation:

$$\text{maximize } \sum_t g_t \hat{C}_t - \chi_{\text{cut}} \sqrt{g^\dagger \Sigma g}$$

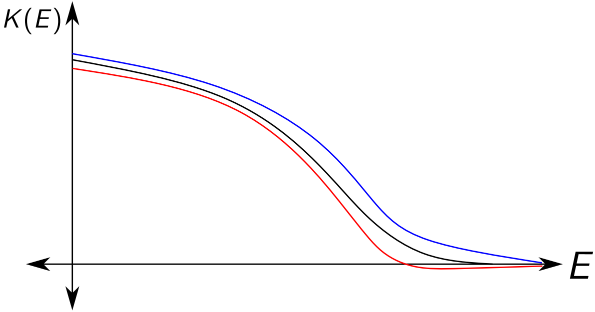
$$\text{subject to } K(E) \geq \sum_t g_t e^{-Et}$$

Inclusive τ decay



Outlook

- Bootstrap ideas link many spectral reconstruction methods
- Promising early results for inclusive τ decay
- Not covered here: finite-dimensional reduction, conformal bootstrap
- Future work: More channels, continuum limit, matrix correlators



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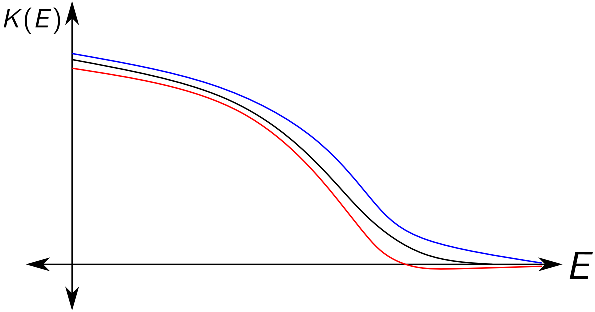
- Thanks for listening! Questions?



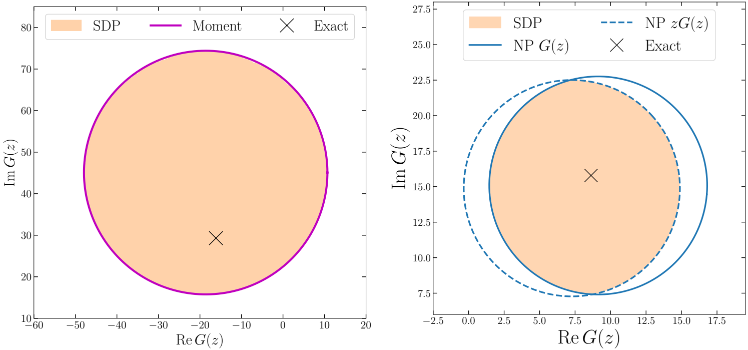
Backup

Wertevorrat vs SDP

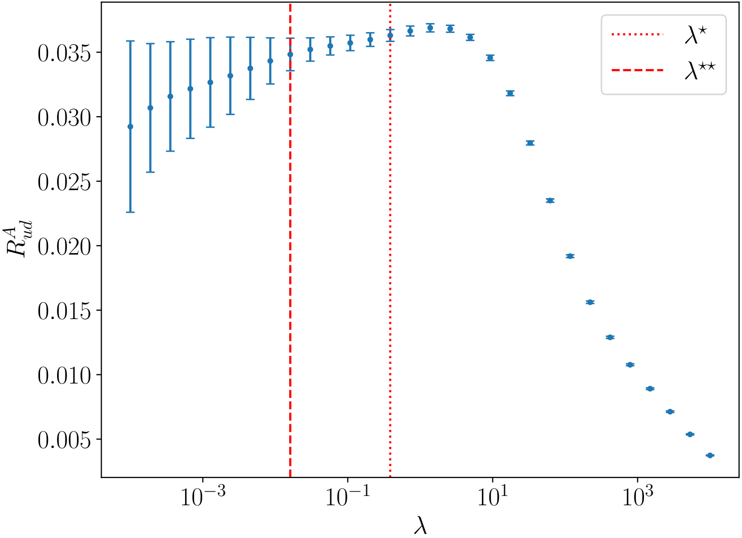
- Bounds on correlation function $G(z)$:

$$G(z) = \int \frac{\rho(E)\, dE}{z - E}$$

- Matches exactly with Nevanlinna-Pick (NP) and Moment problems

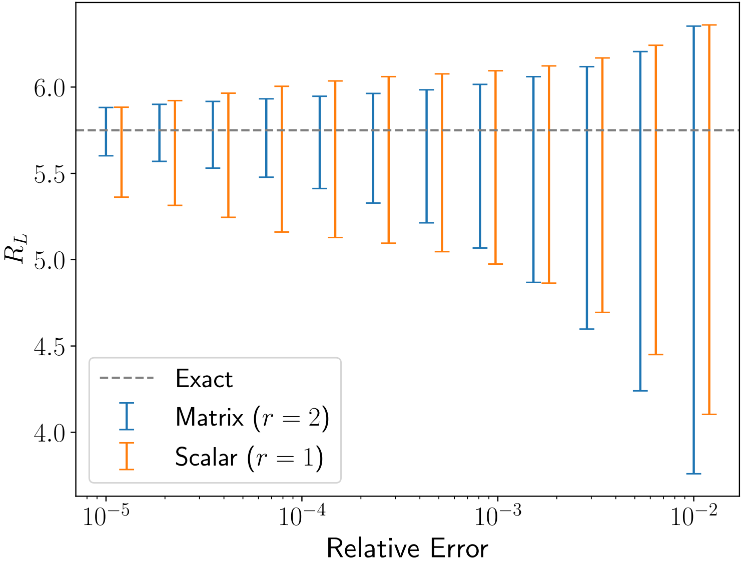


HLT Stability Plot



Toy data

Toy data with noise for semi-inclusive τ decay



Moment Problems

- Distribution $\tilde{\rho} \geq 0 \implies$ moments C_k:

$$C_k = \int \lambda^k \tilde{\rho}(\lambda)\, d\lambda$$

- Moment problem: Given $C_0, \ldots, C_{n-1}$, determine $\tilde{\rho}$
  - Note: $C_k, \tilde{\rho}$ may be matrix-valued
- Extensively studied in mathematical literature
- Closely related to Nevanlinna-Pick interpolation
- Solutions need not exist; e.g. $\mathrm{Var}(\tilde{\rho}) \sim C_2 C_0 - C_1^2 \geq 0$
- More generally

$$\begin{pmatrix} C_0 & C_1 & C_2 \ldots & \\ C_1 & C_2 & C_3 \ldots & \\ C_2 & C_3 & C_4 \ldots & \\ \vdots & \vdots & & \ddots \end{pmatrix} \geq 0$$

Moment problems and Rayleigh-Ritz

- Many equivalent spectroscopy methods
  - Lanczos = Prony = PGEVM = GPOF $\supset$ GEVP $\supset$ m_{eff}
- Two equivalent interpretations: [Abbott et al, 2503.17357]
  - Rayleigh-Ritz: Eigenvalues of Transfer matrix on Krylov space
  - Prony: *minimal* multi-exponential interpolation

$$C(t) = \sum_{k=0}^{N_t/2-1} A_k e^{-E_k t}; \qquad t \in \{0, \ldots, N_t - 1\}$$

- Moment problems capture all possible multi-exponential interpolations
- Direct connections w/ Lanczos also exist