Radiative decay of Kaon into four leptons

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Lattice@College Park, 28 July 2026

Based on

Complete lattice QCD calculation of $K^- \to \ell^- \bar{\nu}_\ell \ell'^+ \ell'^-$ form factors

arXiv:2605.22742 – lattice QCD technical paper, submitted to PRD

Rare kaon decays $K^- \to \ell^- \bar{\nu}_\ell \ell'^+ \ell'^-$: SM predictions from lattice QCD

arXiv:2605.22727 – phenomenological analysis, submitted to PRL

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RM123-SOTON collaboration

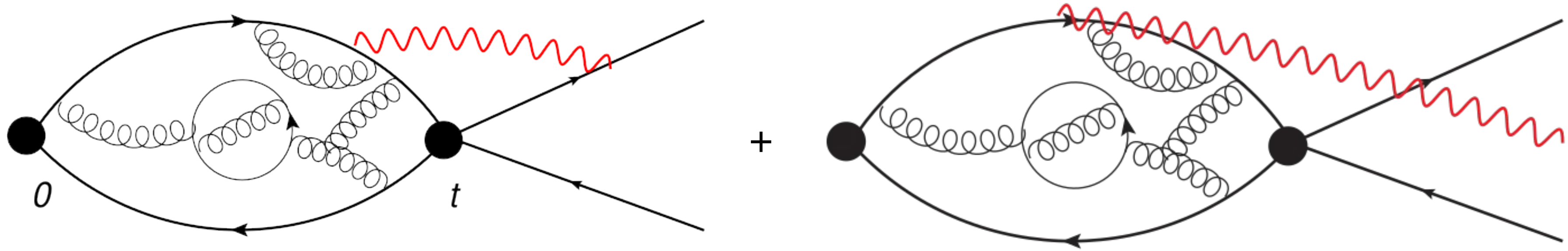
Historical setting

Computing Pion & Kaon decay into leptons beyond percent accuracy

"*QED Corrections to Hadronic Processes in Lattice QCD*," N.Carrasco et al, PRD 91 (2015)

→ QED correction to the $K \to \ell_2$ process are infrared divergent

→ Need to combine virtual correction with (unobserved) soft photon $K \to \ell\nu\gamma(E_\gamma), \quad E_\gamma < \Delta$

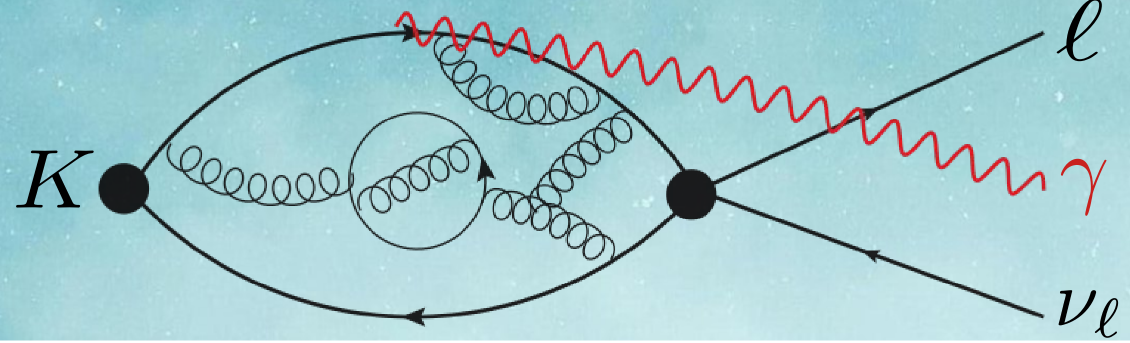


Numerical results

"*First lattice calculation of QED corrections to leptonic decay rates*", D.Giusti et al, PRL 120 (2018)

"*Light-meson leptonic decay rates in lattice QCD+QED*", M.Di Carlo et al, PRD 100 (2019)

Subsequent development: Radiative decay process



$\Gamma[K \to \ell\nu\gamma(E_\gamma)]$ at finite energy is infrared finite! $\quad x_\gamma = 2E_\gamma/M_K \quad x_\ell = 2E_\ell/M_K$

And, experimentally measured, indeed:

- Suppressed by α_{em}
- But helicity suppression is removed!

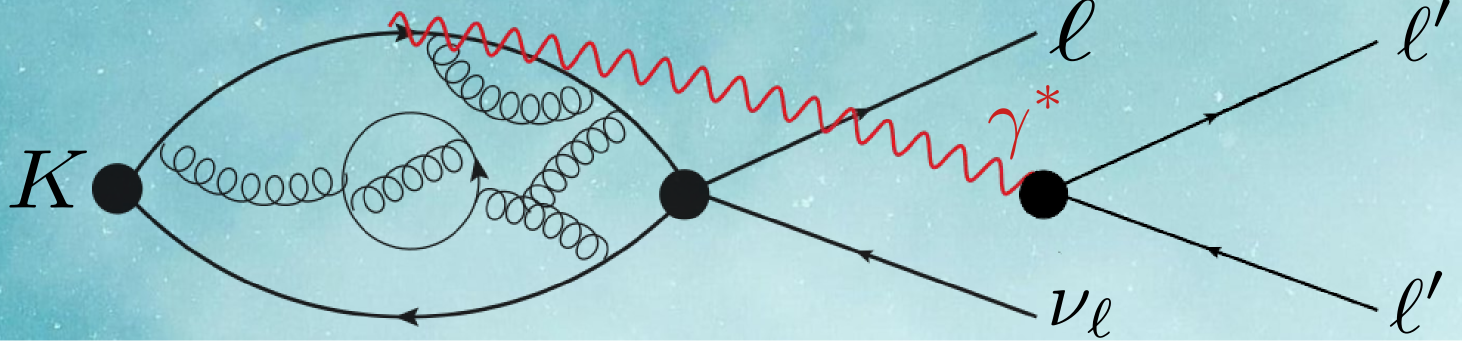
$\Gamma(K^+ \to e^+\nu_e\gamma)/\Gamma(K^+ \to \mu^+\nu_\mu)$ $\quad\Gamma_{17}/\Gamma_2$

VALUE (10^{-5})	*EVTS*	*DOCUMENT ID*		*TECN*	*CHG*	*COMMENT*
$\mathbf{1.62 \pm 0.22}$	OUR AVERAGE Error includes scale factor of 3.9.					
1.98 ± 0.11	95	[1] KOBAYASHI	2023	E36		E_γ in $10 - 250$ MeV, $p_e > 200$ MeV/c
$1.483 \pm 0.066 \pm 0.013$	1.4k	[2] AMBROSINO	2009E	KLOE	$\pm$	E_γ in $10 - 250$ MeV, $p_e > 200$ MeV/c

Preliminary study PRD 103 (2021) - Full study PRD 111 (2025)

[Presented @Lattice 2026]

Novel extension: photon decay into leptonic pair



Motivation 1: starts at $\mathcal{O}\left(G_F^2\alpha_{em}^2\right)$ in the SM, $\mathcal{B}(K_{\ell 2\ell'}) \sim 10^{-8}$ → probe for New Physics

Motivation 2: alternative determination of the Cabibbo angle V_{us}

$$\mathcal{M} = V_{us} G_F \alpha_{em} \left(f_K A^{pert} + B H_w^{\mu\nu} L_{\mu\nu} \right)$$

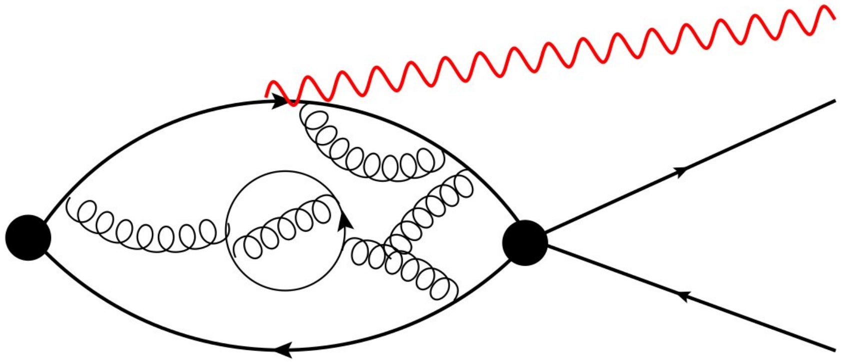
Non-perturbative contribution incorporated in the hadronic tensor $H_w^{\mu\nu}$

Previous exploratory studies at single lattice spacing & 300 MeV pion mass	Tuo et al. Gagliardi et al.	[Phys.Rev.D 105 (2022) 5] [Phys.Rev.D 105 (2022) 11]

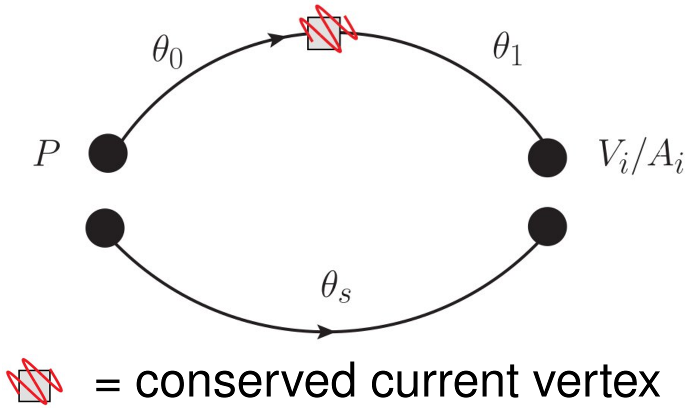
Photon emission ...without a photon

- Photon: external particle, gets completely amputated in the rate
- Only role: to project to the correct polarization the vector vertex

→ We can omit the photon propagator from the beginning



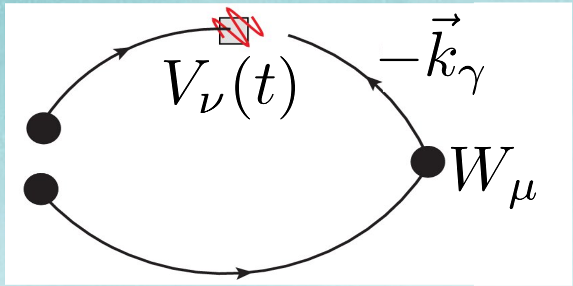
→



No large finite volume effect (no photon at all is present on the lattice)

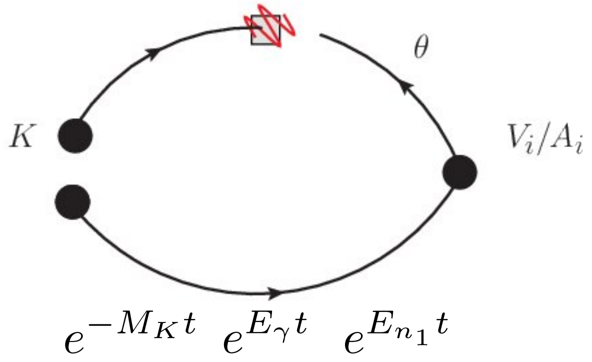
Hadronic tensor & Euclidean correlator

$$H_w^{\mu\nu}(E_\gamma, k_\gamma) = \int_{-\infty}^{+\infty} dt\, e^{E_\gamma t} C_w^{\mu\nu}(t, k_\gamma) \quad , \; C_w^{\mu\nu}(t, k_\gamma) =$$



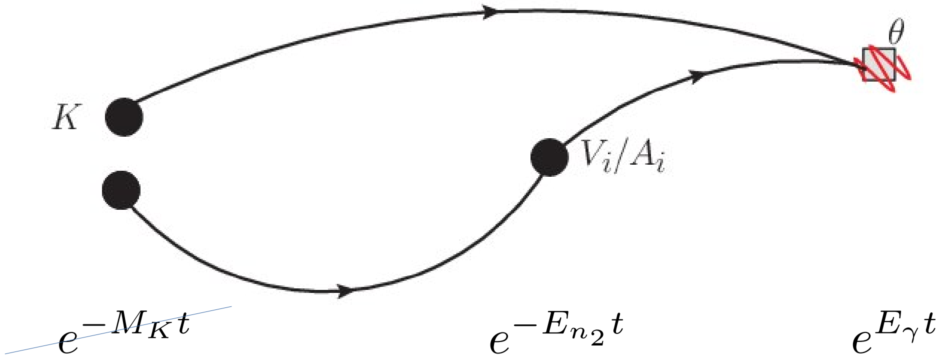
- Photon & leptonic pair reduced to a <u>ν, t dependent kernel</u>
- Sequential propagator over fixed μ and t_w (<u>3D method</u> by Tuo et al., Giusti et al.)
- Allows to take care of the integration over t, <u>taming time-growing error</u>
- All photon virtualities $y_k = 2\,|k_\gamma|\,/M_K$ can be <u>computed at the same time</u>
- Preliminary verification on the choice of t_w to ensure <u>isolation of the ground state</u>
- Four form factors involved: $F_V,\ F_A,\ H_1,\ H_2$ and a pointlike contribution $\propto f_K$

Analytic continuation issue



First time ordering: $t < 0$

- The integration over $t < 0$ converges if
$$E_\gamma + E_{n_1} - M_K > 0$$
- EM current acts on Kaon
- No flavor changes before weak current
- n_1: a stranged-light state with momentum
  → condition always satisfied



Second time ordering: $t > 0$

- The integration over $t > 0$ converges if
$$E_{n_2} - E_\gamma > 0$$
- Flavor changes with weak current
- EM current acts on light state
- n_2: an unflavored state with momentum
  → condition not satisfied if:
$$(2M_\pi)^2 < k^2 < M_K^2$$

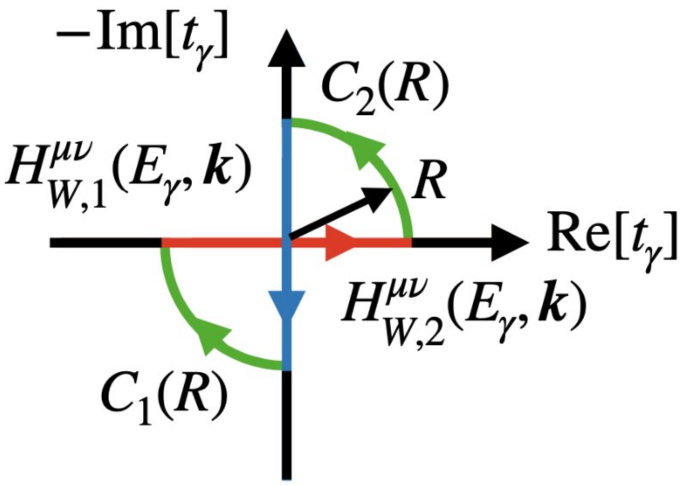
Two-pions threshold

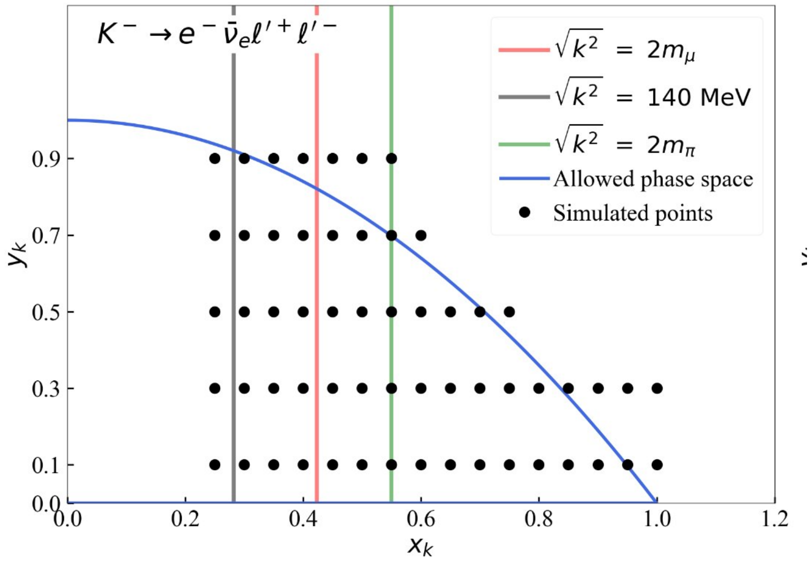
Can be seen as an effect of the non-vanishing of $C_2(R)$ in the complex plane continuation

Issue arises if $(2M_\pi)^2 < k^2 < M_K^2$ which is impossible to achieve if $M_K < 2M_\pi$

On old simulations $M_K \sim 500\,\mathrm{MeV}$ and $M_\pi \sim 300\,\mathrm{MeV}$, no analytic continuation issue

Requires special treatment on physical M_π, on a subset of the phase space





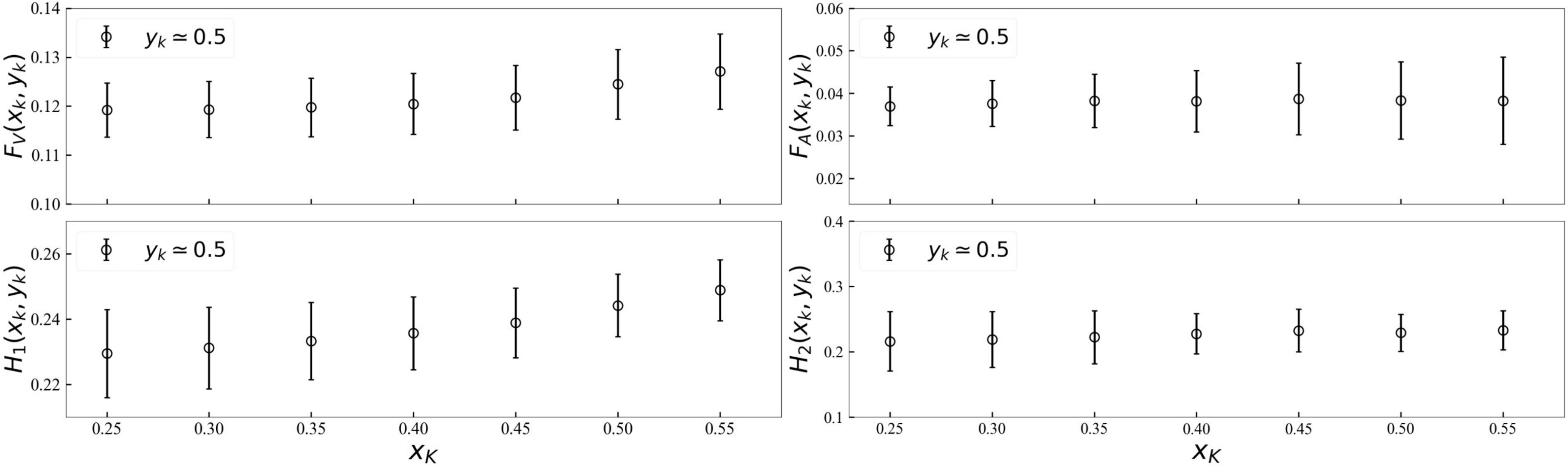
Numerical setup

ID	L/a	M_π [GeV]	a [fm]
B48	48	140.3(3)	0.07948(11)
B64	64	140.2(3)	0.07948(11)
B96	96	140.1(3)	0.07948(11)
C80	80	136.7(3)	0.06819(14)
D96	96	140.8(3)	0.056850(90)

- N_f=2+1+1 simulations of Twisted Mass at maximal twist
- Pion mass close to its physical value
- Three lattice spacing (on a fixed volume) – automatic $\mathcal{O}(a)$ improvement
- Three volumes (on a fixed lattice spacing)

Below the two-pions threshold

Direct integration of the correlation function: $H_w(E_\gamma, k_\gamma) = \int_{-\infty}^{+\infty} e^{E_\gamma t} C_w(t, k_w)$



Example of the four form factors at a fixed virtuality, $y_k \sim 0.5$

Spectral Function Representation (SFR)

[R. Frezzotti et al. PRD 108 (2023)]

By expressing the correlation function as the Laplace transform of the spectral density:

$$C_w\left(t, k_\gamma\right) = \int_0^\infty \frac{dE}{2\pi} e^{-tE} \rho_w\left(E, k_\gamma\right)$$

One can represent the hadronic tensor in terms of the spectral density $\rho\left(E\right)$:

$$H_w(E_\gamma) = \int_0^\infty dt\, e^{iE_\gamma t}\, C_w(t) = \lim_{\epsilon\to 0^+} \int_0^\infty \frac{dE'}{2\pi}\, \rho(E') \int_0^\infty dt\, e^{-i(E'-E_\gamma)t} e^{-\epsilon t}$$

The term $e^{-\epsilon t}$ is introduced to guarantee the time integral convergence, leading to:

$$H_w\left(E_\gamma\right) = \lim_{\epsilon\to 0^+} \int_0^\infty \frac{dE}{2\pi} \frac{\rho\left(E\right)}{E - E_\gamma - i\epsilon} = \mathrm{P.V.} \int_0^\infty \frac{dE}{2\pi} \frac{\rho\left(E\right)}{E - E_\gamma} + \frac{i}{2}\rho\left(E_\gamma\right)$$

Spectral Function Representation (SFR)

[R. Frezzotti et al. PRD 108 (2023)]

$$H_w(E_\gamma) = \lim_{\epsilon \to 0^+} \int_0^\infty \frac{dE}{2\pi} \frac{\rho_w(E)}{E - E_\gamma - i\epsilon}$$

We recognize the convolution of the spectral density with a kernel:

$$H(E,\epsilon) \equiv \int_{E_0}^\infty \frac{dE'}{2\pi} \rho(E')\, K(E'-E,\epsilon) \;, \qquad K(E'-E,\epsilon) \equiv \frac{1}{E'-E-i\epsilon}$$

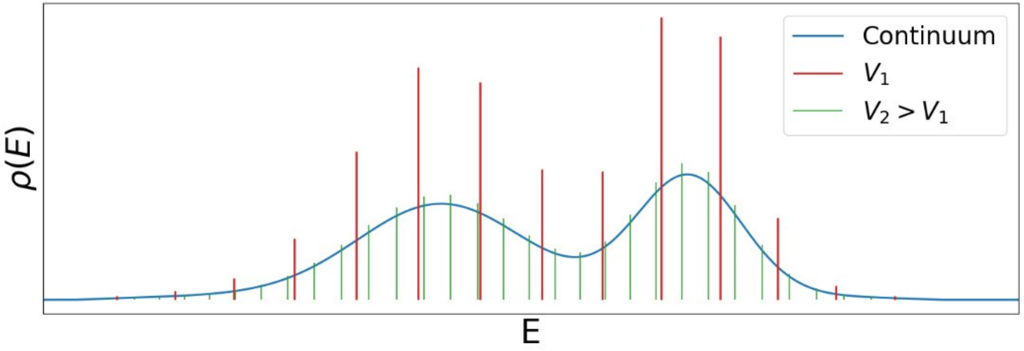
Computable in terms of $C(t)$ using an approximate inverse Laplace transform of $K(E,\epsilon)$

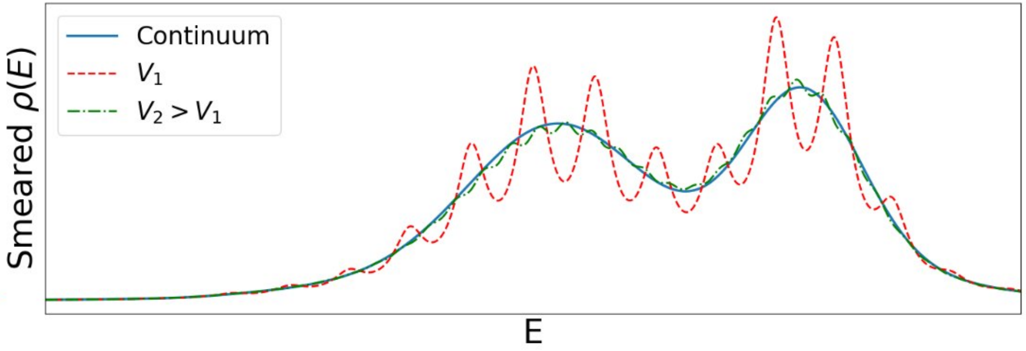
[CFR talk by S.Hashimoto @Mon 11am]

$$\int_{E_0}^\infty \frac{dE'}{2\pi} \rho(E')\, K(E'-E,\epsilon) = \sum_t C_w(t) g_t^K(E,\epsilon) \qquad \sum_t g_t^K(E,\epsilon) e^{-Et} = K(E,\epsilon)$$

We determine g_t via Hansen-Lupo-Tantalo (HLT) method [PRD99 (2019)]

Infinite-volume limit





In the infinite volume limit, the spectral density is a continuous function

At finite volume is a series of delta function associated to finite-volume energy level

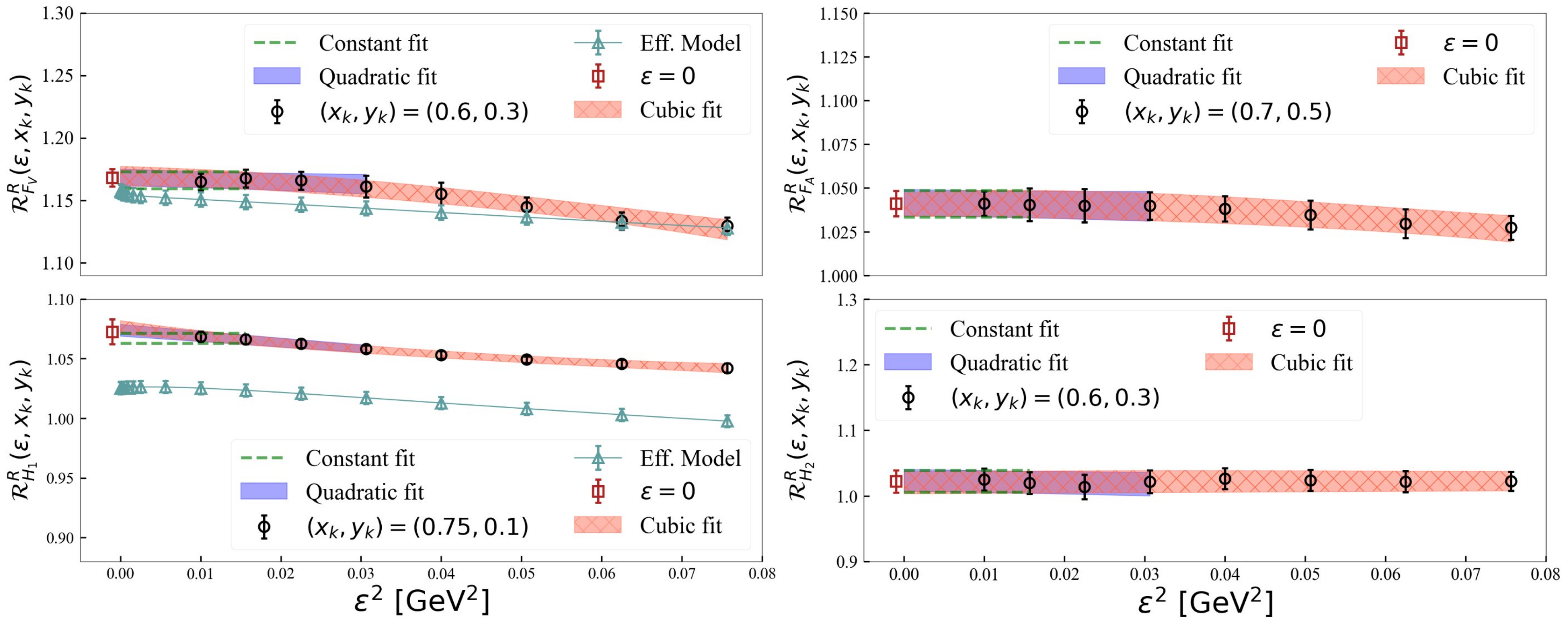
As the volume gets larger the series gets more and more dense

To properly carry out the infinite volume limit we need to take a smeared version of ρ

Smearing bust be larger than the energy level spacing: $\epsilon \gg 1/L$

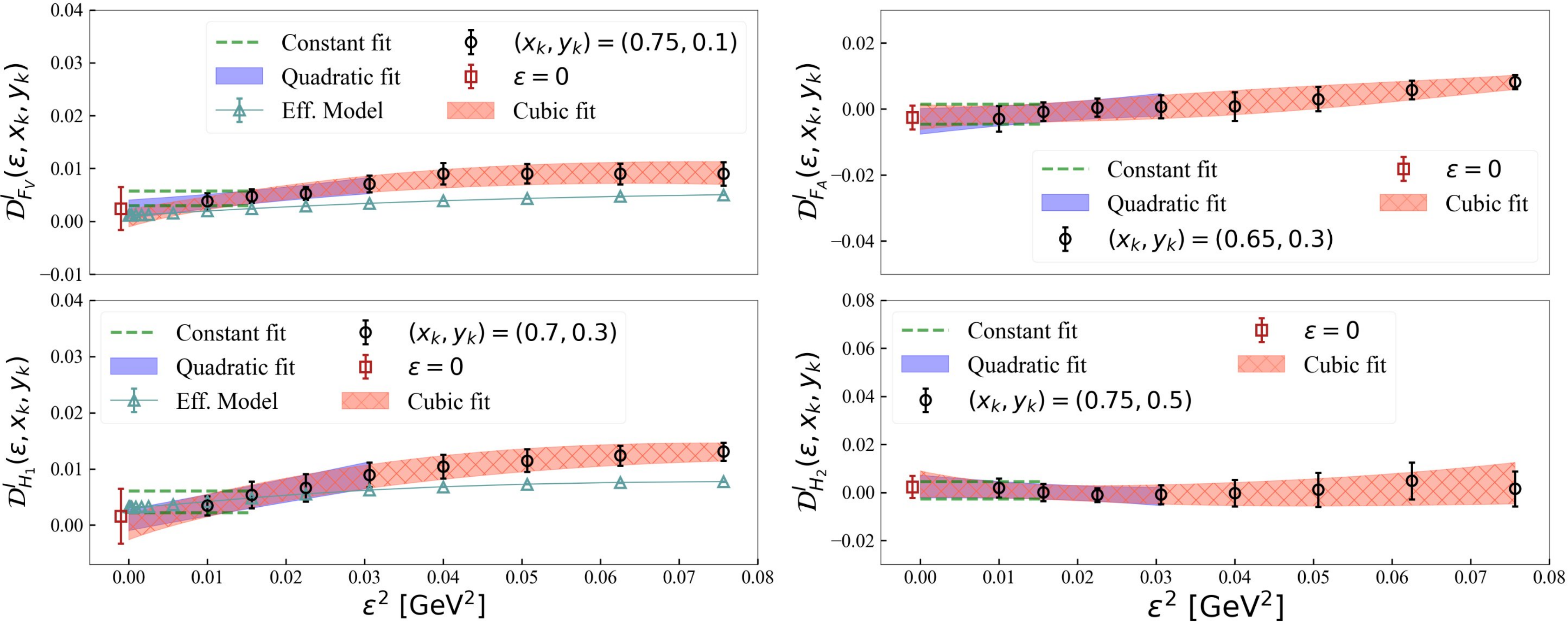
Proper order of limits: $\lim_{\epsilon \to 0} \lim_{L \to \infty}$

ε-extrapolate real part of the f.factors



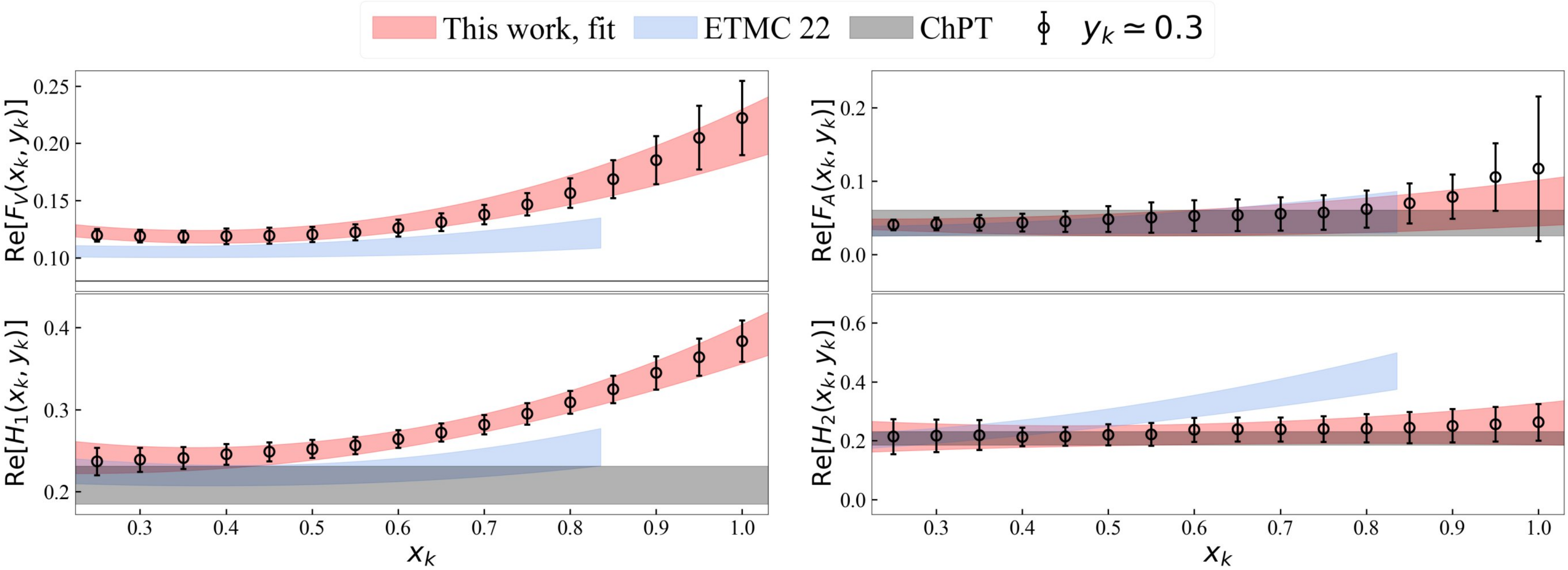
Extrapolated with polynomial & compared with Gounaris-Sakuraki model for $\pi^+\pi^-$ contribution

ε-extrapolate imaginary part of the f.factors



Imaginary part arises above the two pion threshold (see later)

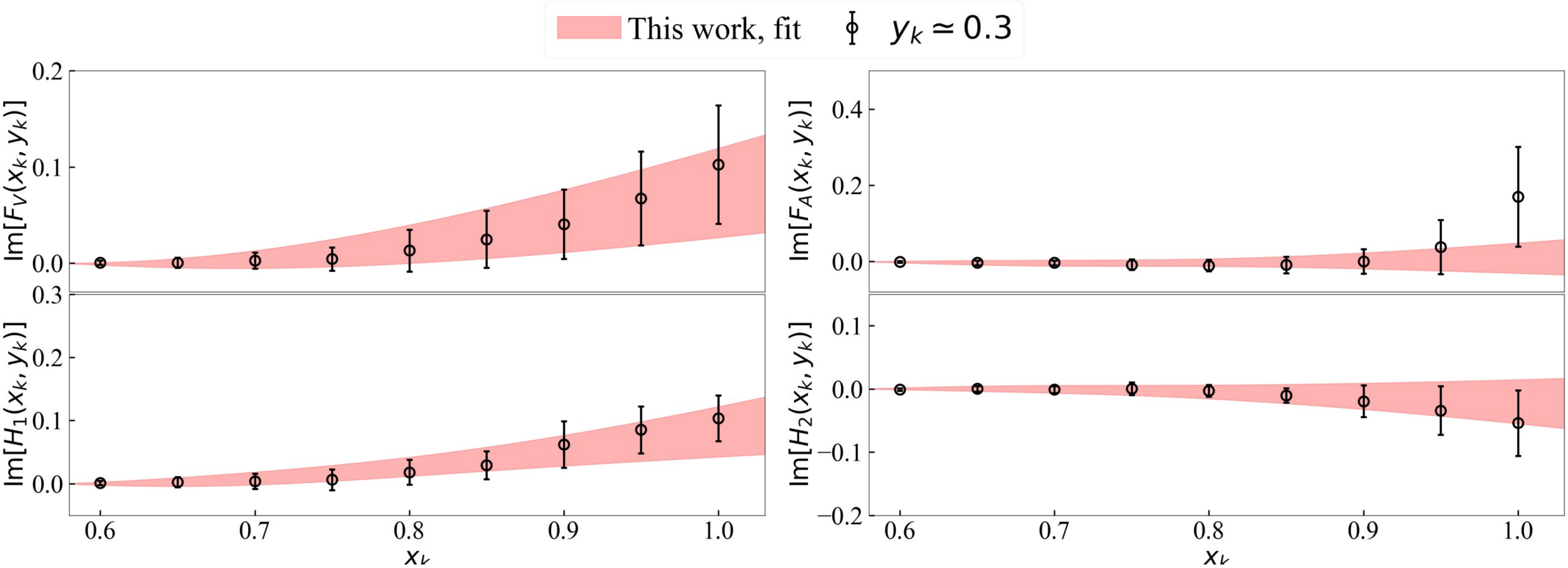
Result for the real part of the f.factors



Substantial improvements: $a \to 0,\ L \to \infty,\ M_\pi^{phys}$

→ Covering the full kinematic range thanks to SFR/HLT

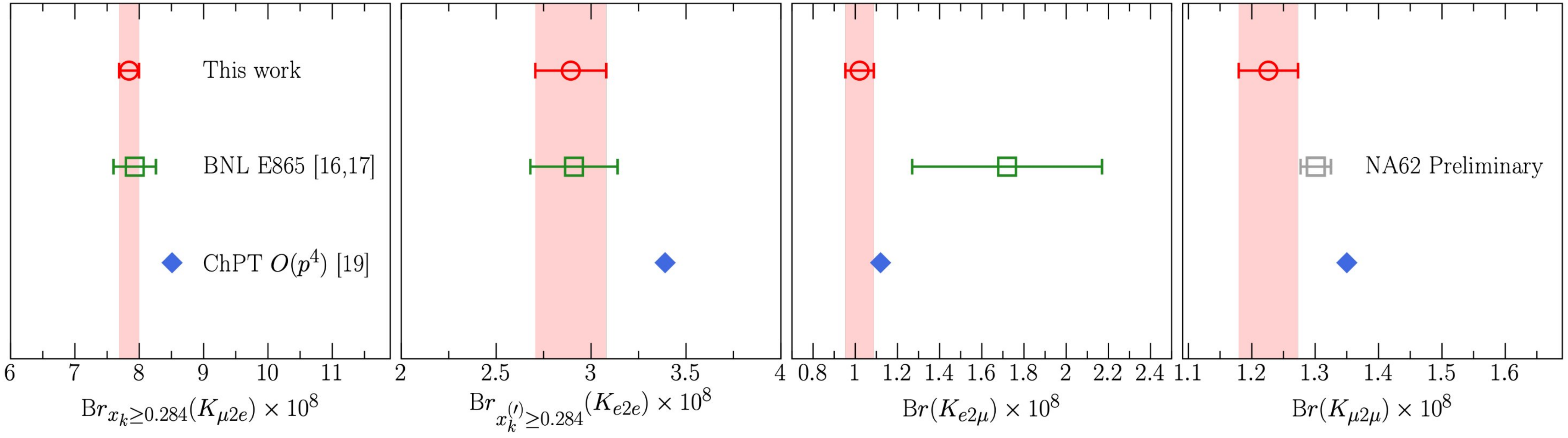
Result for the imaginary part of the f.factors



Tiny (compatible with zero to large extent) but reliably determined

→ First determination of imaginary part of form factors!

Branching fractions



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[17] H. Ma et al., First observation of the decay $K^+ \to e^+ nu(e) mu^+ mu^-$, Phys. Rev. D **73**, 037101 (2006), arXiv:hep-ex/0505011.

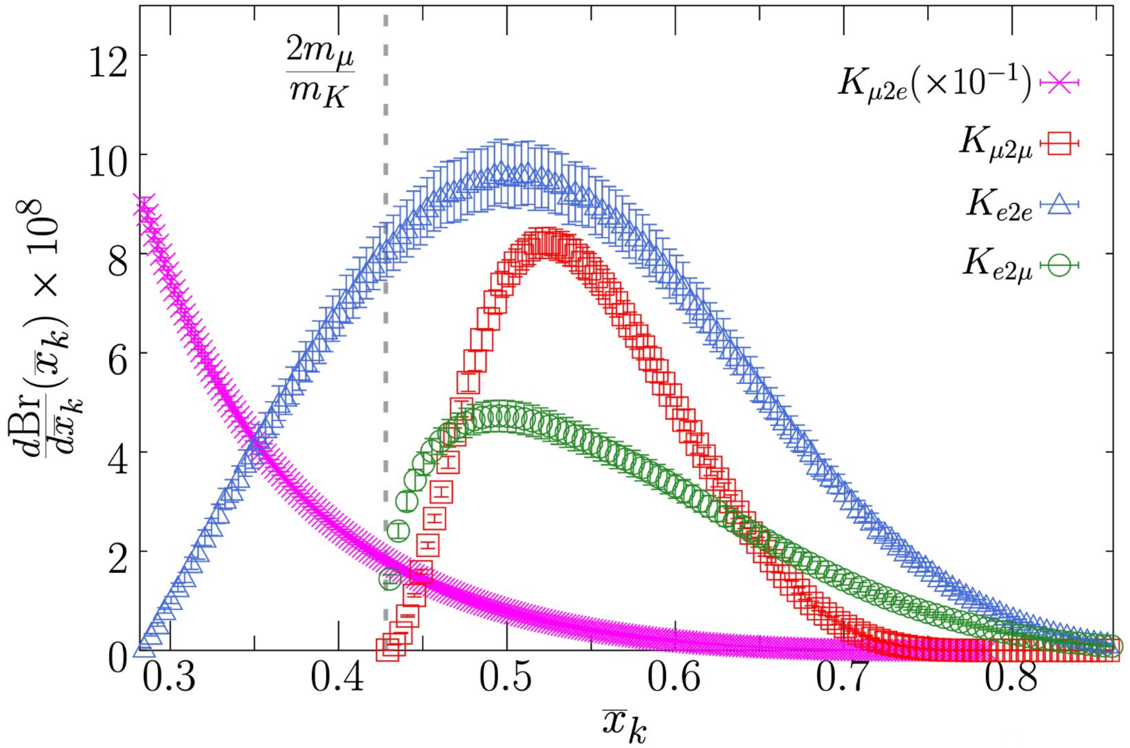
[19] J. Bijnens, G. Ecker, and J. Gasser, Radiative semileptonic kaon decays, Nucl. Phys. B **396**, 81 (1993), arXiv:hep-ph/9209261.

In agreement with ChPT, compatible with available measurement & predictive

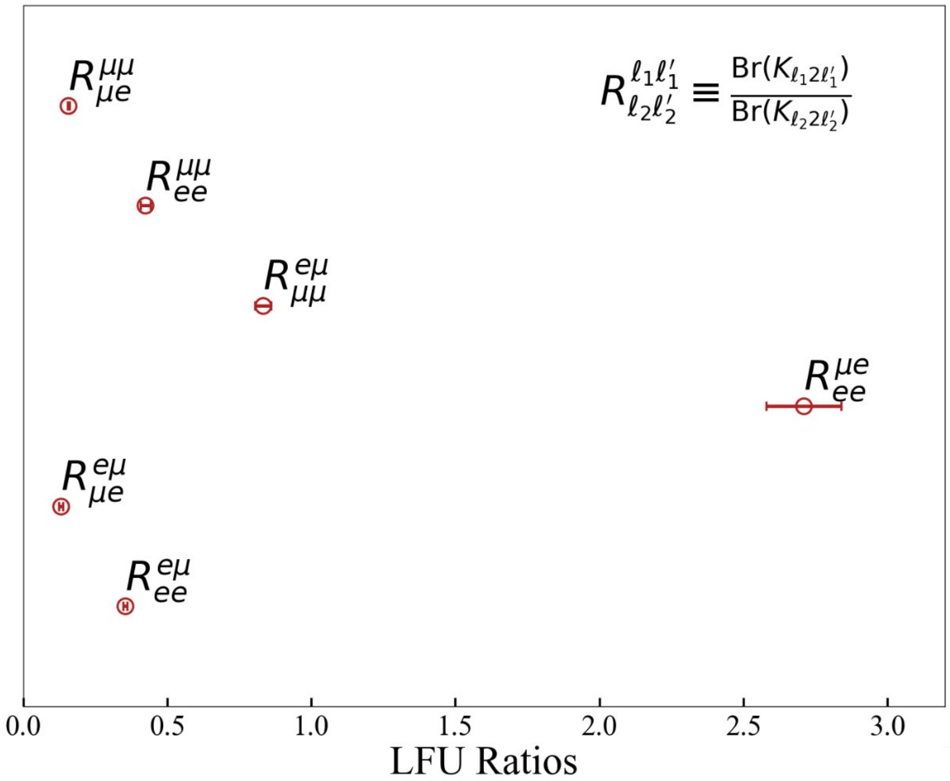
→ NA62 will update also the other channels

Additional phenomenological output

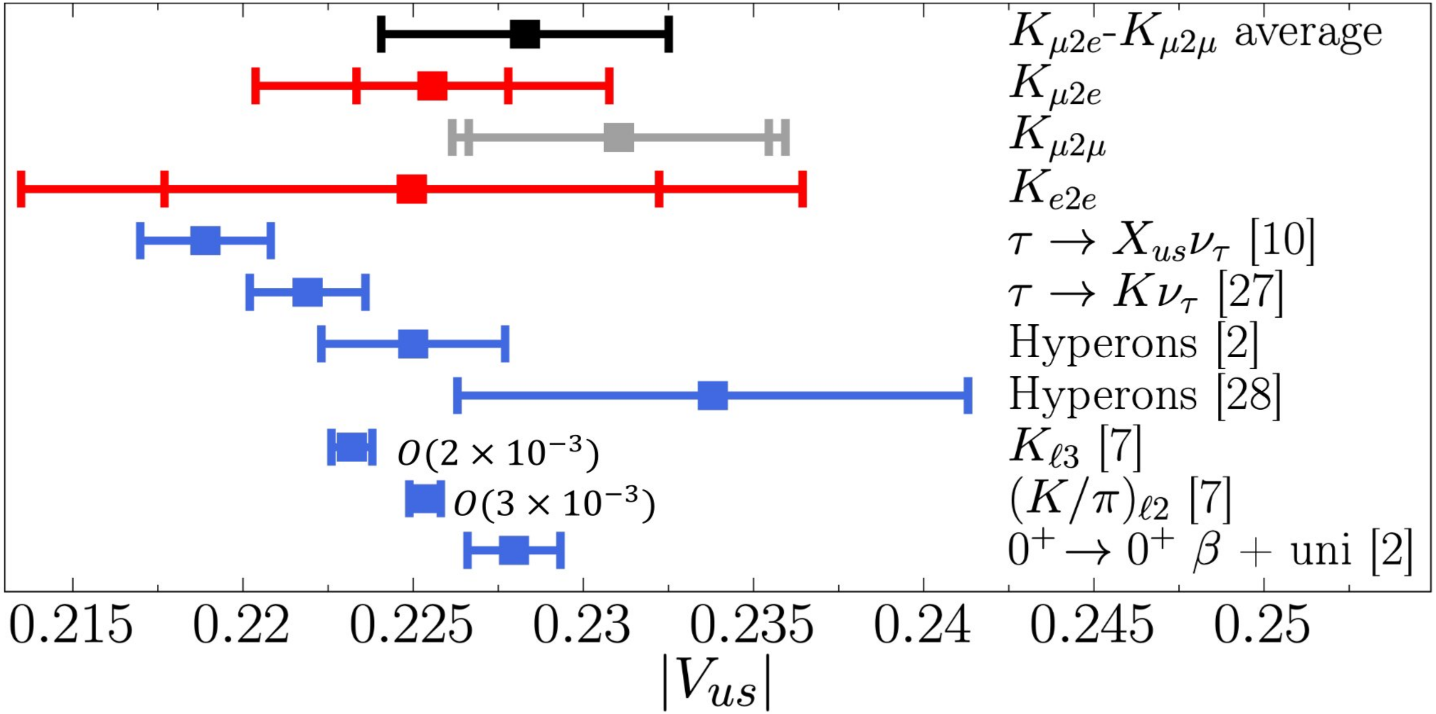
Differential branching fractions



Lepton Flavor Universality ratios



Cabibbo angle prediction



[2] S. Navas et al. (Particle Data Group), Review of particle physics, Phys. Rev. D **110**, 030001 (2024).

[7] Y. Aoki et al. (Flavour Lattice Averaging Group (FLAG)), FLAG Review 2024, (2024), arXiv:2411.04268 [hep-lat].

[10] C. Alexandrou et al. (Extended Twisted Mass), Inclusive Hadronic Decay Rate of the τ Lepton from Lattice QCD: The u¯s Flavor Channel and the Cabibbo Angle, Phys. Rev. Lett. **132**, 261901 (2024), arXiv:2403.05404 [hep-lat].

[27] Y. S. Amhis et al. (HFLAV), Averages of b-hadron, c-hadron, and τ-lepton properties as of 2021, Phys. Rev. D **107**, 052008 (2023), arXiv:2206.07501 [hep-ex].

[28] S. Bacchio and A. Konstantinou, Study of the $\Lambda \to p\ell\bar{\nu}_\ell$ Semileptonic Decay in Lattice QCD, Phys. Rev. Lett. **135**, 231901 (2025), arXiv:2507.09970 [hep-lat].

$\delta|V_{us}| \sim 2\%$ determined from a rare process with $\mathcal{B} \sim \mathcal{O}(10^{-8})$!

Not competitive but complementary to the other determinations
→ Well, let's see if it improves after NA62 updates!

Summarizing

→ **First complete study of $K^- \to \ell^- \bar{\nu}_\ell \ell'^+ \ell'^-$ decays**

State of our art

- Continuum limit, infinite volume limit extrapolation
- Disconnected diagram (see emission) included

Thanks for your attention!

Time integration

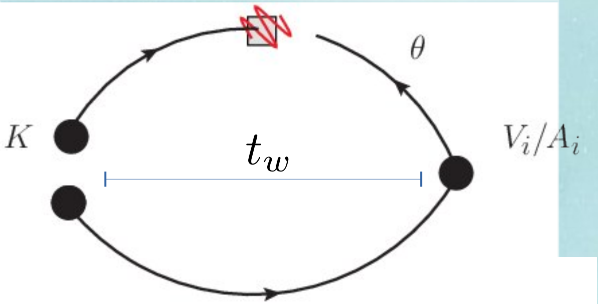
- HLT method to address the second time ordering
- Controlled extrapolation $\varepsilon \to 0$ via GS model
- Improved convergence via modelling the long time behavior of correlators

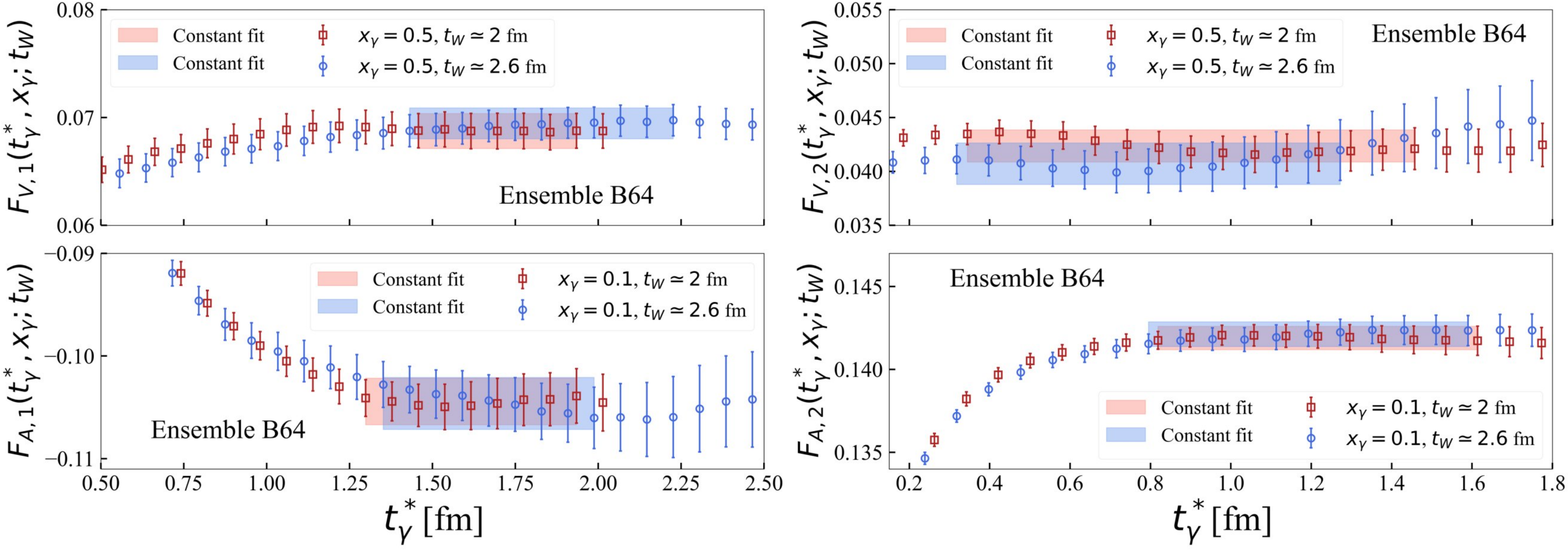
Conclusions & Outlook

- Good agreement with the existing measurements
- Providing predictions which waits to be verified (in progress)
- NA62 will update exp. determinations → will motivate theoretical improvements!

BACKUP SLIDES

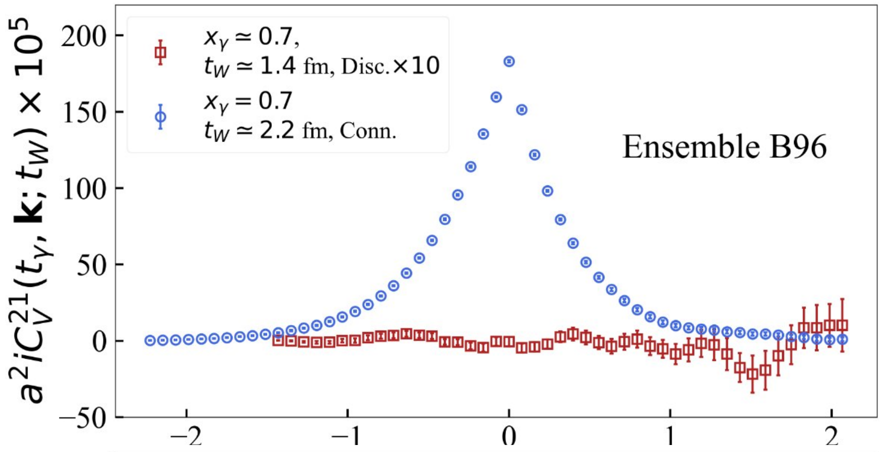
Isolation of Kaon

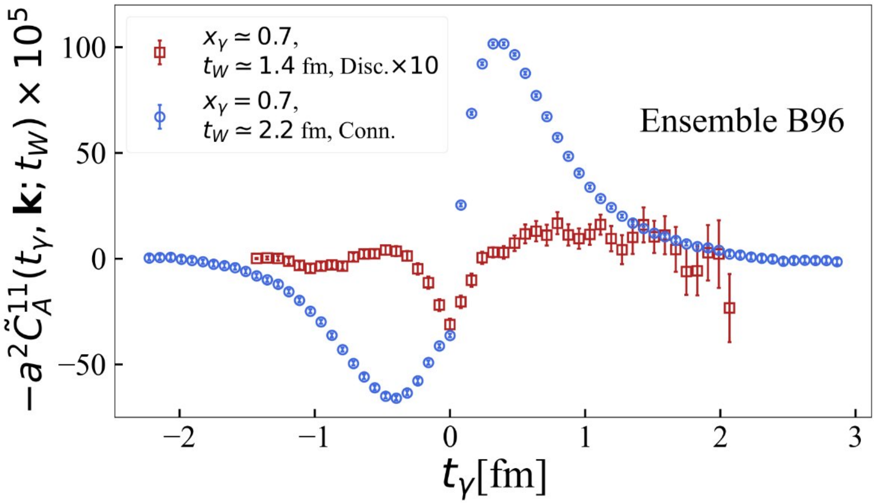


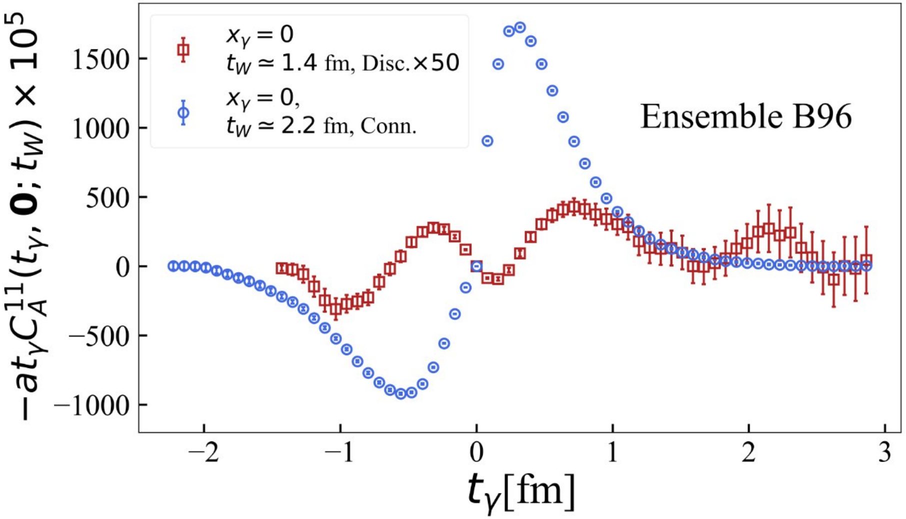


Studying two time separation t_w to verify isolation of the Kaon

Disconnected contribution







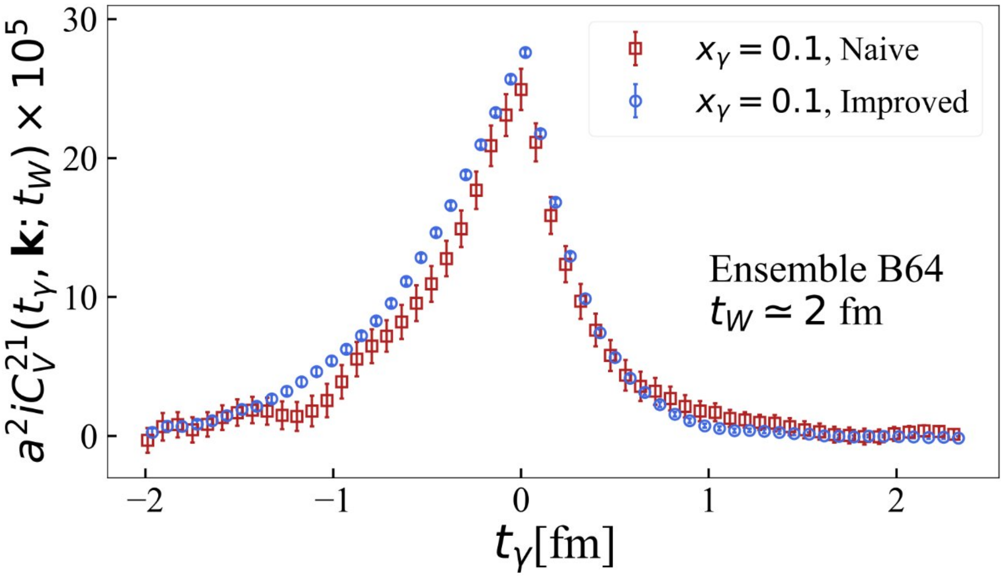
Disconnected emission:

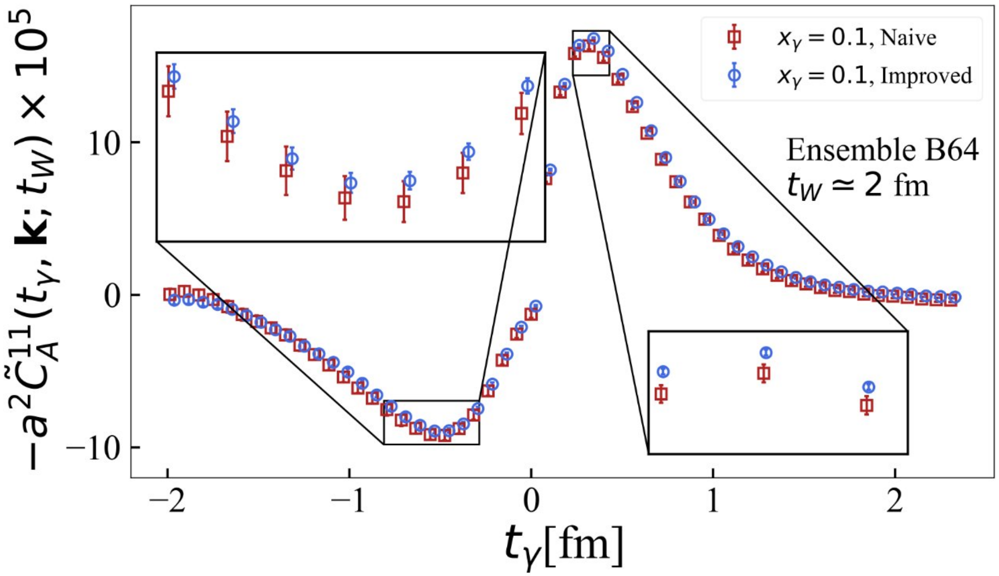
Tested on given x_γ~0.7, B96 ensemble

One-end trick to improve evaluation

(magnification by a factor of 10-50)

Example of integration



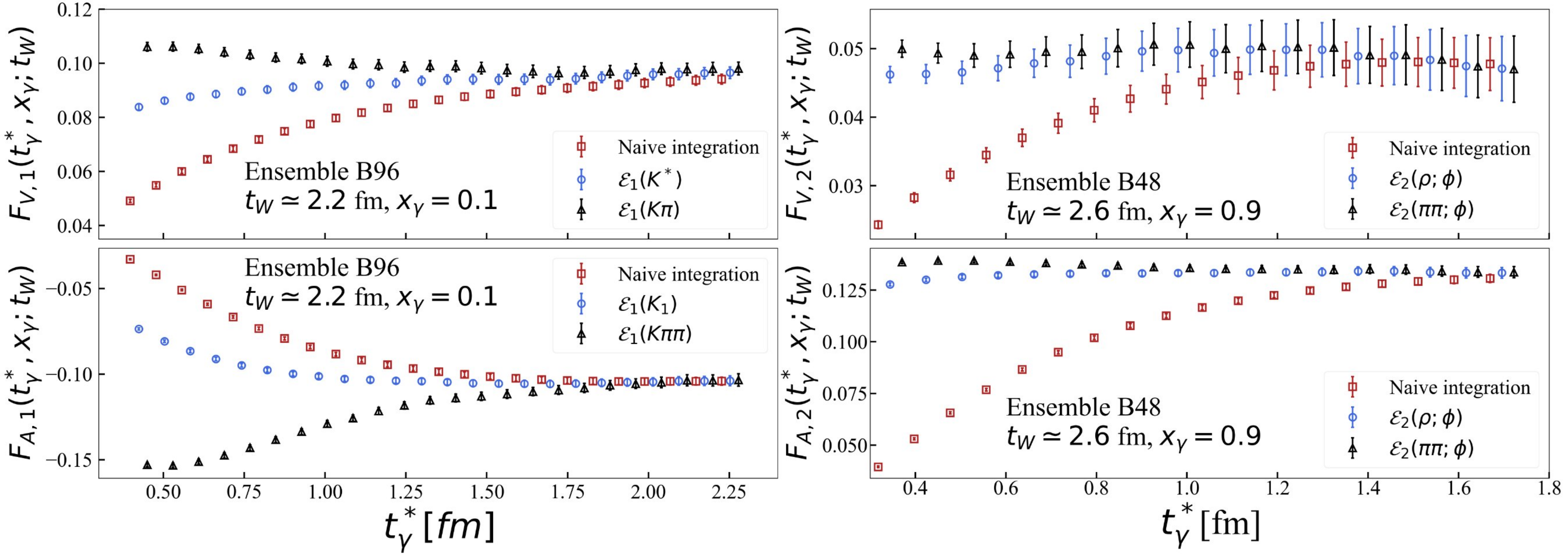


Improved estimators useful at small x_γ

$$\widetilde{C}_A^{11}(t_\gamma, \boldsymbol{k}; t_W) \to \frac{1}{2}\left[\widetilde{C}_A^{11}(t_\gamma, \boldsymbol{k}; t_W) + \widetilde{C}_A^{11}(t_\gamma, -\boldsymbol{k}; t_W)\right]$$ Enforcing parity

$$C_V^{21}(t_\gamma, \boldsymbol{k}; t_W) \to C_V^{21}(t_\gamma, \boldsymbol{k}; t_W) - C_V^{21}(t_\gamma, \mathbf{0}; t_W)e^{E_\gamma t_\gamma}$$ Subtracting “zero”

Analytic integration on t_γ



- Allows to speed up the integration convergence [Tuo et al]
- Projecting on physical on-shell helicity simplifies the continuation

→ Remove $K\gamma$ state, spectral decomposition starts with K^* and K_1