Spectral density reconstruction from lattice QCD correlators: an approach via numerical anti-Laplace transform
(Numerical treatment)

Francesco Di Renzo (University of Parma and INFN)

Lattice 2026

University of Maryland, College Park, 29/07/2026

In collaboration with P. Dimopoulos, M. Aliberti, D. Gavriel and A. Vidal (Parma)

From Shoji's (beautiful) talk …

Details are different, but the broad consensus would be

1. Reconstruction of the local spectrum is indeed an ill-posed problem = no breakthrough (so far)
2. Smeared spectrum can be reconstructed with controlled errors (details matter)

Not a magic…

From Shoji's (beautiful) talk …

Details are different, but the broad consensus would be

1. Reconstruction of the local spectrum is indeed an ill-posed problem = no breakthrough (so far)
2. Smeared spectrum can be reconstructed with controlled errors (details matter)

Not a magic…

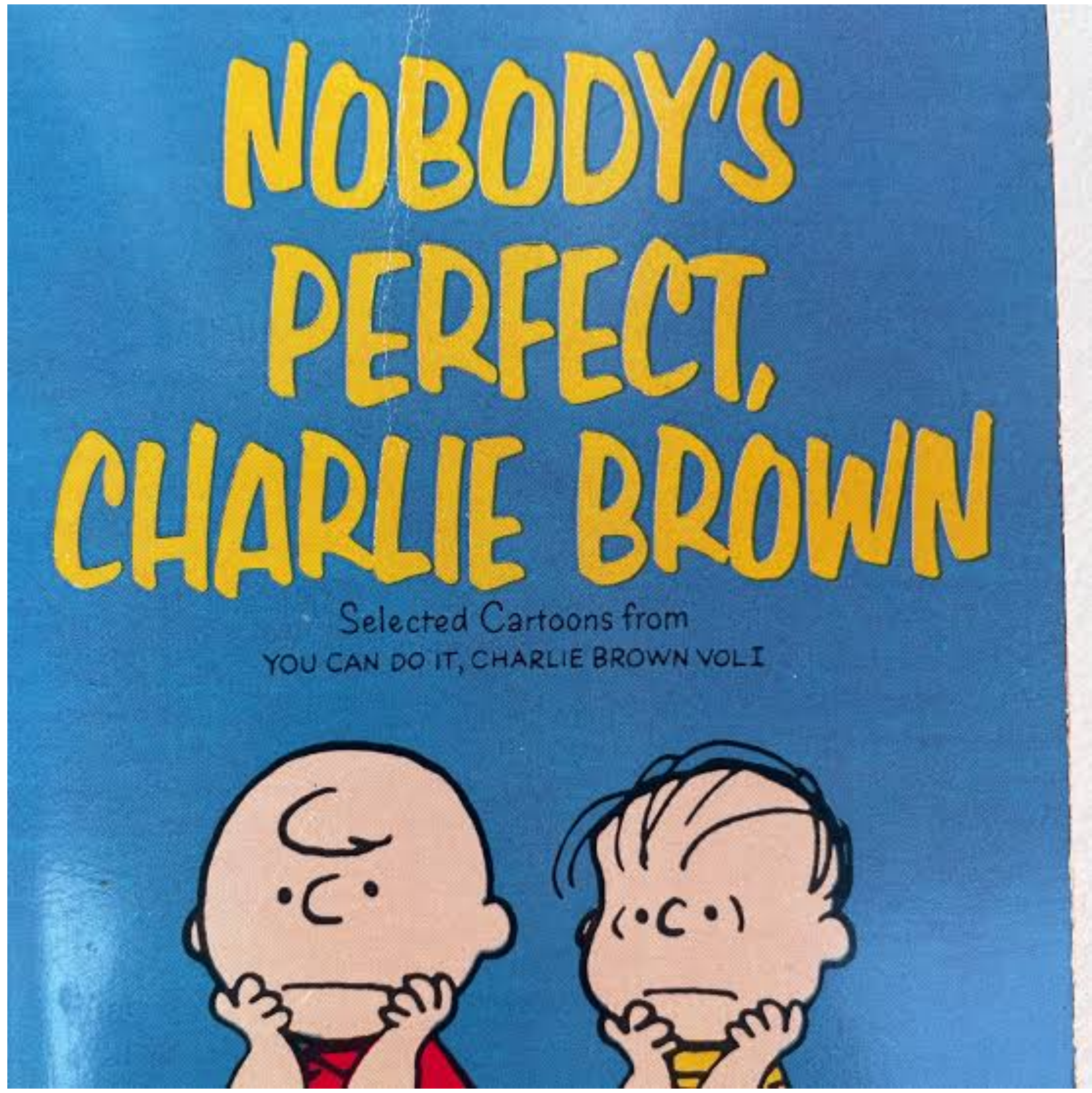
… and this is what we would like to question …

From Shoji's (beautiful) talk ...

Details are different, but the broad consensus would be

1. Reconstruction of the local spectrum is indeed an ill-posed problem = no breakthrough (so far)
2. Smeared spectrum can be reconstructed with controlled errors (details matter)

Not a magic...

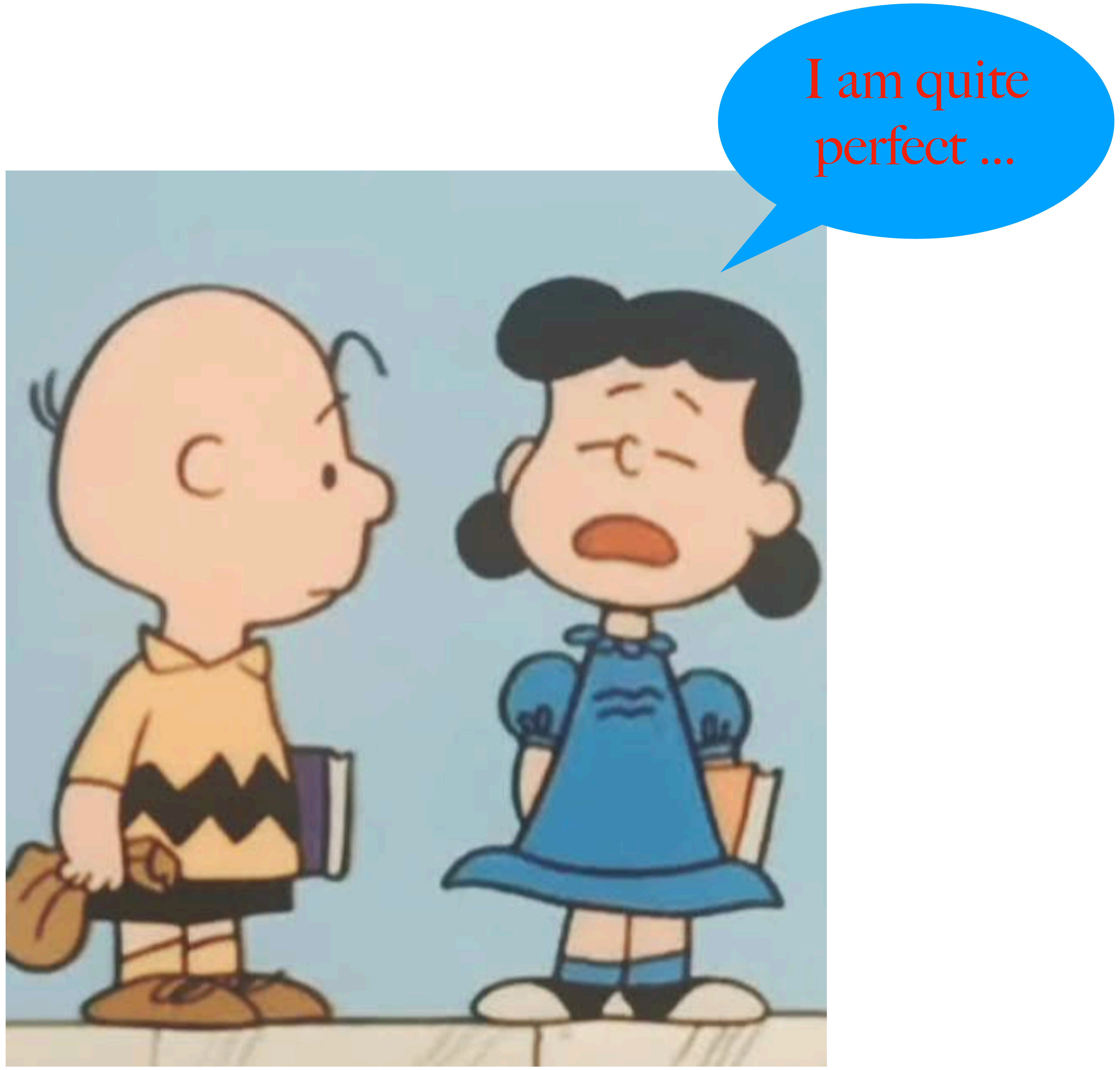


From Shoji's (beautiful) talk …

Details are different, but the broad consensus would be

1. Reconstruction of the local spectrum is indeed an ill-posed problem = no breakthrough (so far)
2. Smeared spectrum can be reconstructed with controlled errors (details matter)

Not a magic…



From Shoji's (beautiful) talk …

> Details are different, but the broad consensus would be
>
> 1. Reconstruction of the local spectrum is indeed an ill-posed problem = no breakthrough (so far)
> 2. Smeared spectrum can be reconstructed with controlled errors (details matter)
>
> Not a magic…

… and this is what we would like to question …

Agenda

- The problem
- How it all began
- Anti Laplace Transforms via quadratures
- Regularisation … that is …
- Regulated delta functions
- Fitting spectral functions
- Conclusions/Outlook

so …

What is the problem?

A lattice correlator is the Laplace Transform of a spectral density

$$C(t_i) = \int_0^\infty \mathrm{e}^{-Et_i}\rho(E)\,dE$$

so …

What is the problem?

And you want to solve the inverse problem …
that is to say …

$$C(t_i) = \int_0^\infty \mathrm{e}^{-Et_i}\rho(E)\,dE$$

so …

What is the problem?

And you want to solve the inverse problem …
that is to say …

$$C(t_i) = \int_0^\infty \mathrm{e}^{-Et_i} \rho(E)\, dE$$

… you know this …

so ...

What is the problem?

And you want to solve the inverse problem ...
that is to say ...

$$C(t_i) = \int_0^\infty \mathrm{e}^{-Et_i}\,\rho(E)\,dE$$

... you know this ...

... and you want to get this ...

so …

What is the problem?

And you want to solve the inverse problem …
that is to say …

$$C(t_i) = \int_0^\infty \mathrm{e}^{-Et_i}\,\rho(E)\,dE$$

… you know this …

… and you want to get this …

YOU CAN'T BE SERIOUS! … this is the prototype of an ill-posed problem!

so ...

What is the problem?

... that is to say ...

$$C(t_i) = \int_0^\infty \mathrm{e}^{-Et_i}\rho(E)\,dE$$

... you know this ...

... and you want to get this ...

YOU CAN'T BE SERIOUS! ... this is the prototype of an ill-posed problem! ... and on top of this ...

$$C(t) = \sum_k c_k\,\mathrm{e}^{-E_k t} \quad \to \quad \rho(E) = \sum_k c_k\,\delta(E - E_k)$$

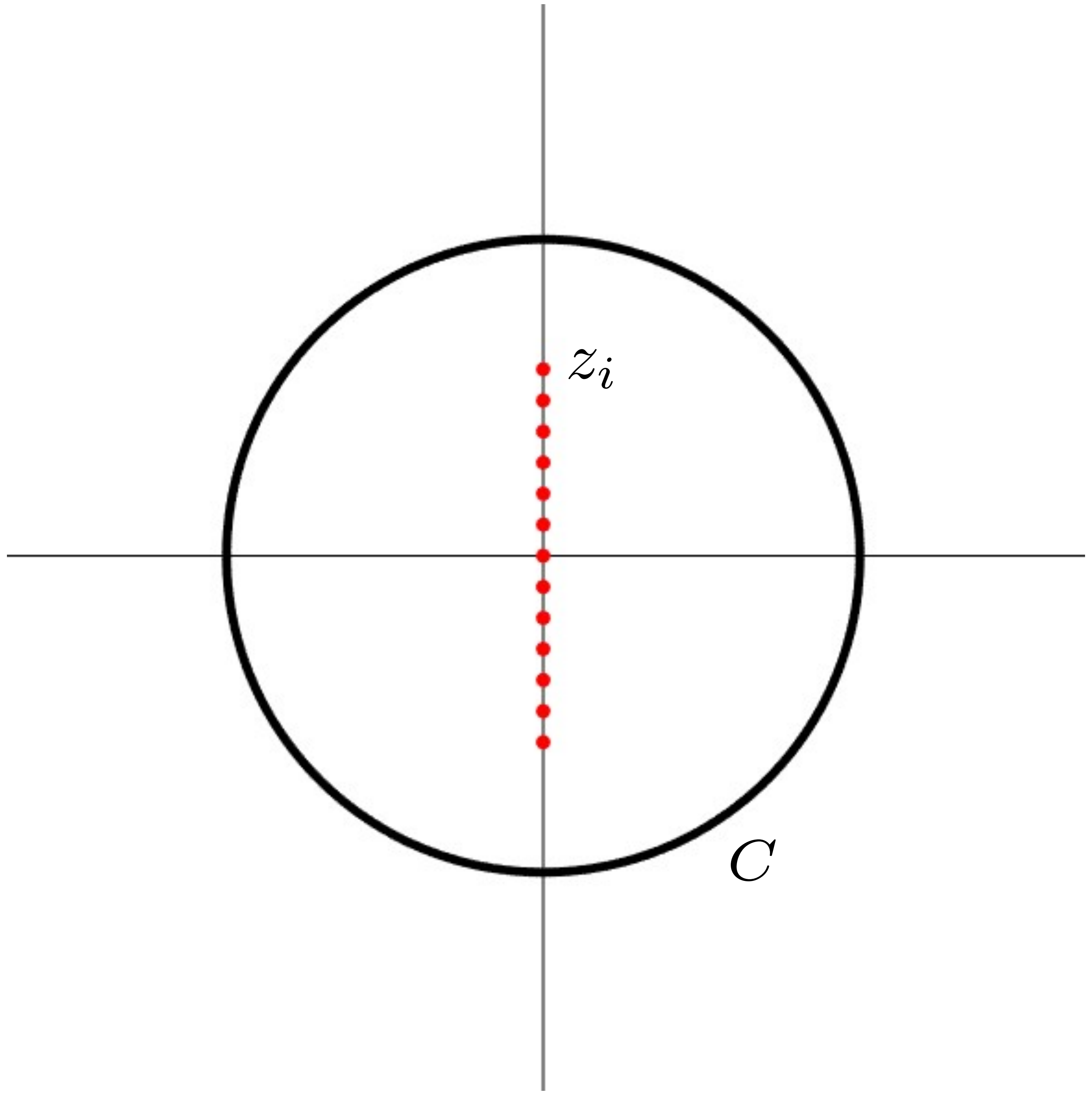
You are looking for an anti Laplace Transform which is a singular distribution!

How it all began…

i.e. another integral inverse problem

How it all began...

i.e. another integral inverse problem



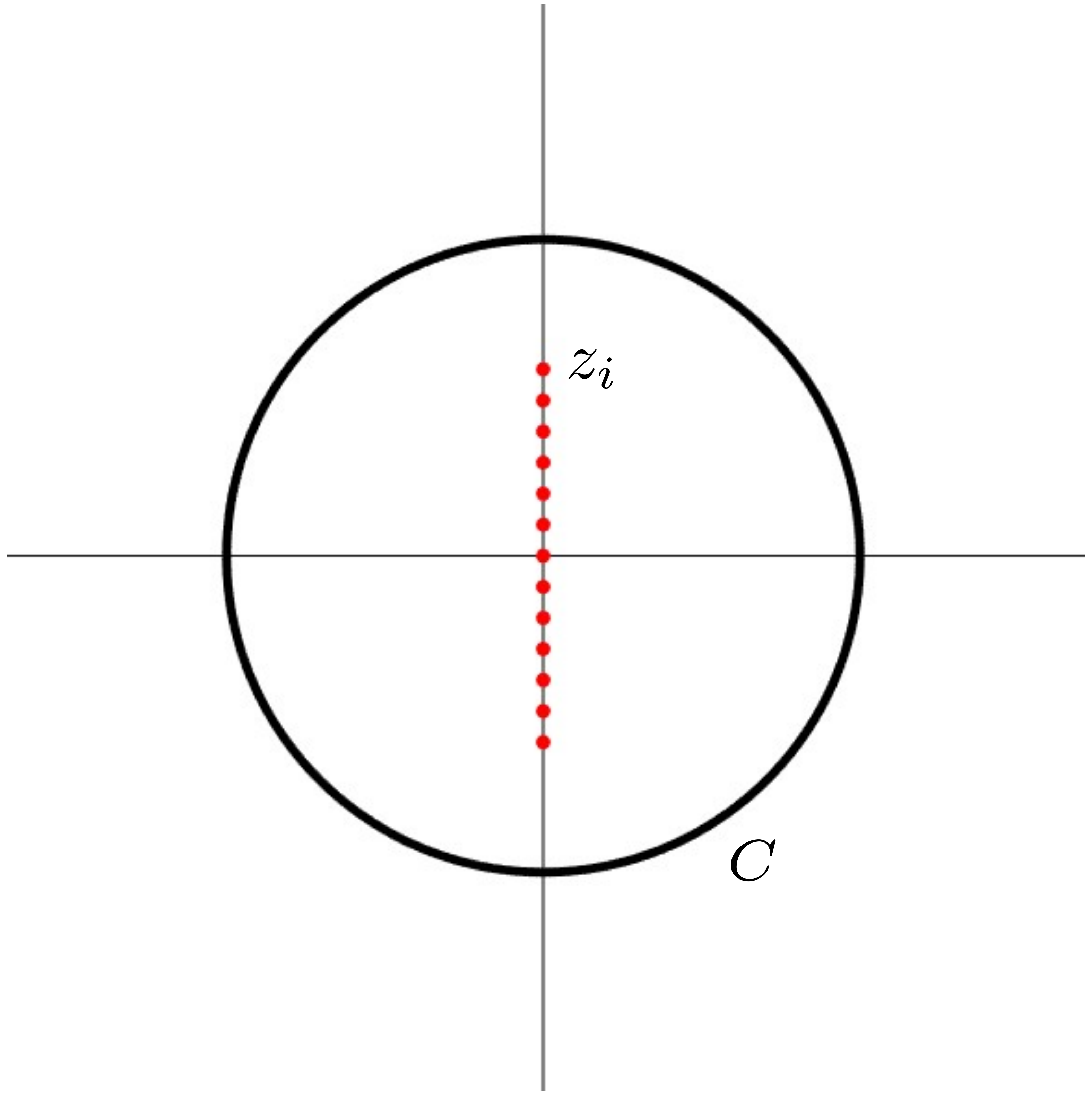
CAUCHY FORMULA

$$f(z_i) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_i} dz$$

If you know the function on the contour, you can compute it at any point inside

How it all began…

i.e. another integral inverse problem



CAUCHY FORMULA

$$f(z_i) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_i} dz$$

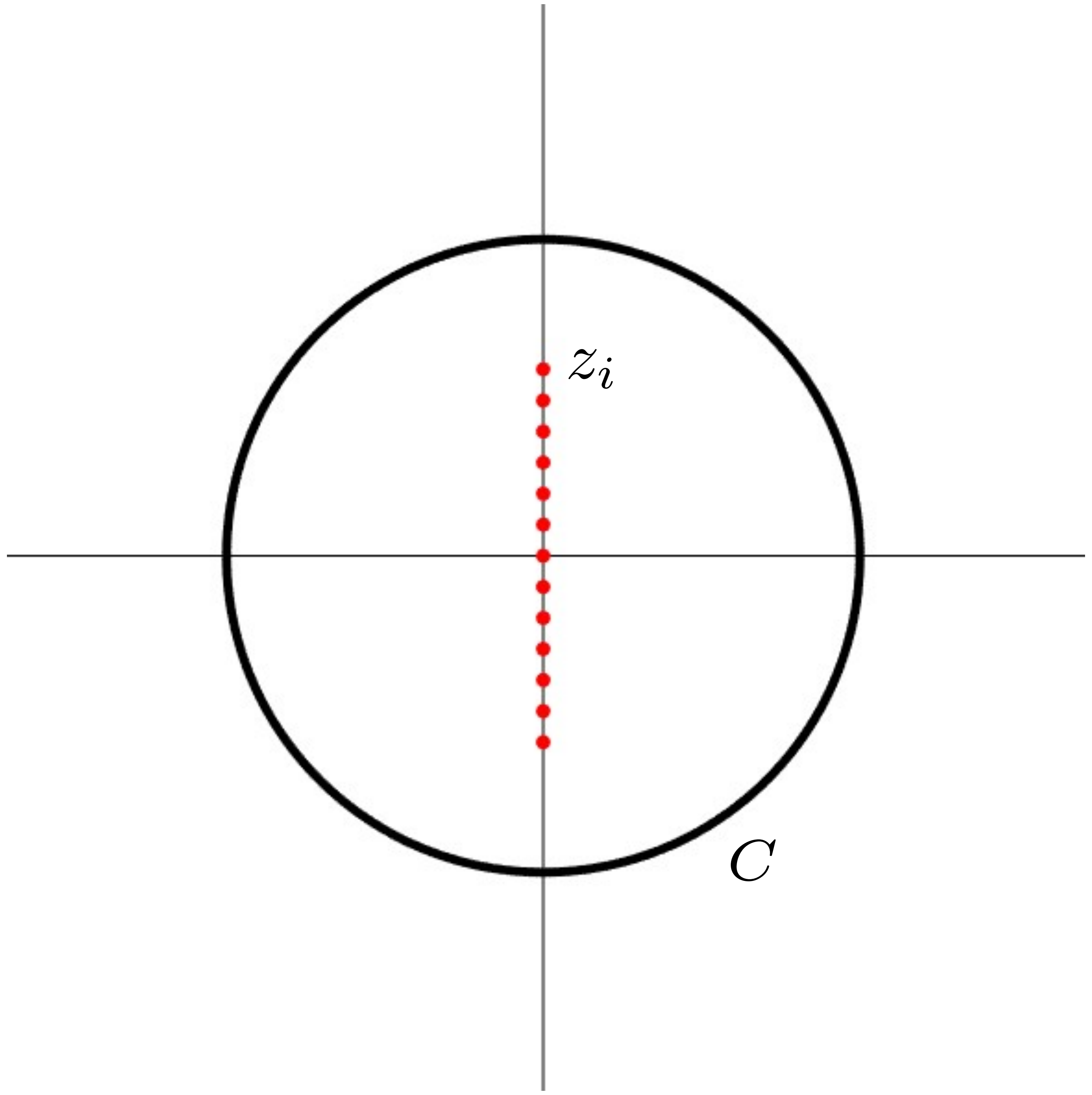
If you know the function on the contour, you can compute it at any point inside

Solving an inverse problem means going the other way around!

… but then if you know the function on the contour, you can compute elsewhere, in the direct way!

How it all began...

i.e. another integral inverse problem

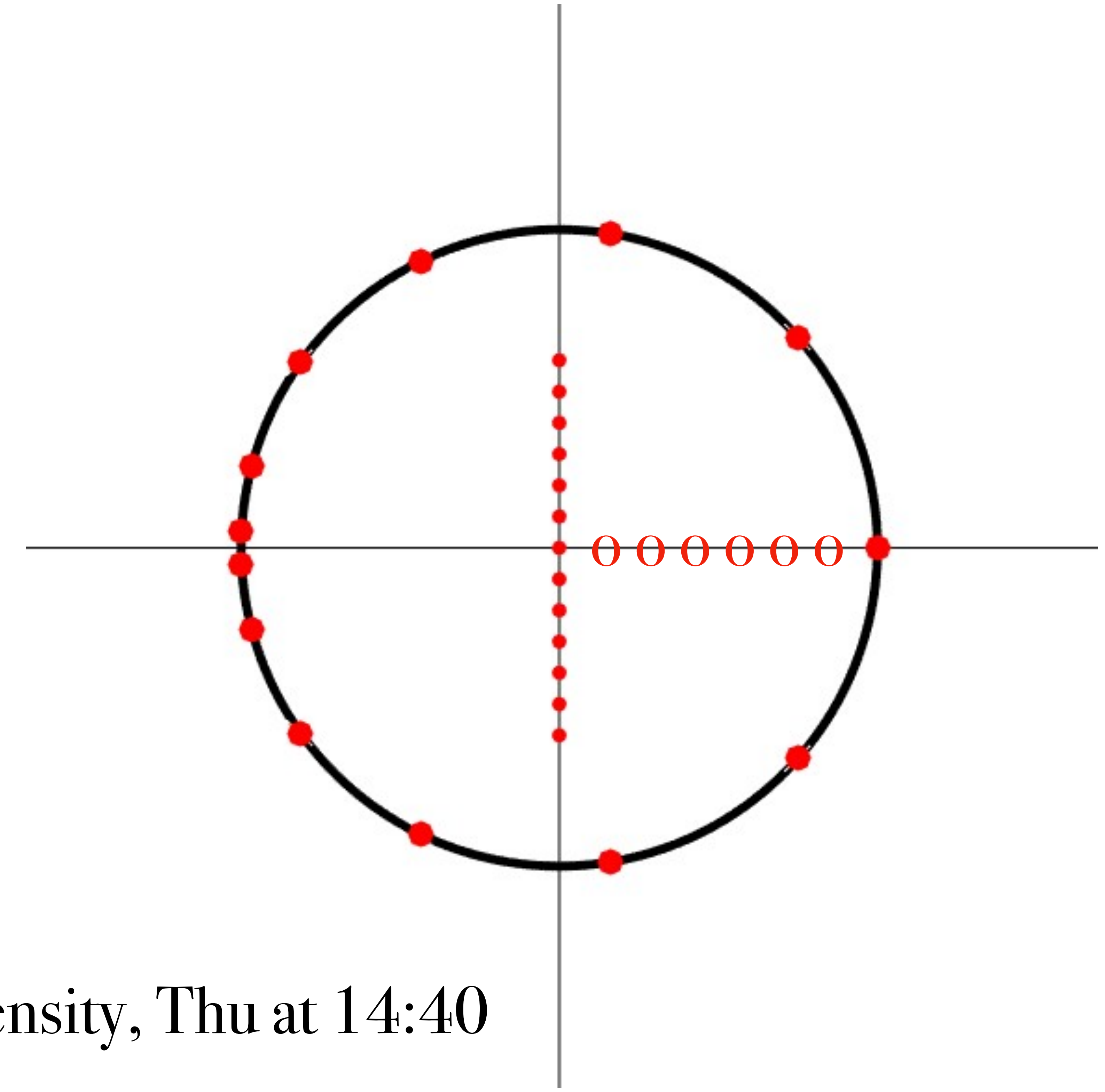


CAUCHY FORMULA

$$f(z_i) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_i} dz$$

If you know the function on the contour, you can compute it at any point inside

Solving an inverse problem means going the other way around!



... but then if you know the function on the contour, you can compute elsewhere, in the direct way!

WHAT YOU WANT TO DO IS TO GO NUMERIC:
given a quadrature scheme, you input n points and solve for n points...

M. Aliberti, Finite Temperature and Density, Thu at 14:40

SO …

What are we going to do?

so …

What are we going to do?

Remember:

You know this …

$$f(s_i) = \int_0^\infty \mathrm{e}^{-ts_i}\, F(t)\, dt$$

You want this …

Remember:

You know this ...

$$f(s_i) = \int_0^\infty \mathrm{e}^{-ts_i}\, F(t)\, dt$$

You want this ...

so ...

What are we going to do?

We slightly rephrase the problem ...

$$f(s_i) = \int_0^\infty \mathrm{e}^{-t}\, \mathrm{e}^{-t(s_i-1)}\, F(t)\, dt$$

so …

What are we going to do?

Remember:

You know this …

$$f(s_i) = \int_0^\infty \mathrm{e}^{-ts_i}\, F(t)\, dt$$

You want this …

We slightly rephrase the problem …

$$f(s_i) = \int_0^\infty \mathrm{e}^{-t}\, \mathrm{e}^{-t(s_i-1)}\, F(t)\, dt$$

… and we can play the numerical game …

$$f(s_i) = \int_0^\infty \mathrm{e}^{-t}\, \mathrm{e}^{-t(s_i-1)}\, F(t)\, dt \sim \sum_j w_j\, \mathrm{e}^{-t_j(s_i-1)}\, F(t_j)$$

You will recognise the Laguerre quadrature scheme

so …

What are we going to do?

Remember:

You know this …

$$f(s_i) = \int_0^\infty \mathrm{e}^{-ts_i}\, F(t)\, dt$$

You want this …

We slightly rephrase the problem …

$$f(s_i) = \int_0^\infty \mathrm{e}^{-t}\, \mathrm{e}^{-t(s_i-1)}\, F(t)\, dt$$

… and we can play the numerical game … $$f(s_i) = \int_0^\infty \mathrm{e}^{-t}\, \mathrm{e}^{-t(s_i-1)}\, F(t)\, dt \sim \sum_j w_j\, \mathrm{e}^{-t_j(s_i-1)}\, F(t_j)$$

You will recognise the Laguerre quadrature scheme

But $f(s_i) = \sum_j \mathrm{e}^{-t_j(s_i-1)}\, F(t_j)$ means $b_i = \sum_j A_{ij}\, x_j$ … and we are done!… a linear system

A fair game: you input n pieces of information and you get back n

... reparametrization!

... In the end ... DOES IT WORK?!

$$f(s_i) = \int_0^\infty \mathrm{e}^{-t s_i}\, F(t)\, dt = t_0 \int_0^\infty \mathrm{e}^{-t\, t_0\, s_i}\, F(t\, t_0)\, dt = t_0 \int_0^\infty \mathrm{e}^{-t}\, \mathrm{e}^{-t(t_0\, s_i - 1)}\, F(t\, t_0)\, dt$$

... we can get quite a number of values! ... actually all those for which the machinery works ...

Not only we can possibly get quite a number of values ... we get a CONSISTENCY ARGUMENT!

$$f(s_i) = t_0 \sum_j w_j\, \mathrm{e}^{-t_j (t_0\, s_i - 1)}\, F(t_0\, t_j)$$

... reparametrization!

... In the end ... DOES IT WORK?!

$$f(s_i) = \int_0^\infty \mathrm{e}^{-t s_i}\, F(t)\, dt = t_0 \int_0^\infty \mathrm{e}^{-t\, t_0\, s_i}\, F(t\, t_0)\, dt = t_0 \int_0^\infty \mathrm{e}^{-t}\, \mathrm{e}^{-t(t_0\, s_i - 1)}\, F(t\, t_0)\, dt$$

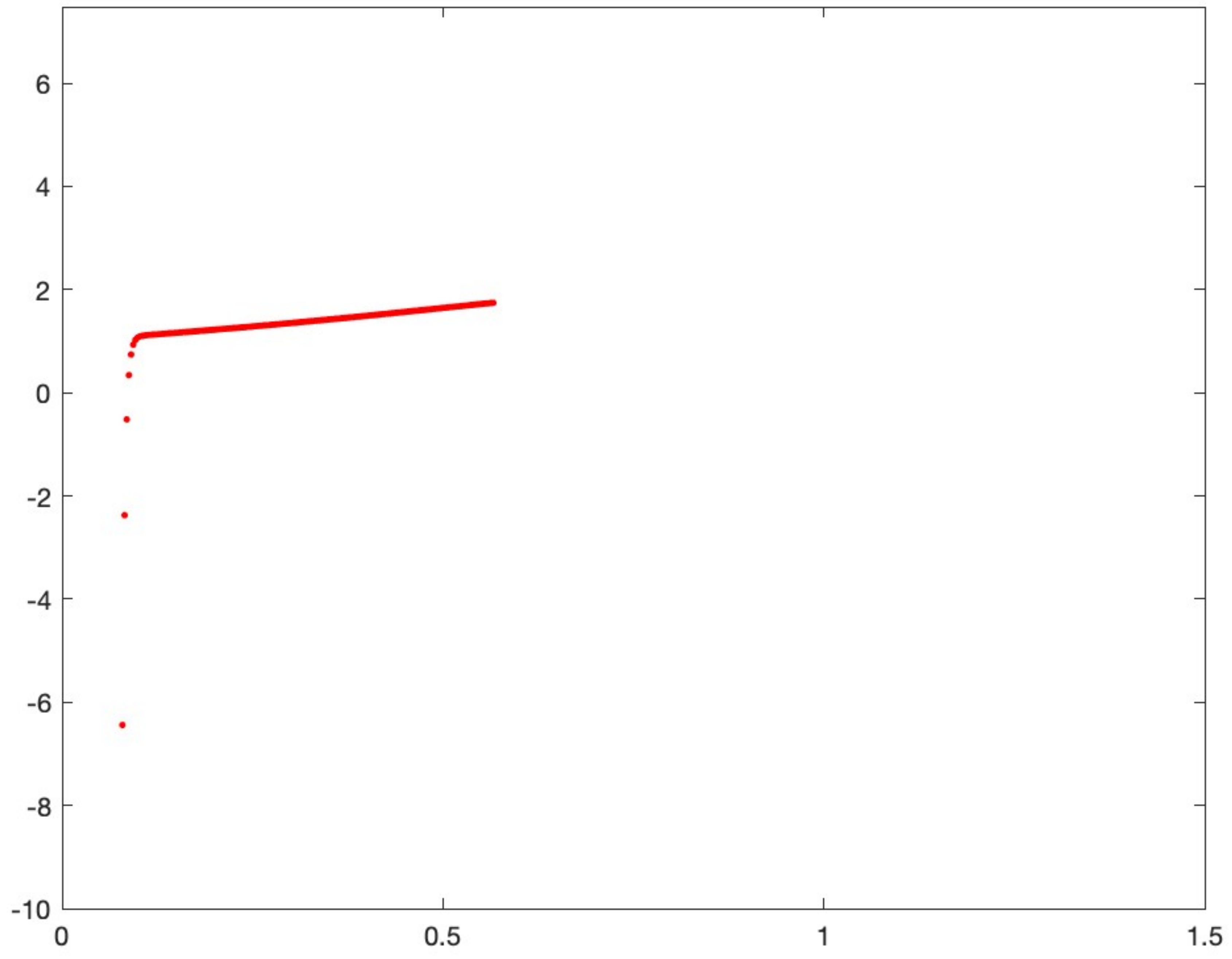
... we can get quite a number of values! ... actually all those for which the machinery works ...

Not only we can possibly get quite a number of values ... we get a CONSISTENCY ARGUMENT!

$$f(s_i) = t_0 \sum_j w_j\, \mathrm{e}^{-t_j (t_0\, s_i - 1)}\, F(t_0\, t_j)$$

$$f(s) = \frac{1}{s-1} \quad \rightarrow \quad F(t) = e^t$$

$$j = 2$$



Some CRAZY points

... and ...

a number falling on

a SMOOTH line

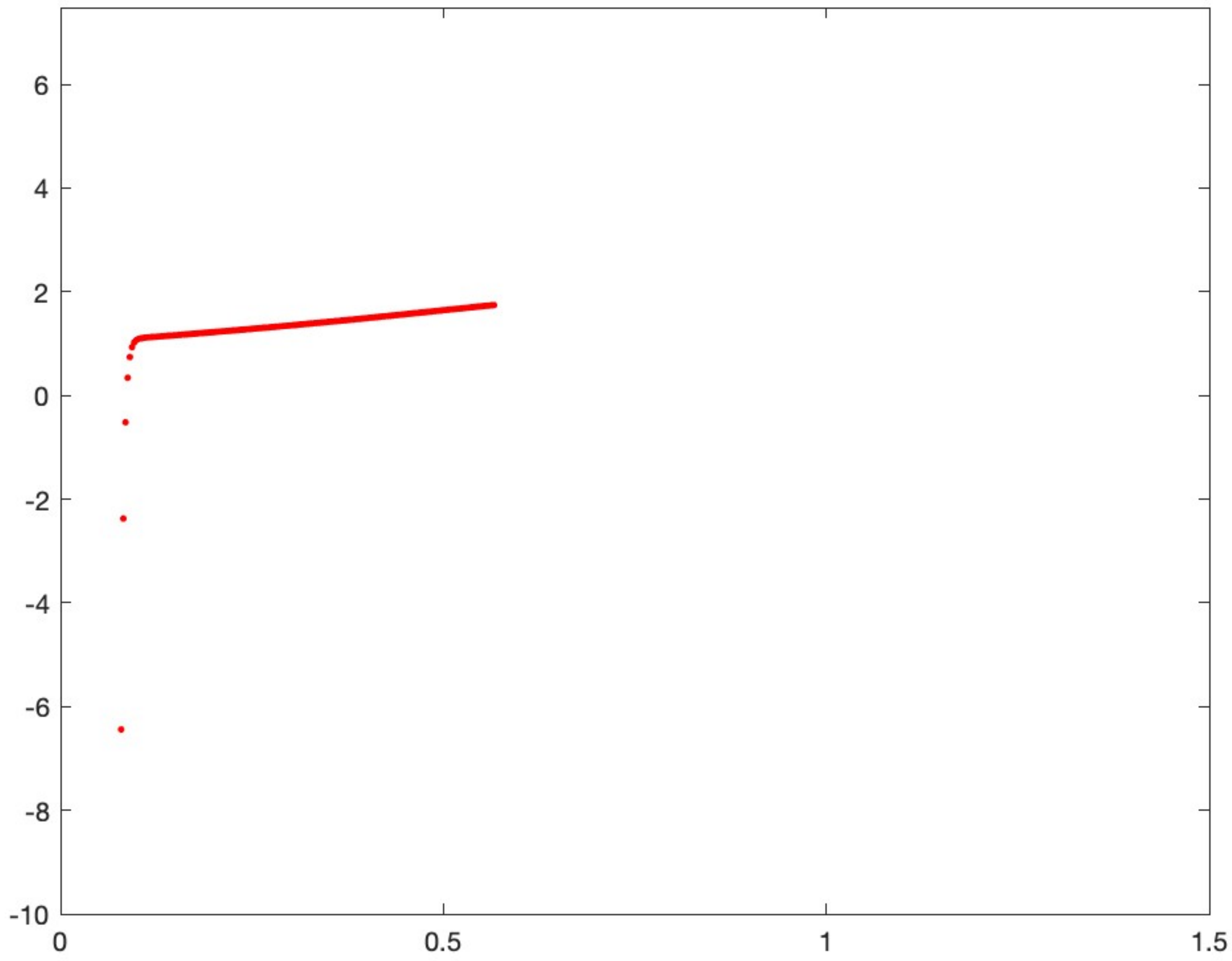
… In the end … DOES IT WORK?!

We said we get a CONSISTENCY ARGUMENT …

$$f(s_i) = t_0 \sum_j w_j\, \mathrm{e}^{-t_j (t_0\, s_i - 1)}\, F(t_0\, t_j)$$

$$f(s) = \frac{1}{s-1} \quad \rightarrow \quad F(t) = e^t$$

$$j = 2$$

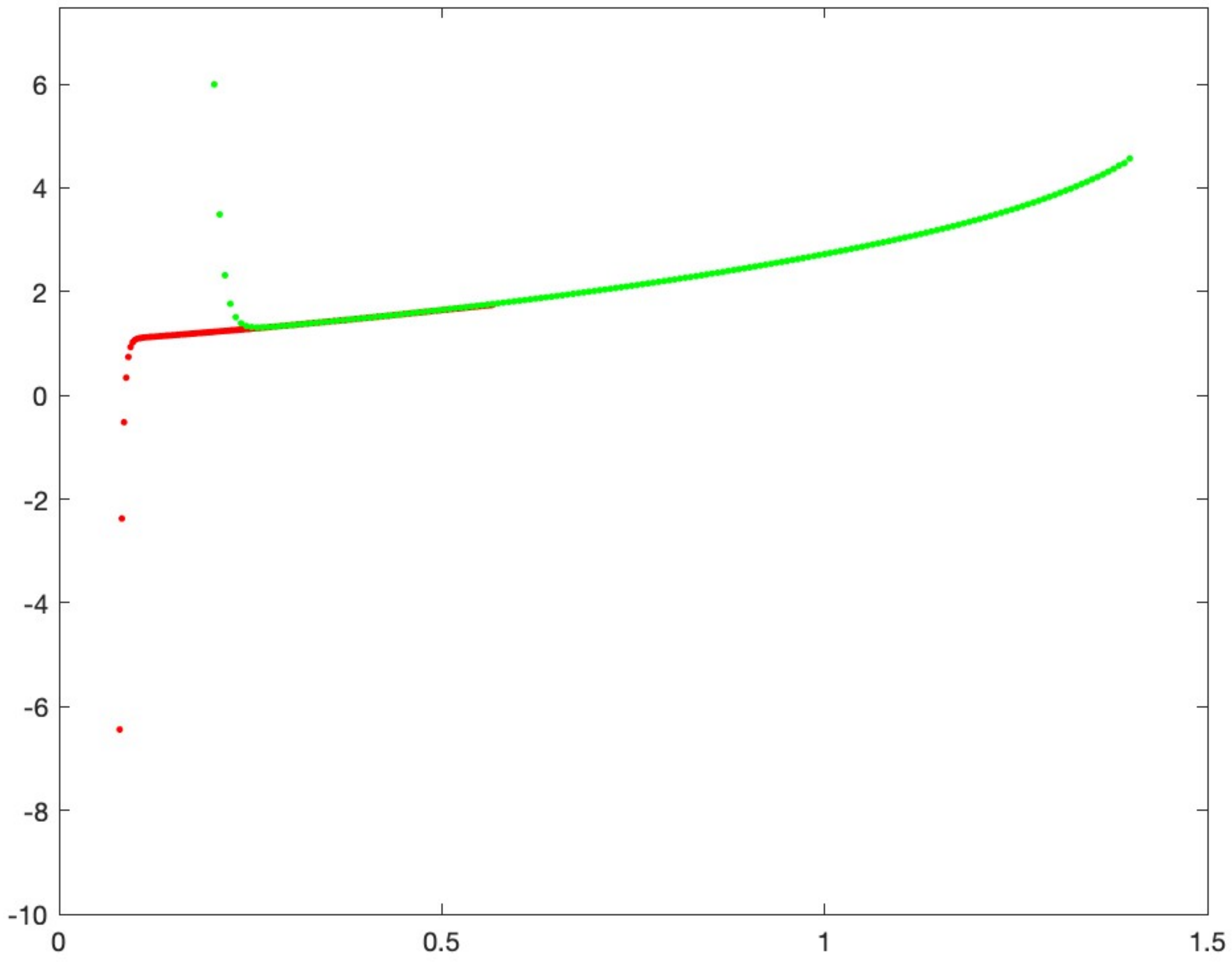


We said we get a CONSISTENCY ARGUMENT ...

$$f(s_i) = t_0 \sum_j w_j \, \mathrm{e}^{-t_j (t_0\, s_i - 1)} \, F(t_0\, t_j)$$

$$f(s) = \frac{1}{s-1} \quad \rightarrow \quad F(t) = e^t$$

$$j = 2, 3$$

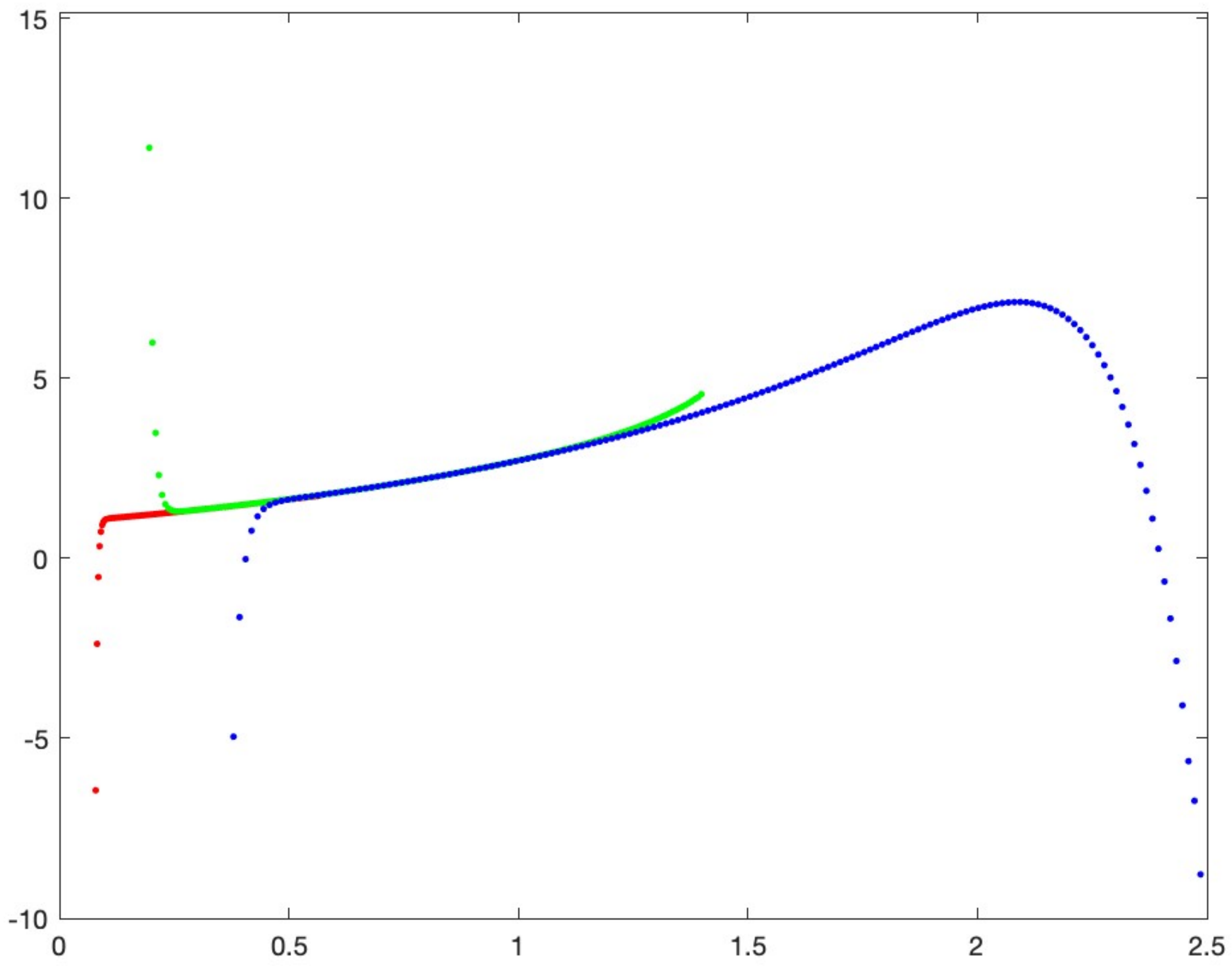


We said we get a CONSISTENCY ARGUMENT ...

$$f(s_i) = t_0 \sum_j w_j \, \mathrm{e}^{-t_j(t_0\, s_i - 1)} \, F(t_0\, t_j)$$

$$f(s) = \frac{1}{s-1} \quad \rightarrow \quad F(t) = e^t$$

$$j = 2, 3, 4$$



… In the end … DOES IT WORK?!

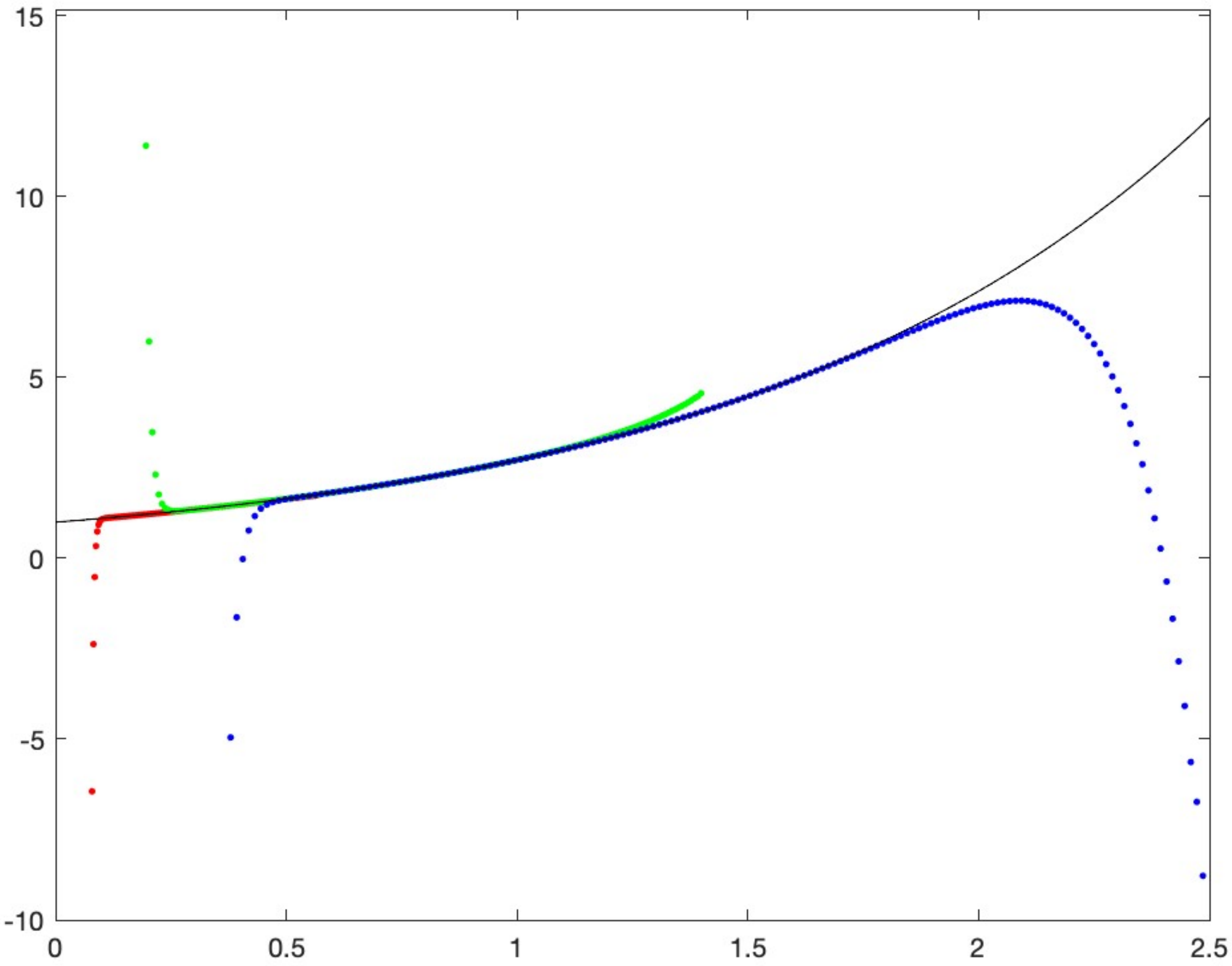
We said we get a CONSISTENCY ARGUMENT …

$$f(s_i) = t_0 \sum_j w_j \, \mathrm{e}^{-t_j (t_0 \, s_i - 1)} \, F(t_0 \, t_j)$$

$$f(s) = \frac{1}{s-1} \quad \rightarrow \quad F(t) = e^t$$

$$j = 2, 3, 4$$

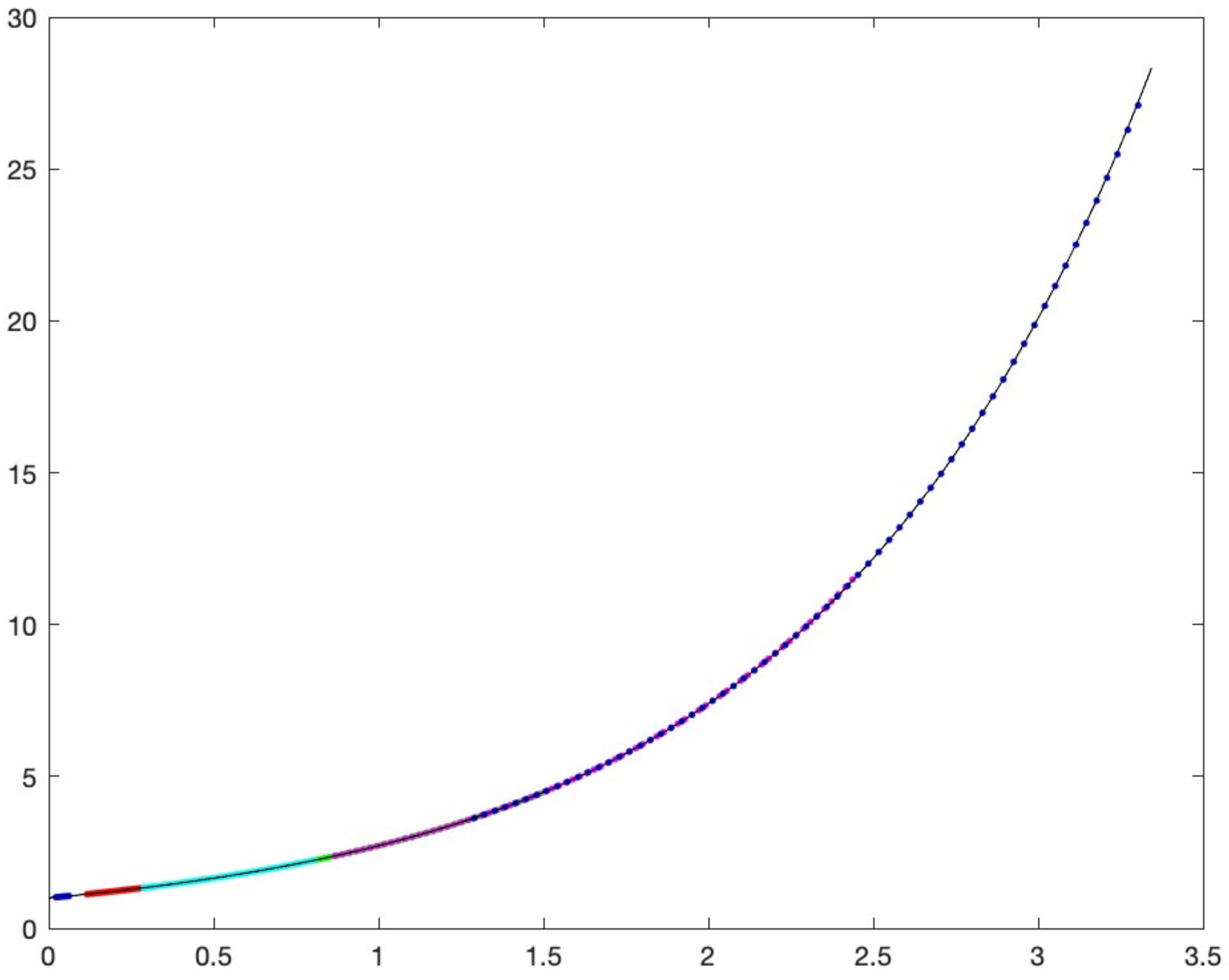
The black line is the EXPONENTIAL …!



… In the end … DOES IT WORK?!

… It is fair to say… IT WORKS!

… provided we look for smooth overlaps …!
(… of course, provided they show up …)

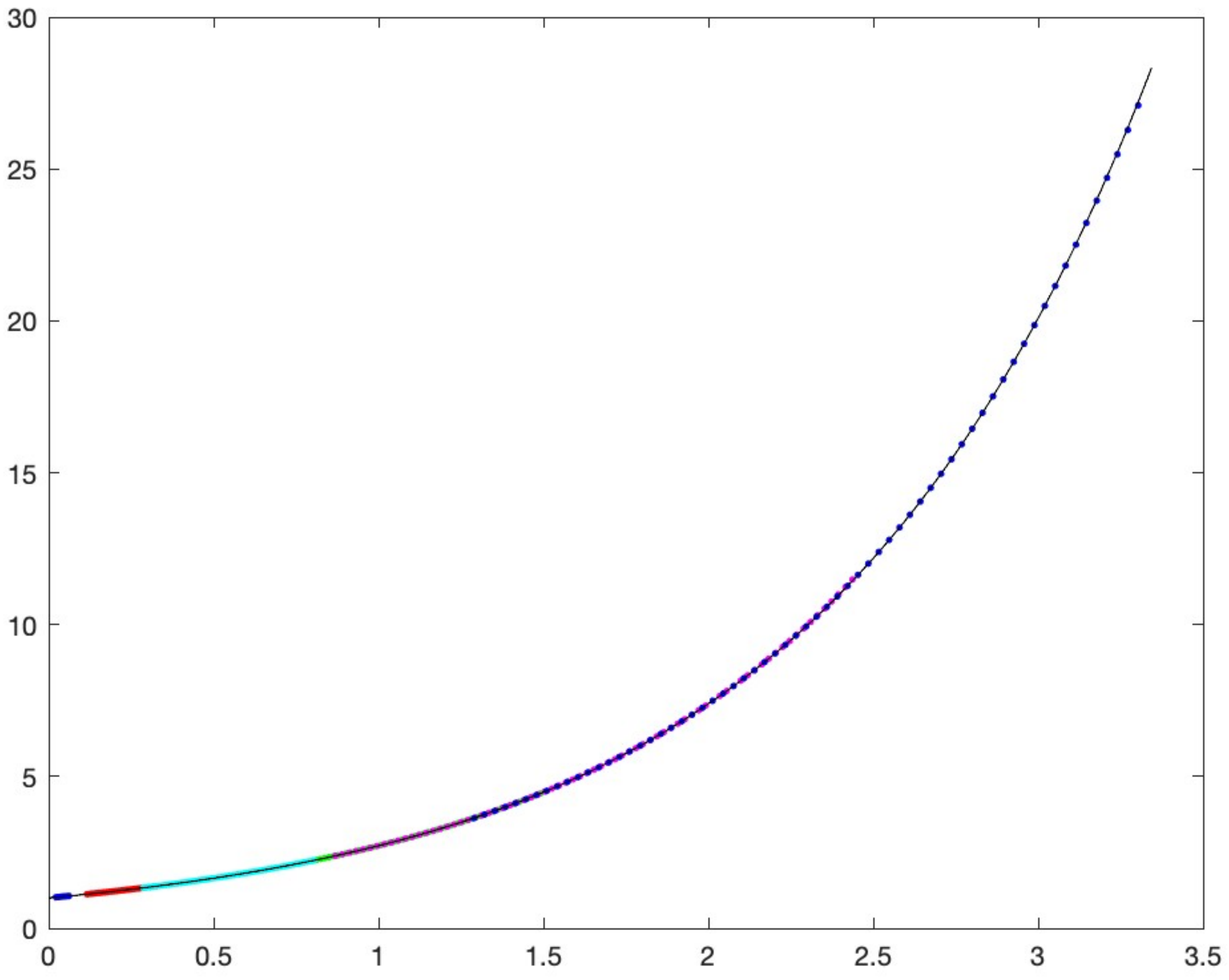


… In the end … DOES IT WORK?!

… It is fair to say… IT WORKS!

… provided we look for smooth overlaps …!
(… of course, provided they show up …)

BEWARE! This is not always the case!
There are *lucky* functions and *unlucky* ones…



… It is fair to say… IT WORKS!

… provided we look for smooth overlaps …!
(… of course, provided they show up …)

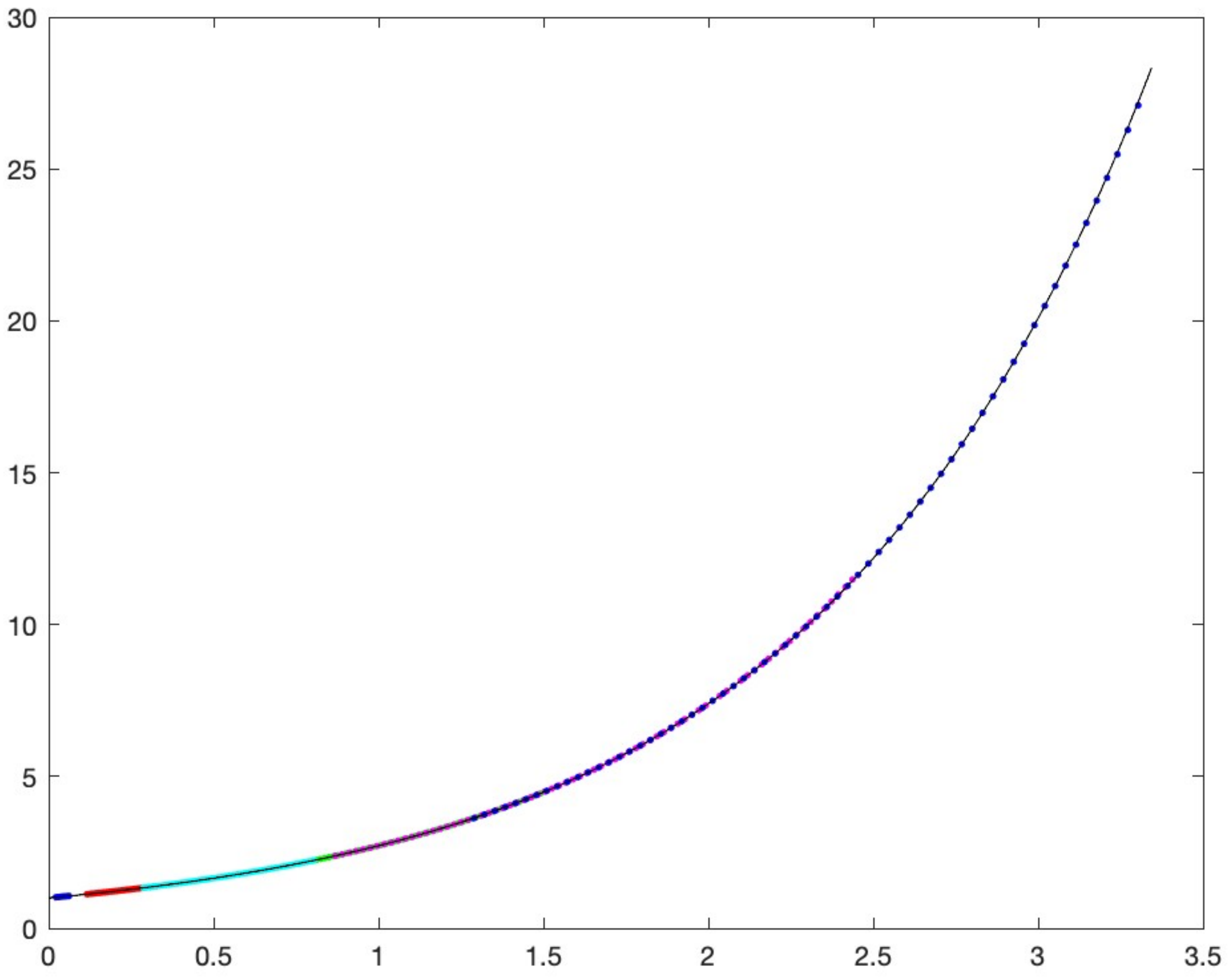
BEWARE! This is not always the case!
There are *lucky* functions and *unlucky* ones…

IMPORTANT OBSERVATION

The most direct (and easy) approach to this self-consistency argument simply asks for changing the input data at fixed quadrature scheme … In fact, in our approach

$$f(s_i) = \sum_j \mathrm{e}^{-t_j(s_i-1)} \, F(t_j) \qquad b_i = \sum_j A_{ij} \, x_j$$

This stays fixed!



YOU CAN’T BE SERIOUS!

… you are not going to play with an exponential …

… you are going to play with a delta function!

YOU CAN'T BE SERIOUS!

... you are not going to play with an exponential ...

... you are going to play with a delta function!

Well, not a a surprise ... if we plug into the game

$$f(s) = \mathrm{e}^{-sE}$$

we get a linear system which is (tremendously) ill-conditioned!

YOU CAN'T BE SERIOUS!

... you are not going to play with an exponential ...

... you are going to play with a delta function!

Well, not a a surprise ... if we plug into the game

$$f(s) = \mathrm{e}^{-sE}$$

we get a linear system which is (tremendously) ill-conditioned!

The obvious thing to do: REGULARISE!

We go for **TYCHONOV REGULARISATION**, i.e.

to solve $A\mathbf{x} = \mathbf{b}$ you minimise $||A\mathbf{x} - \mathbf{b}||^2 + ||\gamma\mathbf{x}||^2$

YOU CAN'T BE SERIOUS!

... you are not going to play with an exponential ...

... you are going to play with a delta function!

Well, not a a surprise ... if we plug into the game

$$f(s) = \mathrm{e}^{-sE}$$

we get a linear system which is (tremendously) ill-conditioned!

The obvious thing to do: REGULARISE!

We go for **TYCHONOV REGULARISATION**, i.e.

to solve $A\mathbf{x} = \mathbf{b}$ you minimise $||A\mathbf{x} - \mathbf{b}||^2 + ||\gamma\mathbf{x}||^2$

... but then ... what is your choice for γ ?

to solve $A\mathbf{x} = \mathbf{b}$

you minimise $||A\mathbf{x} - \mathbf{b}||^2 + ||\gamma\mathbf{x}||^2$

... but then ... what is your choice for γ ?

We play the same game as before!

to solve $A\mathbf{x} = \mathbf{b}$

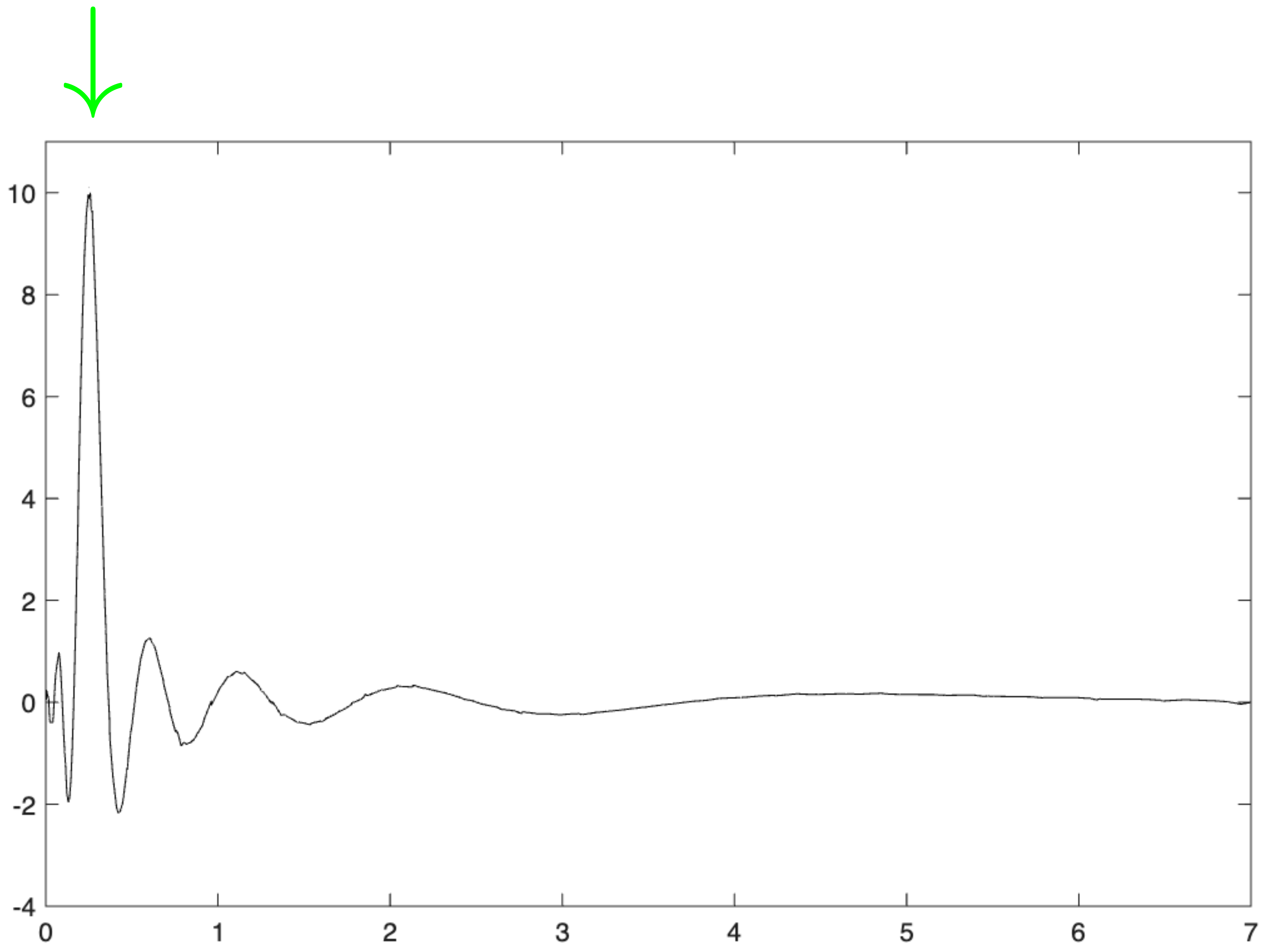
you minimise $||A\mathbf{x} - \mathbf{b}||^2 + ||\gamma \mathbf{x}||^2$

... but then ... what is your choice for γ ?

We play the same game as before!

For a *good* choice of γ
these are two curves going on top of each other!

$$f(s) = \mathrm{e}^{-\,0.25\,s}$$



to solve $\quad A\mathbf{x} = \mathbf{b}$

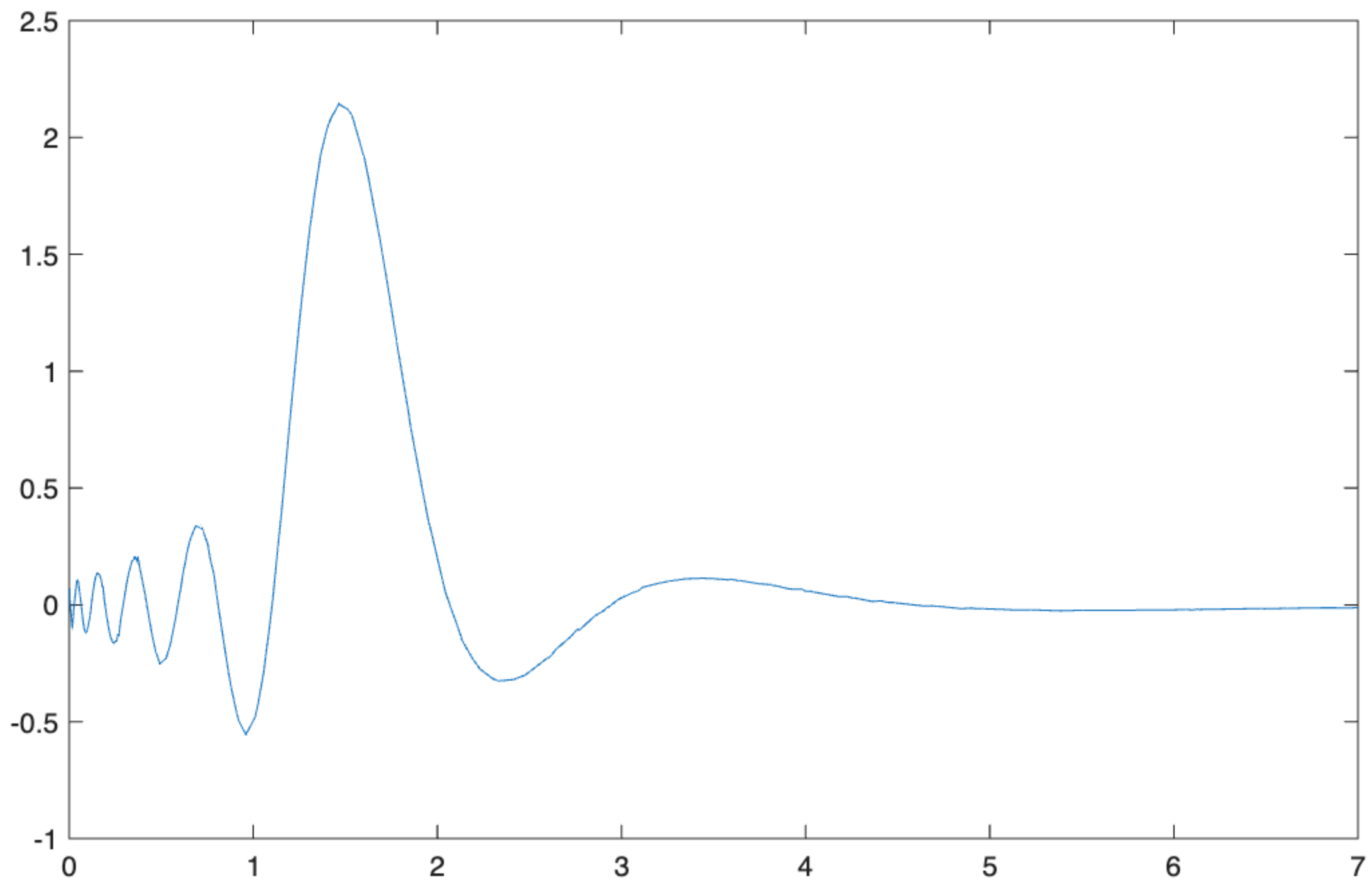
you minimise $\quad ||A\mathbf{x} - \mathbf{b}||^2 + ||\gamma \mathbf{x}||^2$

... but then ... what is your choice for γ ?

We play the same game as before!

For a *good* choice of γ
these are two curves going on top of each other!

$$f(s) = \mathrm{e}^{-\,1.5\,s}$$



to solve $Ax = \mathbf{b}$

you minimise $||A\mathbf{x} - \mathbf{b}||^2 + ||\gamma \mathbf{x}||^2$

... but then ... what is your choice for γ ?

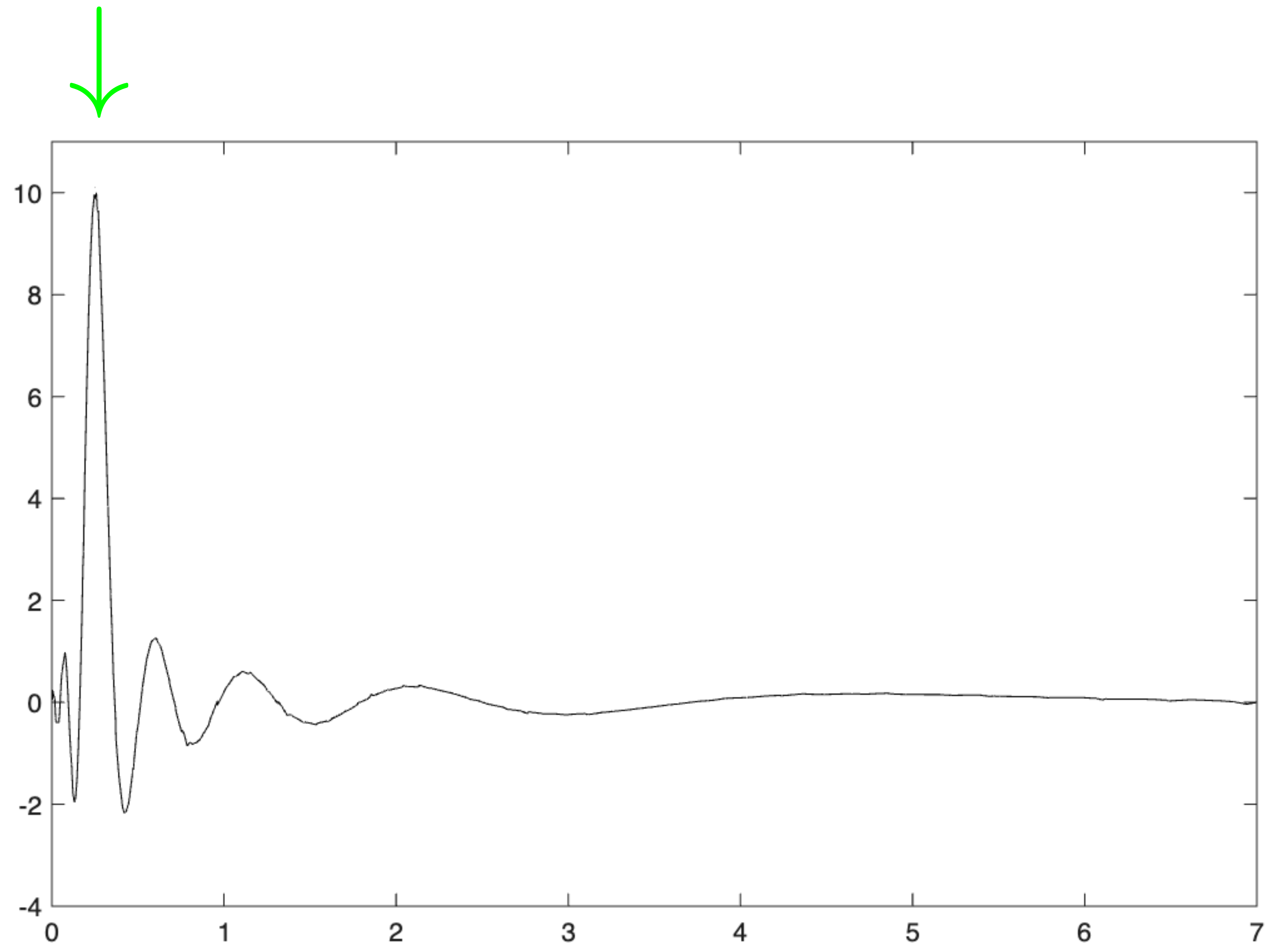
We play the same game as before!

For a *good* choice of γ
these are two curves going on top of each other!

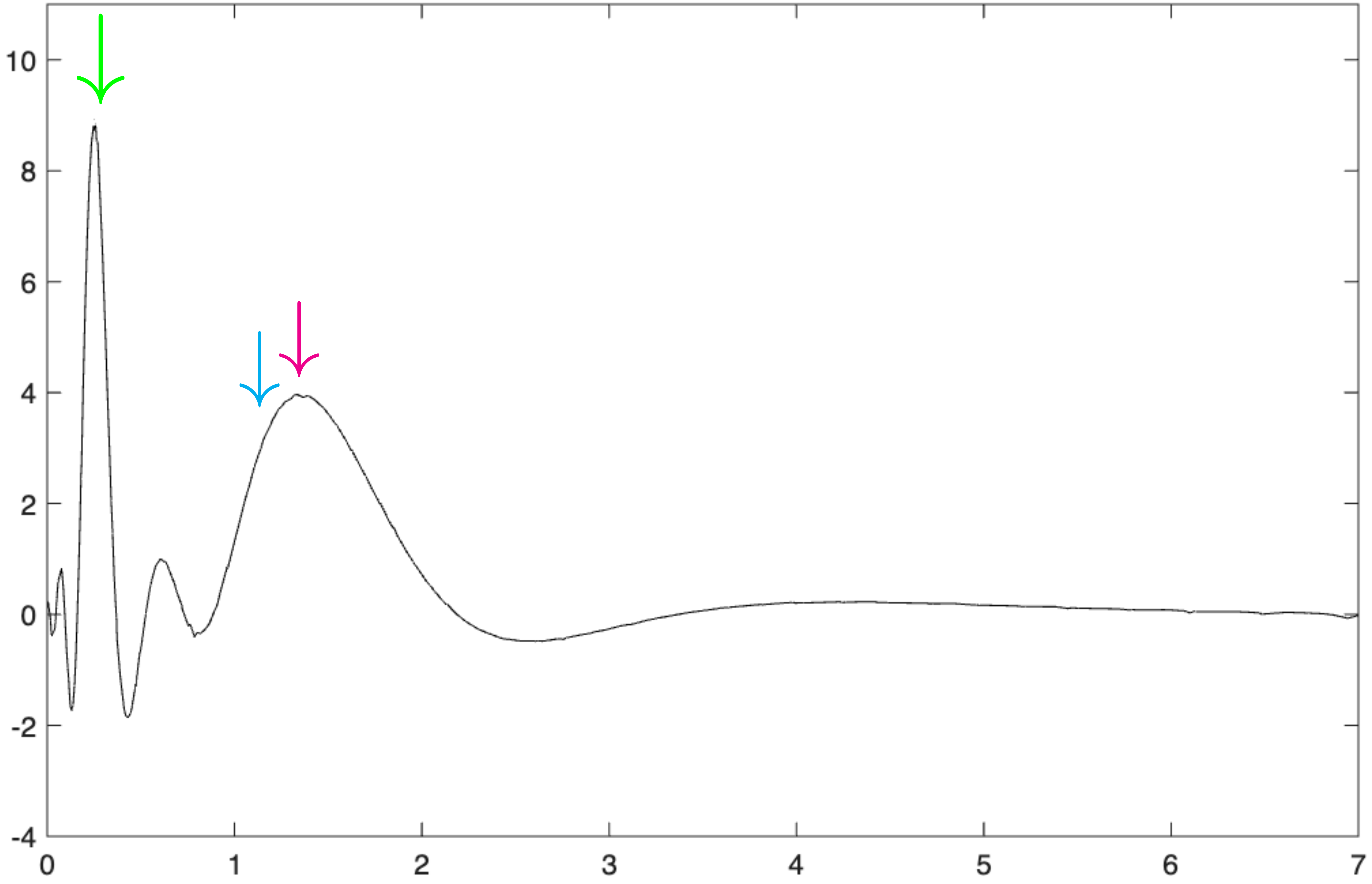
$$f(s) = \mathrm{e}^{-\,0.25\,s}$$

This is actually a regularised delta function!

... which is well defined for a given choice of
- number of points in the quadrature formula
- value of the Tichonov parameter



Now you will guess what is going to happen…



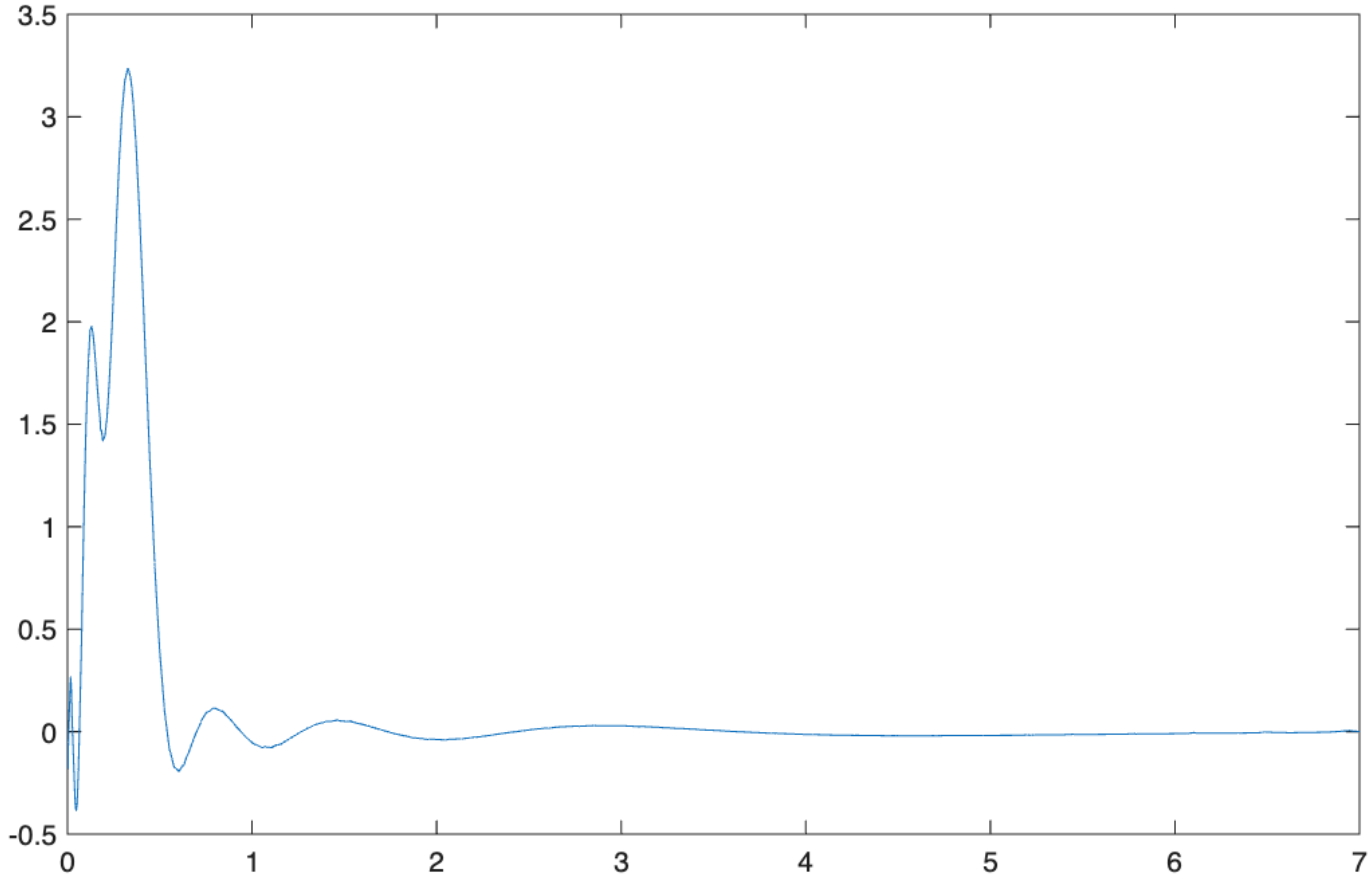
$$f(s) = 0.9\,\mathrm{e}^{-\,0.25\,s} + 0.7\,\mathrm{e}^{-\,1.05\,s} + 2.0\,\mathrm{e}^{-\,1.5\,s}$$

Of course, you cannot let your eye decide … you have to FIT! …   Can we do? YES, WE CAN!

The fitting procedure

- A candidate solution is a superposition of (regularised delta functions) templates.
- The cost function is the norm of the deviation of the candidate from the signal we want to describe.
- The cost function only depends on energies: given the energies, the amplitudes are computed via a (non negative) least squares solution of a linear system.
- We start with an educated guess for one contribution (de facto, the effective mass) and at each iteration a new contribution is selected, trying different starting values in a grid of candidate energies.
- We stop when either the cost function stops improving, or a new contribution falls on top of another one, or the amplitude associated to the new contribution is negligible.
- Once you sit on a solution, you take into account random variations to optimise it.

In Del Debbio, Lupo, Panero, Tantalo, Phys.Rev.D 110 (2024) 7, 074509, a challenging toy model was suggested

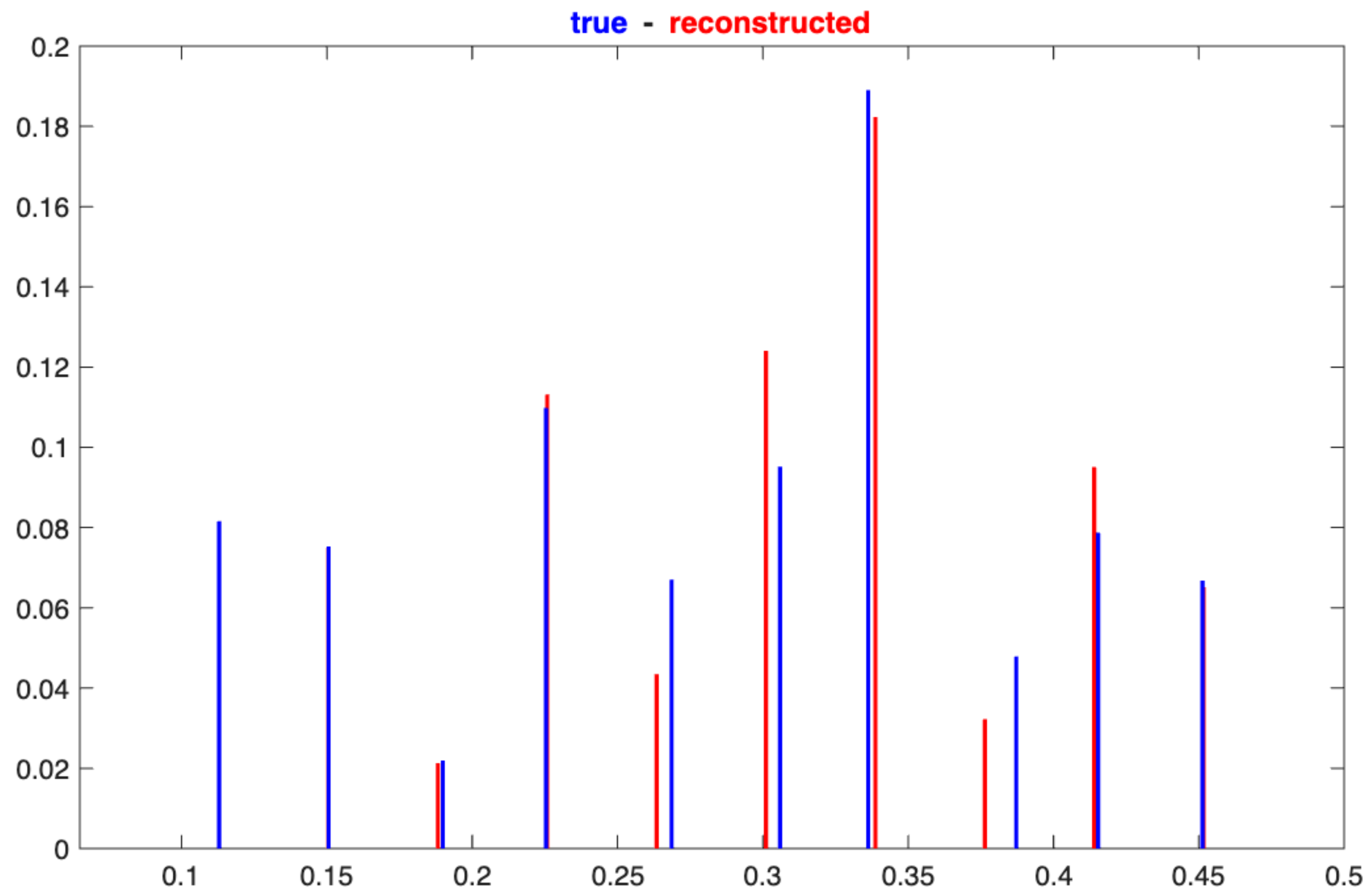


$$f(s) = \sum_{k=1}^{10} c_k \, \mathrm{e}^{-\,E_k\, s}$$

```
goal =

    0.1129    0.1506    0.1882    0.2258    0.2635    0.3011    0.3387    0.3764    0.4140    0.4517
    0.0816    0.0749    0.0213    0.1132    0.0435    0.1241    0.1823    0.0322    0.0951    0.0651


CM10b =

    0.1129    0.1506    0.1899    0.2254    0.2686    0.3059    0.3363    0.3872    0.4153    0.4513
    0.0816    0.0753    0.0220    0.1098    0.0670    0.0952    0.1890    0.0479    0.0787    0.0668


CM10b_over_goal =

    1.0000    1.0003    1.0092    0.9982    1.0196    1.0160    0.9927    1.0287    1.0032    0.9992
    0.9999    1.0048    1.0332    0.9704    1.5414    0.7670    1.0368    1.4866    0.8277    1.0257
```

$$f(s) = \sum_{k=1}^{10} c_k \, \mathrm{e}^{-E_k s}$$

```
goal =

    0.1129    0.1506    0.1882    0.2258    0.2635    0.3011    0.3387    0.3764    0.4140    0.4517
    0.0816    0.0749    0.0213    0.1132    0.0435    0.1241    0.1823    0.0322    0.0951    0.0651


CM10b =

    0.1129    0.1506    0.1899    0.2254    0.2686    0.3059    0.3363    0.3872    0.4153    0.4513
    0.0816    0.0753    0.0220    0.1098    0.0670    0.0952    0.1890    0.0479    0.0787    0.0668


CM10b_over_goal =

    1.0000    1.0003    1.0092    0.9982    1.0196    1.0160    0.9927    1.0287    1.0032    0.9992
    0.9999    1.0048    1.0332    0.9704    1.5414    0.7670    1.0368    1.4866    0.8277    1.0257
```



$$f(s) = \sum_{k=1}^{10} c_k \, e^{-E_k s}$$

Conclusions/Outlook

- Numerical quadratures enable direct computation of anti Laplace Transforms.
- This is viable provided the linear system is regularised in a convenient way.
- Spectral densities can be got by fitting the signal to a superposition of templates, which turn out to be regularisations of delta functions (see next talk).
- Next steps: experiments with noise, proper lattice correlators (we are quite confident…)
- Other integral inverse problem can be addressed as well.