Alleviating the sign problem with contour deformations in heavy-dense QCD

Emmanuel Ortiz Pacheco

ELTE Eötvös Loránd University, Budapest
MTA–ELTE Momentum Strongly Interacting Matter Research Group

with Dávid Pesznyák and Attila Pásztor

Lattice 2026, Maryland, USA

July 30, 2026

Outline

1. Sign problem and reweighting
2. Contour deformation in heavy-dense QCD
3. Manifold optimization
   - Local Fourier ansatz
4. Holomorphic flow
   - Partial flow

The Sign Problem

The path-integral weights are not generally real and positive, so they cannot be used directly as a Monte Carlo probability distribution.

$$Z_w = \int_{\mathcal{M}} \mathcal{D}\phi \, w[\phi], \qquad\qquad w[\phi] = e^{-S[\phi]}.$$

[1] hep-lat/1512.08764

[2] hep-lat/2007.05436

The Sign Problem

The path-integral weights are not generally real and positive, so they cannot be used directly as a Monte Carlo probability distribution.

$$Z_w = \int_{\mathcal{M}} \mathcal{D}\phi \, w[\phi], \qquad\qquad w[\phi] = e^{-S[\phi]}.$$

Phase-quenched reference theory (PQ): sample with the positive measure $r[\phi]$ and restore the complex phase by reweighting.

$$Z_r = \int_{\mathcal{M}} \mathcal{D}\phi \, r[\phi], \qquad\qquad r[\phi] = |w[\phi]| = e^{-\operatorname{Re} S[\phi]}.$$

[1] hep-lat/1512.08764

[2] hep-lat/2007.05436

The Sign Problem

The path-integral weights are not generally real and positive, so they cannot be used directly as a Monte Carlo probability distribution.

$$Z_w = \int_{\mathcal{M}} \mathcal{D}\phi \, w[\phi], \qquad\qquad w[\phi] = e^{-S[\phi]}.$$

Phase-quenched reference theory (PQ): sample with the positive measure $r[\phi]$ and restore the complex phase by reweighting.

$$Z_r = \int_{\mathcal{M}} \mathcal{D}\phi \, r[\phi], \qquad\qquad r[\phi] = |w[\phi]| = e^{-\,\mathrm{Re}\, S[\phi]}.$$

The complex-action problem limits first-principles studies of [1] [2]

- finite-density QCD,
- strongly correlated electrons away from half filling,
- dense nuclear matter,
- theories with topological terms,
- real-time quantum dynamics.

[1] hep-lat/1512.08764

[2] hep-lat/2007.05436

Reweighting and average phase

For any positive reference measure $r[\phi]$,

$$\langle O\rangle_w = \frac{\int_{\mathcal{M}} \mathcal{D}\phi\, O[\phi] w[\phi]}{\int_{\mathcal{M}} \mathcal{D}\phi\, w[\phi]} = \frac{\int_{\mathcal{M}} \mathcal{D}\phi\, O[\phi]\, \dfrac{w[\phi]}{r[\phi]}\, r[\phi]}{\int_{\mathcal{M}} \mathcal{D}\phi\, \dfrac{w[\phi]}{r[\phi]}\, r[\phi]} = \frac{\langle O\, w/r\rangle_r}{\langle w/r\rangle_r}.$$

Reweighting and average phase

For any positive reference measure $r[\phi]$,

$$\langle O\rangle_w = \frac{\int_{\mathcal{M}} \mathcal{D}\phi\, O[\phi] w[\phi]}{\int_{\mathcal{M}} \mathcal{D}\phi\, w[\phi]} = \frac{\int_{\mathcal{M}} \mathcal{D}\phi\, O[\phi]\, \dfrac{w[\phi]}{r[\phi]}\, r[\phi]}{\int_{\mathcal{M}} \mathcal{D}\phi\, \dfrac{w[\phi]}{r[\phi]}\, r[\phi]} = \frac{\langle O\, w/r\rangle_r}{\langle w/r\rangle_r}.$$

For phase reweighting,

$$r[\phi] = |w[\phi]|, \qquad \frac{w[\phi]}{r[\phi]} = e^{-i\ \mathrm{Im}\, S[\phi]} \equiv e^{i\theta[\phi]}.$$

Reweighting and average phase

For any positive reference measure $r[\phi]$,

$$\langle O \rangle_w = \frac{\int_{\mathcal{M}} \mathcal{D}\phi\, O[\phi] w[\phi]}{\int_{\mathcal{M}} \mathcal{D}\phi\, w[\phi]} = \frac{\int_{\mathcal{M}} \mathcal{D}\phi\, O[\phi]\, \dfrac{w[\phi]}{r[\phi]}\, r[\phi]}{\int_{\mathcal{M}} \mathcal{D}\phi\, \dfrac{w[\phi]}{r[\phi]}\, r[\phi]} = \frac{\langle O\, w/r \rangle_r}{\langle w/r \rangle_r}.$$

For phase reweighting,

$$r[\phi] = |w[\phi]|, \qquad \frac{w[\phi]}{r[\phi]} = e^{-i\ \mathrm{Im}\, S[\phi]} \equiv e^{i\theta[\phi]}.$$

$$\boxed{\sigma \equiv \left\langle \frac{w}{r} \right\rangle_r = \left\langle e^{i\theta} \right\rangle_r = \frac{Z_w}{Z_r}}$$

The **average phase** σ measures the severity of the sign problem

Reweighting and average phase

For any positive reference measure $r[\phi]$,

$$\langle O \rangle_w = \frac{\int_{\mathcal{M}} \mathcal{D}\phi\, O[\phi] w[\phi]}{\int_{\mathcal{M}} \mathcal{D}\phi\, w[\phi]} = \frac{\int_{\mathcal{M}} \mathcal{D}\phi\, O[\phi]\, \dfrac{w[\phi]}{r[\phi]}\, r[\phi]}{\int_{\mathcal{M}} \mathcal{D}\phi\, \dfrac{w[\phi]}{r[\phi]}\, r[\phi]} = \frac{\langle O\, w/r \rangle_r}{\langle w/r \rangle_r}.$$

For phase reweighting,

$$r[\phi] = |w[\phi]|, \qquad \frac{w[\phi]}{r[\phi]} = e^{-i\ \mathrm{Im}\ S[\phi]} \equiv e^{i\theta[\phi]}.$$

$$\boxed{\sigma \equiv \left\langle \frac{w}{r} \right\rangle_r = \left\langle e^{i\theta} \right\rangle_r = \frac{Z_w}{Z_r}}$$

The **average phase** σ measures the severity of the sign problem:

$$\sigma \simeq 1 \implies \text{mild phase cancellations,}$$
$$\sigma \simeq 0 \implies \text{severe sign problem.}$$

Contour deformation

We complexify the original variables and deform the integration manifold,

$$\mathcal{M} \longrightarrow \mathcal{M}', \qquad \phi \in \mathcal{M}, \qquad \phi' = \phi'(\phi) \in \mathcal{M}', \qquad \mathcal{J}_{ij}[\phi] = \frac{\partial \phi'_i(\phi)}{\partial \phi_j}.$$

[3] hep-lat/2007.05436

Contour deformation

We complexify the original variables and deform the integration manifold,

$$\mathcal{M} \longrightarrow \mathcal{M}', \qquad \phi \in \mathcal{M}, \qquad \phi' = \phi'(\phi) \in \mathcal{M}', \qquad \mathcal{J}_{ij}[\phi] = \frac{\partial \phi'_i(\phi)}{\partial \phi_j}.$$

Cauchy's theorem gives the same target partition function on any admissible homologous contour[3]:

$$\Longrightarrow Z_w = \int_{\mathcal{M}} \mathcal{D}\phi \, w[\phi] = \int_{\mathcal{M}'} \mathcal{D}\phi' \, \underbrace{w[\phi']}_{e^{-S[\phi']}} = \int_{\mathcal{M}} \mathcal{D}\phi \; \det \mathcal{J}[\phi] \, w[\phi'(\phi)] = \int_{\mathcal{M}} \mathcal{D}\phi \, e^{-\overbrace{S_{\text{eff}}[\phi]}^{S_{\text{eff}}[\phi] \equiv S[\phi'(\phi)] - \log \det \mathcal{J}[\phi]}}$$

[3] hep-lat/2007.05436

Contour deformation

We complexify the original variables and deform the integration manifold,

$$\mathcal{M} \longrightarrow \mathcal{M}', \qquad \phi \in \mathcal{M}, \qquad \phi' = \phi'(\phi) \in \mathcal{M}', \qquad \mathcal{J}_{ij}[\phi] = \frac{\partial \phi_i'(\phi)}{\partial \phi_j}.$$

Cauchy's theorem gives the same target partition function on any admissible homologous contour[3]:

$$\Longrightarrow Z_w = \int_{\mathcal{M}} \mathcal{D}\phi\, w[\phi] = \int_{\mathcal{M}'} \mathcal{D}\phi' \underbrace{w[\phi']}_{e^{-S[\phi']}} = \int_{\mathcal{M}} \mathcal{D}\phi\, \det \mathcal{J}[\phi]\, w[\phi'(\phi)] = \int_{\mathcal{M}} \mathcal{D}\phi\, e^{-\overbrace{S_{\rm eff}[\phi]}^{S_{\rm eff}[\phi] \equiv S[\phi'(\phi)] - \log\det\mathcal{J}[\phi]}}$$

Phase-quenched partition function and phase reweighting on the deformed manifold

$$Z_r^{\rm def} = \int_{\mathcal{M}} \mathcal{D}\phi\, \left|\det \mathcal{J}(\phi) w[\phi'(\phi)]\right| \qquad \Longrightarrow \qquad r^{\rm def}[\phi] = \left|e^{-S_{\rm eff}[\phi]}\right| = |\det \mathcal{J}[\phi]| e^{-\,{\rm Re}\, S[\phi'(\phi)]},$$

$$\frac{e^{-S_{\rm eff}[\phi]}}{|e^{-S_{\rm eff}[\phi]}|} = \frac{\det \mathcal{J}[\phi]}{|\det \mathcal{J}[\phi]|} e^{-i\,{\rm Im}\, S[\phi'(\phi)]} \equiv e^{i\theta_{\rm def}[\phi]},$$

[3] hep-lat/2007.05436

Contour deformation

We complexify the original variables and deform the integration manifold,

$$\mathcal{M} \longrightarrow \mathcal{M}', \qquad \phi \in \mathcal{M}, \qquad \phi' = \phi'(\phi) \in \mathcal{M}', \qquad \mathcal{J}_{ij}[\phi] = \frac{\partial \phi_i'(\phi)}{\partial \phi_j}.$$

Cauchy's theorem gives the same target partition function on any admissible homologous contour[3]:

$$\Longrightarrow Z_w = \int_{\mathcal{M}} \mathcal{D}\phi\, w[\phi] = \int_{\mathcal{M}'} \mathcal{D}\phi' \underbrace{w[\phi']}_{e^{-S[\phi']}} = \int_{\mathcal{M}} \mathcal{D}\phi \,\det \mathcal{J}[\phi]\, w[\phi'(\phi)] = \int_{\mathcal{M}} \mathcal{D}\phi\, e^{-\overbrace{S_{\mathrm{eff}}[\phi]}^{S_{\mathrm{eff}}[\phi] \equiv S[\phi'(\phi)] - \log \det \mathcal{J}[\phi]}}$$

Phase-quenched partition function and phase reweighting on the deformed manifold

$$Z_r^{\mathrm{def}} = \int_{\mathcal{M}} \mathcal{D}\phi\, \left|\det \mathcal{J}(\phi) w[\phi'(\phi)]\right| \qquad \Longrightarrow \qquad r^{\mathrm{def}}[\phi] = \left|e^{-S_{\mathrm{eff}}[\phi]}\right| = |\det \mathcal{J}[\phi]| e^{-\,\mathrm{Re}\, S[\phi'(\phi)]},$$

$$\frac{e^{-S_{\mathrm{eff}}[\phi]}}{|e^{-S_{\mathrm{eff}}[\phi]}|} = \frac{\det \mathcal{J}[\phi]}{|\det \mathcal{J}[\phi]|} e^{-i\,\mathrm{Im}\, S[\phi'(\phi)]} \equiv e^{i\theta_{\mathrm{def}}[\phi]}, \qquad \boxed{\sigma_{\mathrm{def}} = \left\langle \frac{\det \mathcal{J}}{|\det \mathcal{J}|} e^{-i\,\mathrm{Im}\, S[\phi']} \right\rangle_r^{\mathrm{def}} = \frac{Z_w}{Z_r^{\mathrm{def}}}}$$

[3] hep-lat/2007.05436

Contour deformation

We complexify the original variables and deform the integration manifold,

$$\mathcal{M} \longrightarrow \mathcal{M}', \qquad \phi \in \mathcal{M}, \qquad \phi' = \phi'(\phi) \in \mathcal{M}', \qquad \mathcal{J}_{ij}[\phi] = \frac{\partial \phi'_i(\phi)}{\partial \phi_j}.$$

Cauchy's theorem gives the same target partition function on any admissible homologous contour[3]:

$$\Longrightarrow Z_w = \int_{\mathcal{M}} \mathcal{D}\phi\, w[\phi] = \int_{\mathcal{M}'} \mathcal{D}\phi'\, \underbrace{w[\phi']}_{e^{-S[\phi']}} = \int_{\mathcal{M}} \mathcal{D}\phi\, \det \mathcal{J}[\phi]\, w[\phi'(\phi)] = \int_{\mathcal{M}} \mathcal{D}\phi\, e^{-\overbrace{S_{\text{eff}}[\phi]}^{S_{\text{eff}}[\phi] \equiv S[\phi'(\phi)] - \log \det \mathcal{J}[\phi]}}$$

Phase-quenched partition function and phase reweighting on the deformed manifold

$$Z_r^{\text{def}} = \int_{\mathcal{M}} \mathcal{D}\phi\, \left|\det \mathcal{J}(\phi) w[\phi'(\phi)]\right| \qquad \Longrightarrow \qquad r^{\text{def}}[\phi] = \left|e^{-S_{\text{eff}}[\phi]}\right| = |\det \mathcal{J}[\phi]| e^{-\,\text{Re}\, S[\phi'(\phi)]},$$

$$\frac{e^{-S_{\text{eff}}[\phi]}}{|e^{-S_{\text{eff}}[\phi]}|} = \frac{\det \mathcal{J}[\phi]}{|\det \mathcal{J}[\phi]|} e^{-i\,\text{Im}\, S[\phi'(\phi)]} \equiv e^{i\theta_{\text{def}}[\phi]}, \qquad \boxed{\sigma_{\text{def}} = \left\langle \frac{\det \mathcal{J}}{|\det \mathcal{J}|} e^{-i\,\text{Im}\, S[\phi']} \right\rangle_r^{\text{def}} = \frac{Z_w}{Z_r^{\text{def}}}}$$

Z_w is contour invariant, whereas Z_r^{def} depends on the contour.

[3] **hep-lat/2007.05436**

Heavy-dense QCD in Polyakov gauge

- The Wilson gauge action is

$$S_g = -\frac{\beta}{6}\sum_x \sum_{\mu<\nu}\left[\mathrm{Tr}\,U_{\mu\nu}(x) + \mathrm{Tr}\,U_{\mu\nu}(x)^{-1}\right], \qquad U_{\mu\nu}^{-1} = U_{\mu\nu}^{\dagger} \quad \text{only for} \quad SU(3).$$

[4] hep-lat/2007.05436

[5] hep-th/2311.06343

Heavy-dense QCD in Polyakov gauge

- The Wilson gauge action is

$$S_g = -\frac{\beta}{6}\sum_x \sum_{\mu<\nu}\left[\mathrm{Tr}\, U_{\mu\nu}(x) + \mathrm{Tr}\, U_{\mu\nu}(x)^{-1}\right], \qquad U_{\mu\nu}^{-1} = U_{\mu\nu}^{\dagger} \quad \text{only for} \quad SU(3).$$

- The heavy-dense fermion action is

$$S_f = -2N_f \sum_{\vec{x}} \log\left[\det\left(\mathbf{1} + he^{\hat{\mu}}P(\vec{x})\right)\det\left(\mathbf{1} + he^{-\hat{\mu}}\left[P(\vec{x})\right]^{-1}\right)\right], \qquad h = (2\kappa)^{N_t},\ \hat{\mu} = \mu a N_t = \frac{\mu}{T}.$$

In **Polyakov gauge**, before and after deformation

$$P(\vec{x}) = \mathrm{diag}\left(e^{i\theta_1(\vec{x})}, e^{i\theta_2(\vec{x})}, e^{-i[\theta_1(\vec{x})+\theta_2(\vec{x})]}\right) \in SU(3) \quad \longrightarrow \quad P(\vec{x}) \in SL(3,\mathbb{C}), \quad \theta_i(\vec{x}) \in \mathbb{C}.$$

***Only the Polyakov angles are deformed.**

[4] hep-lat/2007.05436

[5] hep-th/2311.06343

Heavy-dense QCD in Polyakov gauge

- The Wilson gauge action is

$$S_g = -\frac{\beta}{6} \sum_x \sum_{\mu<\nu} \left[\mathrm{Tr}\, U_{\mu\nu}(x) + \mathrm{Tr}\, U_{\mu\nu}(x)^{-1} \right], \qquad U_{\mu\nu}^{-1} = U_{\mu\nu}^{\dagger} \quad \text{only for} \quad SU(3).$$

- The heavy-dense fermion action is

$$S_f = -2N_f \sum_{\vec{x}} \log\left[\det\left(\mathbf{1} + h e^{\hat{\mu}} P(\vec{x}) \right) \det\left(\mathbf{1} + h e^{-\hat{\mu}} \left[P(\vec{x}) \right]^{-1} \right) \right], \qquad h = (2\kappa)^{N_t},\ \hat{\mu} = \mu a N_t = \frac{\mu}{T}.$$

In **Polyakov gauge**, before and after deformation

$$P(\vec{x}) = \mathrm{diag}\left(e^{i\theta_1(\vec{x})}, e^{i\theta_2(\vec{x})}, e^{-i[\theta_1(\vec{x})+\theta_2(\vec{x})]} \right) \in SU(3) \quad \longrightarrow \quad P(\vec{x}) \in SL(3,\mathbb{C}), \quad \theta_i(\vec{x}) \in \mathbb{C}.$$

***Only the Polyakov angles are deformed.**

- Haar/Vandermonde contribution

$$S_V = -\sum_{\vec{x}} \log\left[\sin^2\left(\frac{\theta_1(\vec{x}) - \theta_2(\vec{x})}{2} \right) \sin^2\left(\frac{\theta_1(\vec{x}) + 2\theta_2(\vec{x})}{2} \right) \sin^2\left(\frac{2\theta_1(\vec{x}) + \theta_2(\vec{x})}{2} \right) \right].$$

***Regularize the logarithmic singularities of S_V inside the argument w of $sin(w)$ via $w \to w + \epsilon$, with $\epsilon \to 0$.**

[4] hep-lat/2007.05436

[5] hep-th/2311.06343

Heavy-dense QCD in Polyakov gauge

- The Wilson gauge action is

$$S_g = -\frac{\beta}{6} \sum_x \sum_{\mu<\nu} \left[\mathrm{Tr}\, U_{\mu\nu}(x) + \mathrm{Tr}\, U_{\mu\nu}(x)^{-1} \right], \qquad U_{\mu\nu}^{-1} = U_{\mu\nu}^{\dagger} \quad \text{only for} \quad SU(3).$$

- The heavy-dense fermion action is

$$S_f = -2N_f \sum_{\vec{x}} \log\left[\det\left(\mathbf{1} + h e^{\hat{\mu}} P(\vec{x}) \right) \det\left(\mathbf{1} + h e^{-\hat{\mu}} \left[P(\vec{x}) \right]^{-1} \right) \right], \qquad h = (2\kappa)^{N_t},\ \hat{\mu} = \mu a N_t = \frac{\mu}{T}.$$

In **Polyakov gauge**, before and after deformation

$$P(\vec{x}) = \mathrm{diag}\left(e^{i\theta_1(\vec{x})}, e^{i\theta_2(\vec{x})}, e^{-i[\theta_1(\vec{x})+\theta_2(\vec{x})]} \right) \in SU(3) \quad \longrightarrow \quad P(\vec{x}) \in SL(3,\mathbb{C}), \quad \theta_i(\vec{x}) \in \mathbb{C}.$$

***Only the Polyakov angles are deformed.**

- Haar/Vandermonde contribution

$$S_V = -\sum_{\vec{x}} \log\left[\sin^2\left(\frac{\theta_1(\vec{x}) - \theta_2(\vec{x})}{2} \right) \sin^2\left(\frac{\theta_1(\vec{x}) + 2\theta_2(\vec{x})}{2} \right) \sin^2\left(\frac{2\theta_1(\vec{x}) + \theta_2(\vec{x})}{2} \right) \right].$$

***Regularize the logarithmic singularities of S_V inside the argument w of $sin(w)$ via $w \to w + \epsilon$, with $\epsilon \to 0$.**

- The effective action is [4,5]

$$S_{\text{eff}} = S_g + S_V + S_f + S_{\mathcal{J}},$$

[4] hep-lat/2007.05436

[5] hep-th/2311.06343

Manifold optimization

The sign problem is weaker when $\sigma_{\text{def}} = Z_w / Z_r^{\text{def}}$ is larger. Since Z_w does not depend on the manifold parameters, maximizing σ_{def} is equivalent to minimizing the deformed phase-quenched partition function.

Parameterize a family of manifolds by a finite set $\{p\}$,

$$\mathcal{M} \longrightarrow \mathcal{M}'(\{p\}),$$

and optimize $\{p\}$ using the cost function [6] [7]

$$\mathcal{F}(\{p\}) = -\log \sigma(\{p\}) = -\log Z_w + \log Z_r^{\text{def}}(\{p\}).$$

[6] **hep-lat/1705.05605**

[7] **hep-lat/2311.06343**

Manifold optimization

The sign problem is weaker when $\sigma_{\text{def}} = Z_w / Z_r^{\text{def}}$ is larger. Since Z_w does not depend on the manifold parameters, maximizing σ_{def} is equivalent to minimizing the deformed phase-quenched partition function.

Parameterize a family of manifolds by a finite set $\{p\}$,

$$\mathcal{M} \longrightarrow \mathcal{M}'(\{p\}),$$

and optimize $\{p\}$ using the cost function [6] [7]

$$\mathcal{F}(\{p\}) = -\log \sigma(\{p\}) = -\log Z_w + \log Z_r^{\text{def}}(\{p\}).$$

Only the phase-quenched partition function changes, so

$$\nabla_p \mathcal{F}(\{p\}) = \nabla_p \log Z_r^{\text{def}}(\{p\}) = -\left\langle \nabla_p \operatorname{Re} S_r^{\text{eff}}(\{p\}) \right\rangle_r^{\text{def}}.$$

Consequently,

$$\min_{\{p\}} \mathcal{F}(\{p\}) \quad \Longleftrightarrow \quad \max_{\{p\}} \sigma(\{p\}).$$

[6] **hep-lat/1705.05605**

[7] **hep-lat/2311.06343**

Manifold optimization

Local Fourier ansatz[1 2]

$$\widetilde{\theta}_k(\vec{x}) = \theta_k(\vec{x}) + i \sum_{m,n=0}^{N} \lambda_{mn}^{(k)} \cos\big[m\theta_1(\vec{x}) + n\theta_2(\vec{x})\big]$$

$k = 1, 2$. There are $2(N+1)^2$ real parameters. The same coefficients $\lambda_{mn}^{(k)}$ are used at every spatial site.

PQ sample on $\mathcal{M}(\lambda_r) \longrightarrow \nabla_\lambda \mathcal{F} = -\left\langle \nabla_\lambda S_r^{\mathrm{eff}} \right\rangle_r \longrightarrow \lambda_{r+1}$

N_{opt}: number of successive parameter updates.

[1] **arXiv:1705.05605 [hep-lat]**.
[7] **arXiv:2311.06343 [hep-lat]**.

Manifold optimization

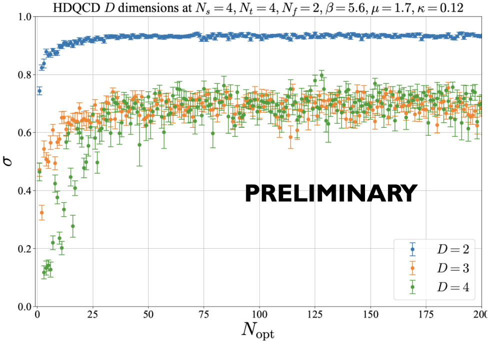
Local Fourier ansatz[1,2]

$$\widetilde{\theta}_k(\vec{x}) = \theta_k(\vec{x}) + i \sum_{m,n=0}^{N} \lambda_{mn}^{(k)} \cos\big[m\theta_1(\vec{x}) + n\theta_2(\vec{x})\big]$$

$k = 1, 2$. There are $2(N+1)^2$ real parameters. The same coefficients $\lambda_{mn}^{(k)}$ are used at every spatial site.

PQ sample on $\mathcal{M}(\lambda_r) \longrightarrow \nabla_\lambda \mathcal{F} = -\left\langle \nabla_\lambda S_r^{\text{eff}} \right\rangle_r \longrightarrow \lambda_{r+1}$

N_{opt}: number of successive parameter updates.



Rapid increase followed by a plateau after roughly 30–50 updates: $D = 2 \simeq 0.93$, while $D = 3, 4 \simeq 0.70$.

[1] arXiv:1705.05605 [hep-lat].

[7] arXiv:2311.06343 [hep-lat].

Holomorphic Flow Equations

1. The fields evolve according to,

$$\frac{\mathrm{d}\phi'_i}{\mathrm{d}t} = \left(\frac{\partial S[\phi']}{\partial \phi'_i}\right)^*, \qquad \phi'_i(t=0) = \phi_i.$$

2. The Jacobian obeys

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}t} = (\mathcal{H}\mathcal{J})^*, \qquad \mathcal{H}_{ij} = \frac{\partial^2 S}{\partial \phi'_i \partial \phi'_j},$$

with $\phi' = \phi$, $\mathcal{J} = \mathbf{1}$, and $\det \mathcal{J} = 1$ at $t = 0$. [1]

The flow defines a family of deformed integration manifolds parameterized by the fictitious flow time t, with

$$S_{\text{eff}}[\phi] = S[\phi'(\phi)] - \log\det\mathcal{J}[\phi].$$

[1] hep-lat/1512.08764

Holomorphic Flow Equations

1. The fields evolve according to,

$$\frac{\mathrm{d}\phi_i'}{\mathrm{d}t} = \left(\frac{\partial S[\phi']}{\partial \phi_i'}\right)^*, \qquad \phi_i'(t=0) = \phi_i.$$

2. The Jacobian obeys

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}t} = (\mathcal{H}\mathcal{J})^*, \qquad \mathcal{H}_{ij} = \frac{\partial^2 S}{\partial \phi_i' \partial \phi_j'},$$

with $\phi' = \phi$, $\mathcal{J} = \mathbf{1}$, and $\det \mathcal{J} = 1$ at $t = 0$. [1]

The flow defines a family of deformed integration manifolds parameterized by the fictitious flow time t, with

$$S_{\text{eff}}[\phi] = S[\phi'(\phi)] - \log\det \mathcal{J}[\phi].$$

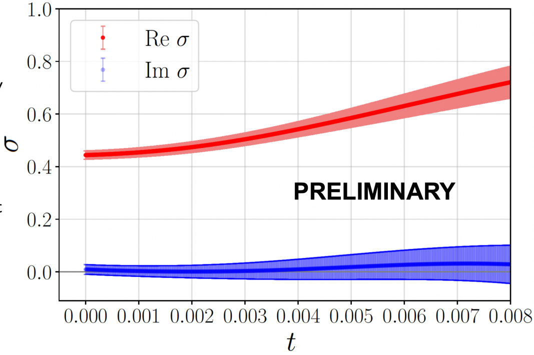
Holomorphic-flow identities:

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re} S[\phi'(t)] = \sum_i \left|\frac{\partial S}{\partial \phi_i'}\right|^2 \geq 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im} S[\phi'(t)] = 0.$$

[1] **hep-lat/1512.08764**

Partial flow (Polyakov-loop eigenvalues)

- Only the Polyakov-loop eigenangles are flowed.
- The real part of the average phase increases with flow time. The imaginary part remains consistent with zero.
- At finite flow time, the phase cancellations are reduced and σ improves.
- For now, all configurations are gather at zero flowtime and reweighted to finite flowtime (leading to the errors growing with flowtime).
- Eventually, we plan to use a cheap approximation of the finite flowtime Jacobian in the accept-reject step, and later reweight to the exact Jacobian.



$D = 4,\ N_s = 4,\ N_t = 4,\ N_f = 2,\ \Delta t = 10^{-8},\ \mu = 1.0,\ \kappa = 0.12,\ \epsilon = 10^{-2},\ \tau = 150,\ \beta = 5.6.$

* Local Fourier gives a plateau average phase around $\sigma \approx 0.6$ for this particular set of parameters.

Conclusions

Contour deformation changes the sampling problem without changing the target theory.

For the present application to a four-dimensional lattice gauge theory in the heavy-dense limit of QCD:

- The local Fourier manifold provides a computational improvement with a block-local Jacobian.
- The holomorphic field and Jacobian flows provide an independent, action-driven deformation.
- We found that at large-flow region the data becomes statistically noisy due to reweighting. To-do: Finite-flow simulations.
- Both optimized and flowed manifolds reduce phase cancellations at finite deformation in 4D HDQCD.

Conclusions

Contour deformation changes the sampling problem without changing the target theory.

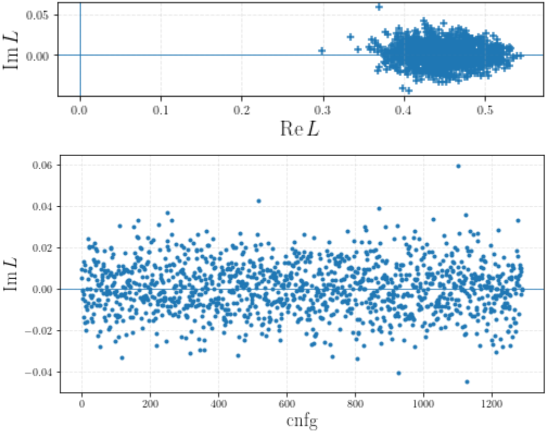
For the present application to a four-dimensional lattice gauge theory in the heavy-dense limit of QCD:

- The local Fourier manifold provides a computational improvement with a block-local Jacobian.
- The holomorphic field and Jacobian flows provide an independent, action-driven deformation.
- We found that at large-flow region the data becomes statistically noisy due to reweighting. To-do: Finite-flow simulations.
- Both optimized and flowed manifolds reduce phase cancellations at finite deformation in 4D HDQCD.

Thanks!

Backup

Polyakov-loop diagnostics



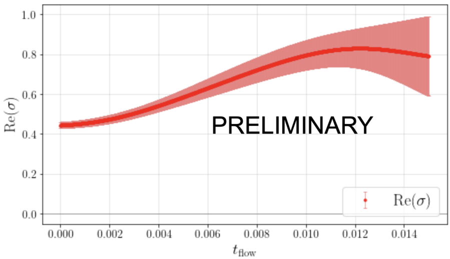
The volume-averaged traced Polyakov loop is

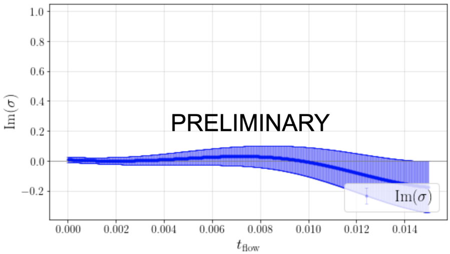
$$L = \frac{1}{3N_s^3} \sum_{\vec{x}} \mathrm{Tr}\, P_{\vec{x}}.$$

The complex-plane distribution and Monte Carlo history provide compact diagnostics of phase fluctuations and balance configurations.

- Currently 2000 configurations.

Partial flow at large t_{flow}





Parameters: $D = 4$, $N_s = 4$, $N_t = 4$, $N_f = 2$, $\Delta t = 10^{-8}$, $\mu = 1.0$, $\kappa = 0.12$, $\epsilon = 10^{-2}$, $\tau = 150$, $\beta = 5.6$;