Isospin-breaking effects in inclusive hadronic tau data for the muon $(g-2)$

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2607.00564, 2606.04729
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Lattice Conference 2026
University of Maryland, USA, July 31st

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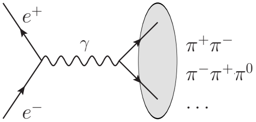
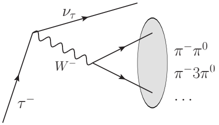
τ DECAYS FOR THE MUON $g-2$



Isospin breaking corrections
connect τ with HVP
for muon $g-2$
[Alemani et al. '98]



Strategy from Lattice QCD+QED [Bruno et al '26]

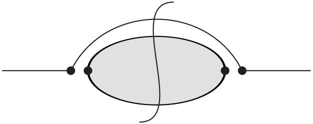
- delicate short [Boyle et al' 26] and long distance effects
- shed light given current tensions in e^+e^- panorama

Lattice QCD(+QED): first-principle systematically-improvable approach

- a fully inclusive strategy
- taking care of short-distance renormalization
- discussing non-factorizable effects (aka G_{EM})

$$\tau^-(P) \to \nu(q)\,\mathrm{had}_V^-(p)$$

$$\mathcal{M}_f(P,q,p_1\cdots p_{n_f}) = 2G_\mathrm{F}V_\mathrm{ud}\,\bar{u}(q)\gamma_\mu^L u(P)\ \langle \mathrm{out}, p_1\cdots p_{n_f}|\mathcal{J}_\mu^-(0)|0\rangle$$



$$d\Gamma = \frac{1}{4m}d\Phi_q\sum_f d\Phi_f\sum_{\mathrm{spin}}|\mathcal{M}_f|^2$$

Isovector spectral density isospin limit $= \rho$

$$\left[d\Phi_q = \frac{d^3q}{(2\pi)^3 2\omega_q}\right]$$

$$\frac{d\Gamma}{ds} = \frac{G_\mathrm{F}^2|V_\mathrm{ud}|^2 m_\tau^3}{8\pi}\kappa(\hat{s})\,\rho(s)\,,\quad \kappa(\hat{s}) = (1+2\hat{s})\,(1-\hat{s})^2\quad \hat{s} = \frac{s}{m_\tau^2}$$

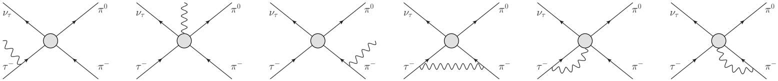
from experiment we get $\dfrac{1}{\Gamma}\dfrac{d\Gamma}{ds} \to \dfrac{\Gamma}{\Gamma_e}\dfrac{1}{\Gamma}\dfrac{d\Gamma}{ds} = \dfrac{1}{\Gamma_e}\dfrac{d\Gamma}{ds}$

$$\Gamma_e = \Gamma(\tau\to e\bar{\nu}\nu) = \frac{\mathcal{B}_e\Gamma}{\mathcal{B}} = \frac{G_\mathrm{F}^2 m_\tau^5}{192\pi^3}$$

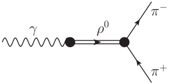
Overview of current status

Addressed in various models: ChPT, meson dominance models

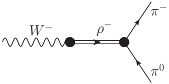
[Cirigliano et al 00 01][Flores-Talpa et al '06 '07]



Difference between neutral and charged "form factors"



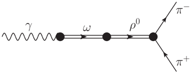
vs



Sources of IB breaking in phenomenological models

$m_{\rho^0} \neq m_{\rho^\pm}$, $\Gamma_{\rho^0} \neq \Gamma_{\rho^\pm}$, $m_{\pi^0} \neq m_{\pi^\pm}$

$\delta_{\rho\omega} \simeq O(m_\mathrm{u} - m_\mathrm{d}) + O(e^2)$



Hadronic τ decays. Short distance renormalization

Wilson coefficient: matching between SM and EFT at $O(\alpha)$

W-regularization [Sirlin '78][Sirlin '82][Marciano, Sirlin '88][Braaten, Li '90]

[Brod, Gorbahn '08][Gorbahn et al '23][Bigi et al '23][Cirigliano et al '23]

$$C^{\overline{\mathrm{MS}}}(\mu) = U(\mu, \mu_W) C^{\overline{\mathrm{MS}}}(\mu_W) \quad \gamma_W = \{\alpha, \alpha\alpha_s\} + O(\alpha\alpha_s^2, \alpha^2)$$

Regularization Invariant Momentum schemes [Martinelli et al '94]

suitable for Lattice QCD = off-shell (massless) quark states

$C^{\overline{\mathrm{MS}} \to \mathrm{RI}}(\mu) = \{1, \alpha, \alpha\alpha_s\} + O(\alpha\alpha_s^2, \alpha^2)$ [Gorbahn et al '23]

Physical effective Hamiltonian $H_W \propto C^{\mathrm{RI}}(\mu) Z^{\mathrm{RI}}(\mu, a) \mathcal{O}^{\mathrm{latt}}(a)$

$C^{\mathrm{RI}}(\mu) = C^{\overline{\mathrm{MS}} \to \mathrm{RI}}(\mu) U(\mu, \mu_W) C^{\overline{\mathrm{MS}}}(\mu_W)$

$\rightarrow$ scheme + scale + gauge dependence cancels $C^{\mathrm{RI}} \leftrightarrow Z^{\mathrm{RI}}$

Wilson coefficient: matching between SM and EFT at $O(\alpha)$

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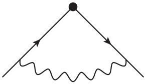
$\to$ scheme + scale + gauge dependence cancels $C^{\mathrm{RI}} \leftrightarrow Z^{\mathrm{RI}}$

Renormalization at $O(\alpha)$

$$Z_{\mathcal{O}}(\mu^2) = \sqrt{Z_\tau(\mu^2)}Z_{V,\mathrm{ud}}(\mu^2)Z_{\mathrm{tri}}(\mu^2)$$

$Z_{\mathrm{ud}}(\mu)$ = QCD factorized radiative corrections



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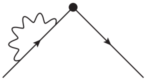
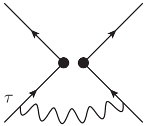


$+\cdots$

Λ_μ = quark bilinear amputated GF

four-fermion operator $\Lambda_\mu \otimes \gamma_\mu^L$



$Z_{\mathrm{tri}}^{\mathrm{RI}}(\mu)$ = triangle short-distance radiative corrections

Ward identities

A brief detour

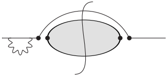
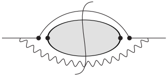
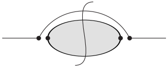
Preserving Ward Identity in QCD bilinear very important
  prevents $C(\mu) = 1 + O(\alpha_s^n) + \cdots$ [Gorbahn et al '23]
  simpler assessment of syst errors
  $C^{\mathrm{RI}}(\mu)$ calculated w/ single-trace convention in [Gorbahn et al '23]

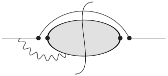
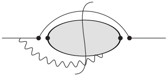
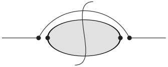
[MB, Boyle et al '26]

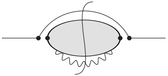
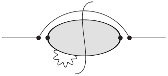
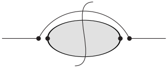
1. Extension of RI-SMOM scheme [Sturm et al '08] in pure QCD
   $P_\mu^{\mathrm{RI/SMOM}_y} = \frac{1}{2N_c p^2}(y\not{p} + (y-1)\not{p}')q_\mu$
2. Natural embedding in semileptonic operator
   double trace convention $\Lambda_\mu \otimes \gamma_\mu$
3. equivalent to [Gorbahn et al '23] via Fierz identities

A strategy for the HVP prediction from τ decays

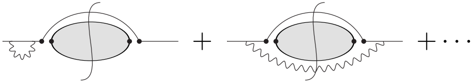
Three IR safe classes, each w/ separate UV counterterms

 $+$  $+ \cdots + (Z_\tau(\mu) - 1)$ 

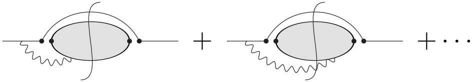
 $+$  $+ \cdots + 2(Z_{\rm tri}^{\rm RI}(\mu) - 1)$ 

 $+$  $+ \cdots + 2(Z_{V,\rm ud}(\mu) - 1)$ 

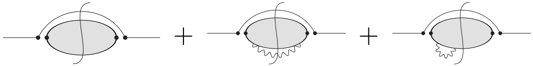
1. Start from experimental data $\left[\frac{1}{\Gamma_e}\frac{d\Gamma}{ds}\right]^{\exp}$
2. subtract initial state radiative corrections (analytic)



3. subtract non-factorizable radiative corrections



4. divide Wilson coefficient and Laplace transform
5. we are left with scale+scheme+gauge dependent

$G^{W,\exp}(t,\mu) \simeq Z^2_{V,\mathrm{ud}}$ 

$[Z_{V,\mathrm{ud}} = 1 + O(\alpha)]$

Neutral correlator $G^{\gamma}(t) \equiv -\frac{1}{3}\sum_k \int d^3\boldsymbol{x}\ \langle j_k^{\gamma}(t,\boldsymbol{x}) j_k^{\gamma}(0)\rangle$

Isospin decomposition $G^{\gamma}(t) \equiv G_{00}^{\gamma}(t) + 2G_{01}^{\gamma}(t) + G_{11}^{\gamma}(t)$

Charged correlator $G^{W}(t,\mu) = [Z_{\rm ud}^{\rm RI}(\mu)]^2 \langle j^{+}(t,\mathbf{0}) j^{-}(0)\rangle_{\rm QCD(1+O(\alpha))}$

charged vector current $j_{\mu}^{-}(x) = \frac{1}{\sqrt{2}}\bar{u}(x)\gamma_{\mu} d(x)$

$$\begin{aligned} G^{\gamma}(t) = \quad & G_{00}^{\gamma}(t) + 2G_{01}^{\gamma}(t) && && \text{Lattice QCD+QED} \\ & + G_{11}^{\gamma}(t) - G^{W}(t,\mu) && = \delta G_{11}(t,\mu) && \text{Lattice QCD+QED} \\ & + G^{W}(t,\mu) - G^{W,\rm exp}(t,\mu) && && \text{Exp. data or theory} \\ & + G^{W,\rm exp}(t,\mu) && && \text{Exp. data and theory} \end{aligned}$$

Calculate a_{μ}, windows etc... of any of the above

e.g. $\int dt\, w(t,m_{\mu})\, G^{\gamma}(t)\, \Theta^{\rm win}(t)$

Neutral correlator $G^{\gamma}(t) \equiv -\frac{1}{3}\sum_k \int d^3\boldsymbol{x}\ \langle j_k^{\gamma}(t,\boldsymbol{x}) j_k^{\gamma}(0)\rangle$

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charged vector current $j_{\mu}^{-}(x) = \frac{1}{\sqrt{2}}\bar{u}(x)\gamma_{\mu}d(x)$

$G^{\gamma}(t) =$	$G_{00}^{\gamma}(t) + 2G_{01}^{\gamma}(t)$		Lattice QCD+QED
	$+\,G_{11}^{\gamma}(t) - G^{W}(t,\mu)$	$= \delta G_{11}(t,\mu)$	Lattice QCD+QED
	$+\,G^{W}(t,\mu) - G^{W,\mathrm{exp}}(t,\mu)$		Exp. data or theory
	$+\ G^{W,\mathrm{exp}}(t,\mu)$		Exp. data and theory

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$G^\gamma(t) = \quad G_{00}^\gamma(t) + 2G_{01}^\gamma(t)$		Lattice QCD+QED
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$+\,G^{W,\exp}(t,\mu)$		Exp. data and theory

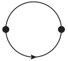
Calculate a_μ, windows etc... of any of the above

e.g. $\int dt\, w(t,m_\mu)\, G^\gamma(t)\, \Theta^{\rm win}(t)$

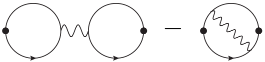
$$\delta G_{11}(t,\mu) = G_{11}^{\gamma}(t) - G^{W}(t,\mu) = O(\alpha)$$

calculable in Euclidean space from Lattice QCD
fully inclusive both in channels and energy
scheme + scale + gauge dependent
large cancellations, all diagrams cancel out except for V and F

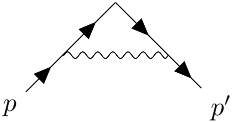
$$\delta G_{11}(t,\mu) = Z_V^{\mathrm{QCD}}(Z_{\gamma,1} - Z_{V,\mathrm{ud}}(\mu^2))(c)(t)$$

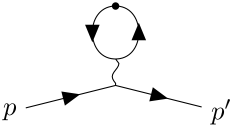
$Z(\mu)\times$ 

$$+e^2 \frac{(Q_{\mathrm{u}}-Q_{\mathrm{d}})^2}{4}(Z_V^{\mathrm{QCD}})^4((F)-(V))(t)$$



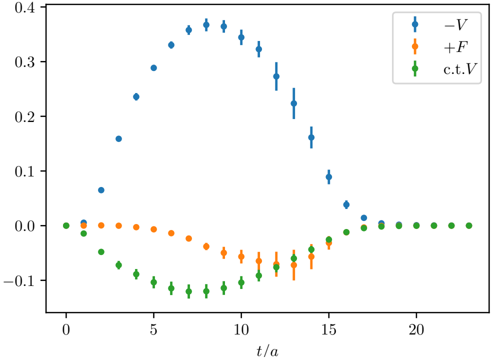
Renormalized by (amputated)





Preliminary

[Counter-term for V] + [$F - V$] (counter-term for F ongoing)



blinded analysis

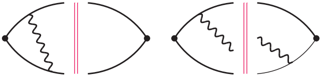
$m_\pi \simeq 280$ MeV,
$a^{-1} = 1.73$ GeV

fourth moment t^4 with smooth cutoff at ≈ 1.6 fm

counter-term at $\mu = m_\tau$ (twisted BC)

G^W calculated from Lattice QCD+QED differs from $G^{W,\mathrm{exp}}$

syst effect $G^W - G^{W,\mathrm{exp}} = O(\alpha^0) + O(\alpha^1)$

$\delta G_{11}(t,\mu) \to$ 

lattice fully inclusive vs experiment $\theta(m_\tau - \sqrt{s})$

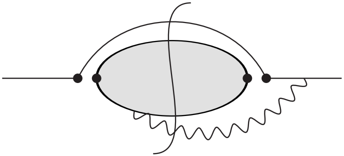
- vector spectrum $\sqrt{s} > m_\tau$ at $O(\alpha^0)$
- vector spectrum $\sqrt{s} > m_\tau$ also has $O(\alpha)$
- suppressed in intermediate and long-dist windows

mixing between axial and vector states

- $A \to V + \gamma$ inside exp.data, known effect
  - estimated small in EFT [Cirigliano et al '01][Colangelo et al' 25]
- $V \to A + \gamma$ inside lattice data, must be taken in account

1. ampl $\mathcal{M}_f^{\text{real,had}}$ from time-ordered prod
   $\int_x e^{ikx} \langle V | T\{\mathcal{J}_\alpha^\gamma(x)\mathcal{J}_\mu^{L,-}(0)\}|0\rangle$

2. interference $\mathcal{M}_f^{\text{real,had}} \left[\mathcal{M}_f^{\text{real},\tau}\right]^\dagger$
   $\int_x e^{ikx} \langle 0|\mathcal{J}_\mu^+(0)|V\rangle\langle V|T\{\mathcal{J}_\alpha^\gamma(x)\mathcal{J}_\nu^{L,-}(0)\}|0\rangle$

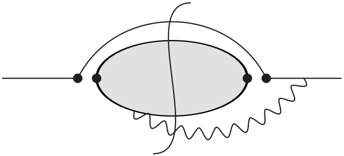


3. hadronic phase-space + sum over all vector states
   $$\int_x e^{ikx} \langle 0|\mathcal{J}_\mu^+(0)\,\delta(\hat{H}-\sqrt{s})\,T\{\widetilde{\mathcal{J}}_\alpha^\gamma(x)\mathcal{J}_\nu^{L,-}(0)\}|0\rangle$$

4. calculable from Euclidean correlator $\langle \widetilde{j}_\mu^+(t,\mathbf{0})\,\widetilde{j}_\alpha^\gamma(\tau,\boldsymbol{k})\,j_\nu^{L,-}(0)\rangle$?
   there is δ-function $\rightarrow$ first inverse problem?
   there is T product $\rightarrow$ second inverse problem?

Notation: $\mathcal{J}^{L,-}$ charged weak, $\mathcal{J}^+$ charged vector, $\mathcal{J}$: Minkowski
$\widetilde{j}$: Fourier trafo

1. ampl $\mathcal{M}_f^{\text{real,had}}$ from time-ordered prod
   $\int_x e^{ikx}\,\langle V|T\{\mathcal{J}_\alpha^\gamma(x)\mathcal{J}_\mu^{L,-}(0)\}|0\rangle$



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   there is δ-function $\to$ first inverse problem?
   there is T product $\to$ second inverse problem?

Notation: $\mathcal{J}^{L,-}$ charged weak, $\mathcal{J}^+$ charged vector, $\mathcal{J}$: Minkowski
$\widetilde{j}$: Fourier trafo

$$\int_x e^{ikx}\, \langle 0|\mathcal{J}^{+}_{\mu}(0)\, \delta(\hat{H}-\sqrt{s})\, T\{\widetilde{\mathcal{J}}^{\gamma}_{\alpha}(x)\mathcal{J}^{L,-}_{\nu}(0)\}|0\rangle$$

integral $x_0 < 0$
$\rightarrow$ directly calculable from Lattice QCD

integral ds
$\rightarrow$ calculable via exact reconstruction tail

integral over $x_0 > 0$ and ds

$\rightarrow \boldsymbol{k} = 0$ calculable via extract reconstruction tail
$\rightarrow |\boldsymbol{k}| > 0$ double inverse problem

Strategy is now complete [Bruno et al '26]

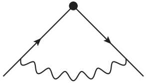
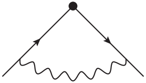
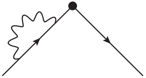
- short-distance in RI-SMOM scheme
  - first results renormalization of charged current
- initial-state corrections analytic
- final-state radiation in Euclidean space
- first results for spectrum above m_τ [J Parrino Lattice 26]
- non-factorizable still pose severe inverse problem
  - sinergy with dispersive approach likely in short run

Thanks for your attention

Renormalization of neutral current

1. same diagrams but different charge prefactors
2. gauge invariance relates wave-function w/ vertex corrections

Gauge inv. of neutral current simplifies renormalization of charged current

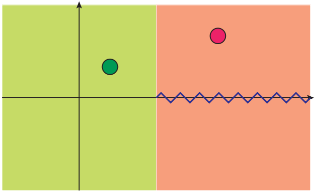
Neutral		-Charged		$=$ Total
$\frac{1}{2}Q_{\mathrm{u}}^2 + \frac{1}{2}Q_{\mathrm{d}}^2$		$-Q_{\mathrm{u}}Q_{\mathrm{d}}$		$\to (Q_{\mathrm{u}} - Q_{\mathrm{d}})^2$
$Q_{\mathrm{u}}^2 + Q_{\mathrm{d}}^2$		$-Q_{\mathrm{u}}^2 - Q_{\mathrm{d}}^2$		$\to 0$

Inverse Laplace Transform

[MB et al '26]

$\kappa(\omega) = \dfrac{1}{\omega - \bar{\omega} + i\sigma} \to f(t)$ such that $\rho_\kappa = \int dt\, f(t)\, C(t)$?

$$\rho_\kappa = \lim_{T\to\infty} \int d\omega \int_0^T dt\, e^{-(\omega-\bar{\omega}-i\sigma)t}\, \rho(\omega) = \lim_{T\to\infty} \int_0^T dt\, e^{(\bar{\omega}+i\sigma)t}\, C(t)$$



Limit $T \to \infty$ exists if $\bar{\omega}$ below support of ρ

interplay between pole $\bar{\omega} + i\sigma$ vs branch cut

ρ has branch cut starting at multi-particle thresholds $E_{\rm thr}$

if $\bar{\omega} \leq E_{\rm thr}$ $\quad \to \forall t\ \exists M > 0 \mid f(t)C(t) < e^{-Mt}$

e.g. HVP contribution to $(g-2)_\mu$ [Blum '02][Bernecker-Meyer '11]

if $\bar{\omega} > E_{\rm thr}$ $\quad \to$ inverse problem, no "direct" analytic continuation