Comparison of various extrapolation schemes for the equation of state at finite density

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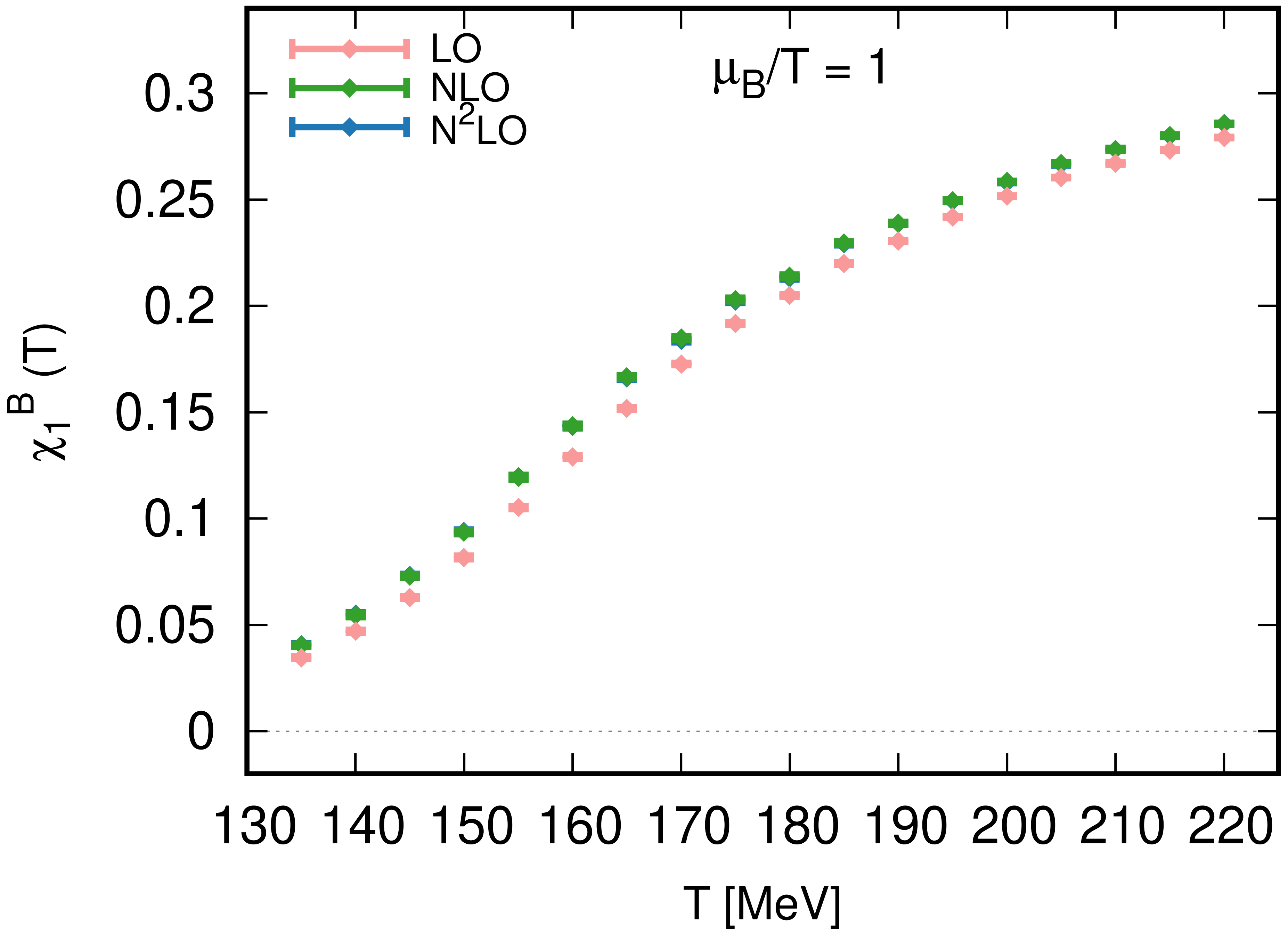
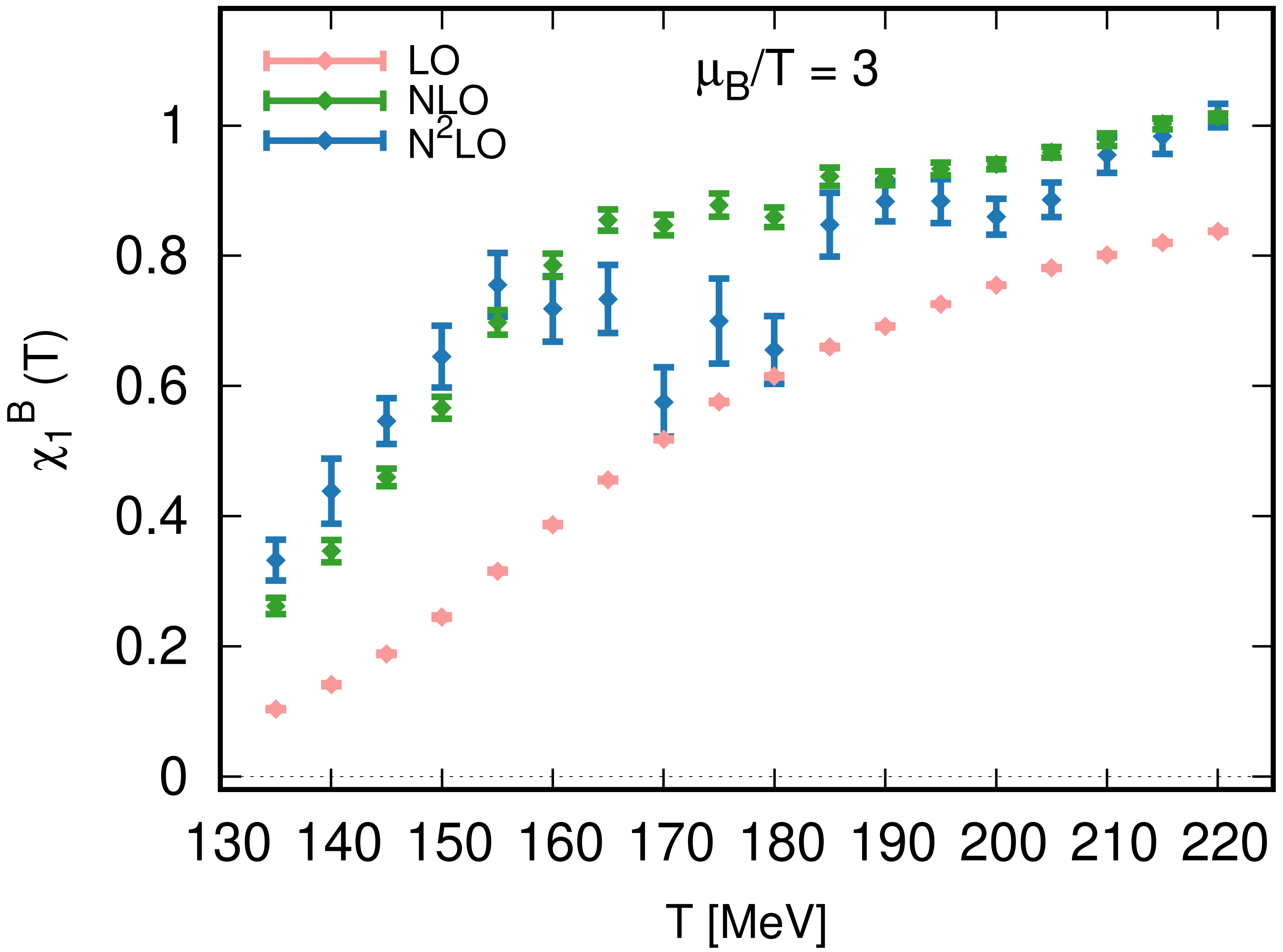
Motivation

- Equation of State (EoS): crucial input for hydrodynamical simulations of heavy ion collisions and Neutron stars.
- Known with good precision at zero chemical potential through lattice QCD simulations.
- Extrapolation schemes for EoS at finite density:
  - Taylor series (PRD 66, 074507 (2002), PRD 71, 054508 (2005))
  - T'-Expansion (PRL 126, 232001 (2021))
  - Entropy Contour based expansion (PRC 113, L012201 (2026))
- With high precision data set for $16^3 \times 8$ lattice (≈ 2 million configurations per temperature), we compare the schemes at higher orders than those explored in the original works where continuum extrapolated fluctuations were used.

Taylor Series

(PRD 66, 074507 (2002), Figs. taken from PRL 126, 232001 (2021))

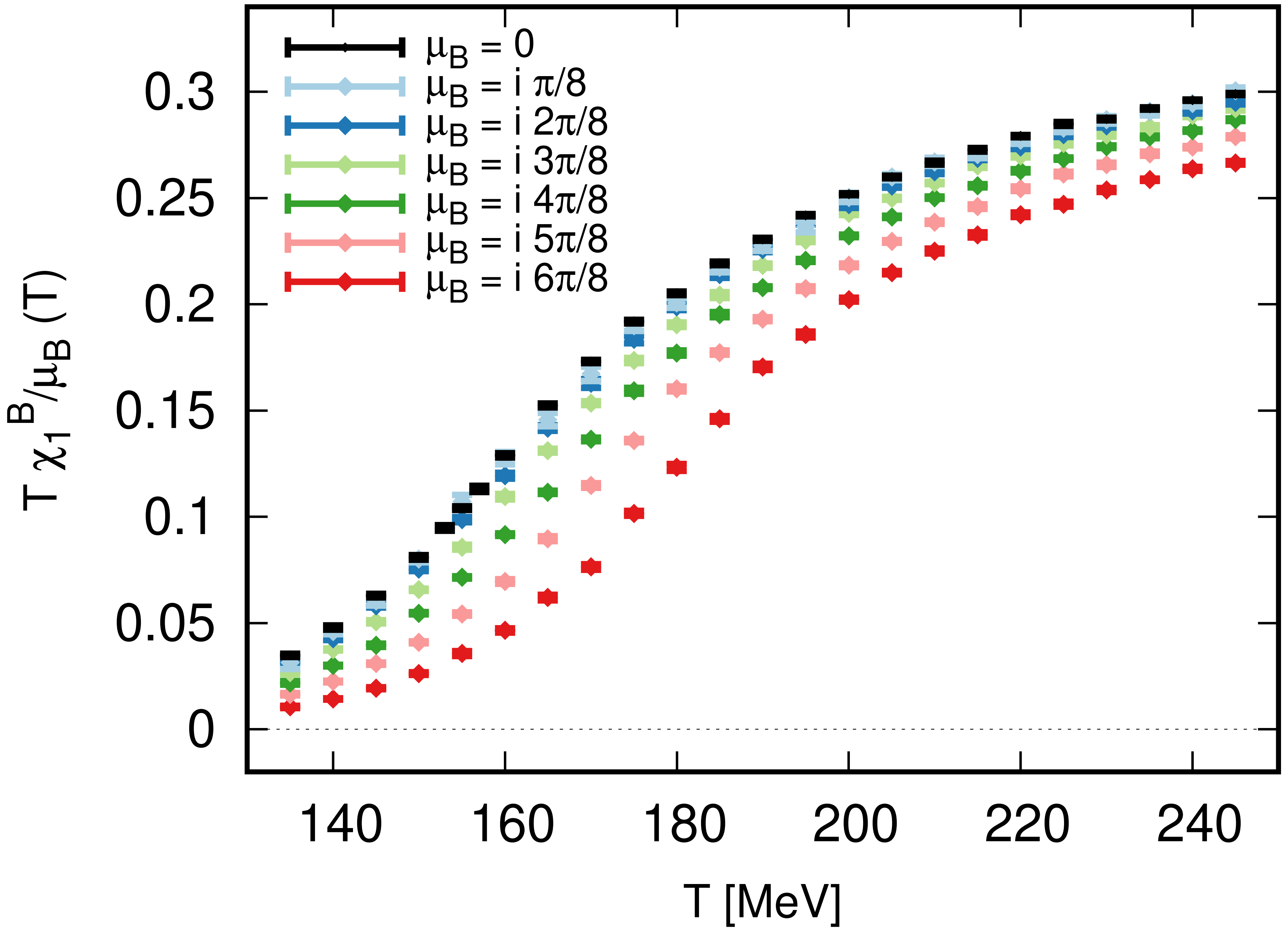


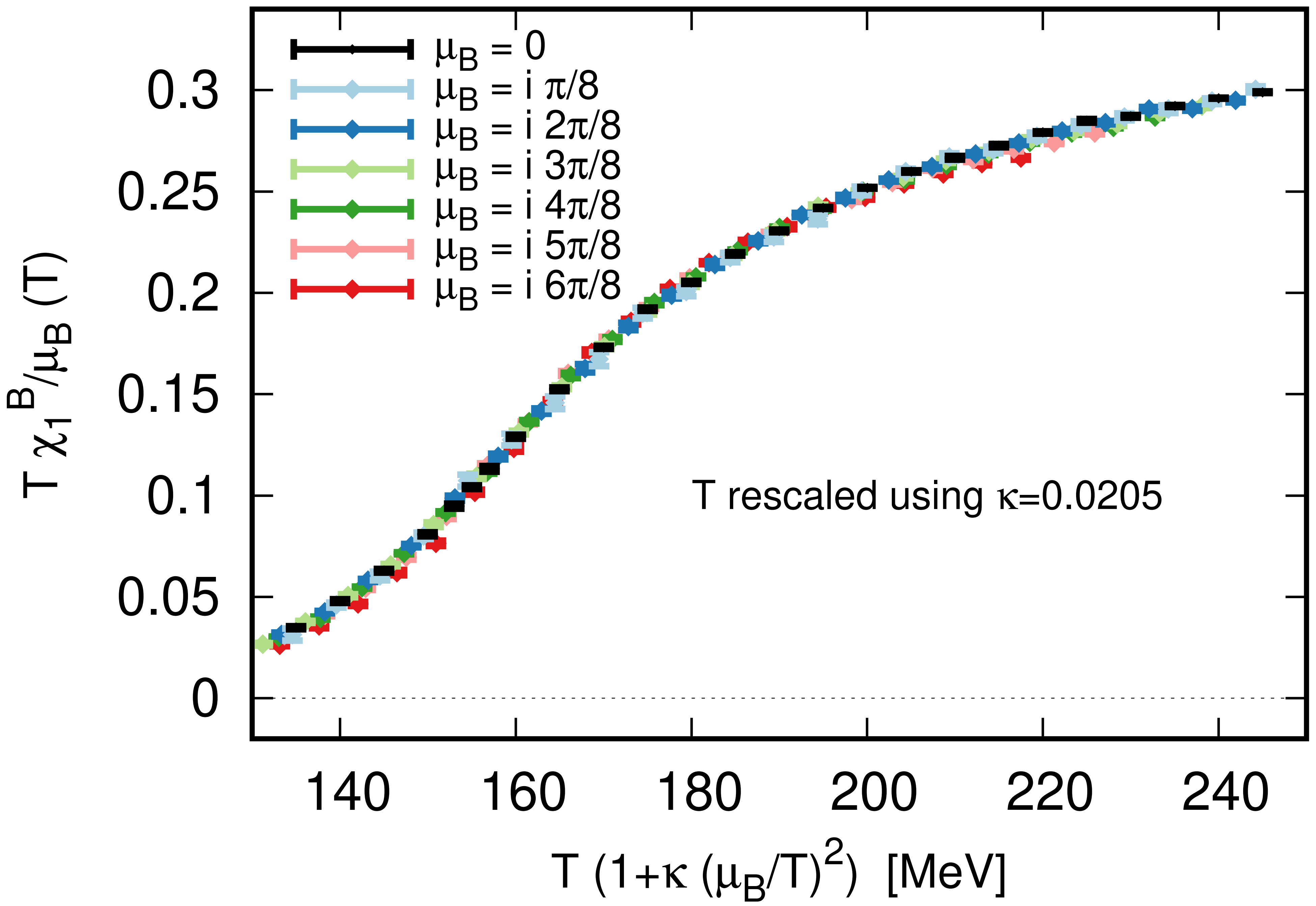


- $\dfrac{p(T,\mu_B)}{T^4} = \displaystyle\sum_{n=0} \frac{1}{(2n)!}\chi_{2n}^B(T,\,0)\left(\frac{\mu_B}{T}\right)^{2n}; \quad \chi_1^B(T,\mu_B) = \left(\frac{\partial}{\partial \mu_B/T}\right)\frac{p(T,\mu_B)}{T^4} = \sum_{n=1}\frac{1}{(2n-1)!}\chi_{2n}^B(T,\,0)\left(\frac{\mu_B}{T}\right)^{2n-1}$

T'-Expansion

(PRL 126, 232001 (2021))

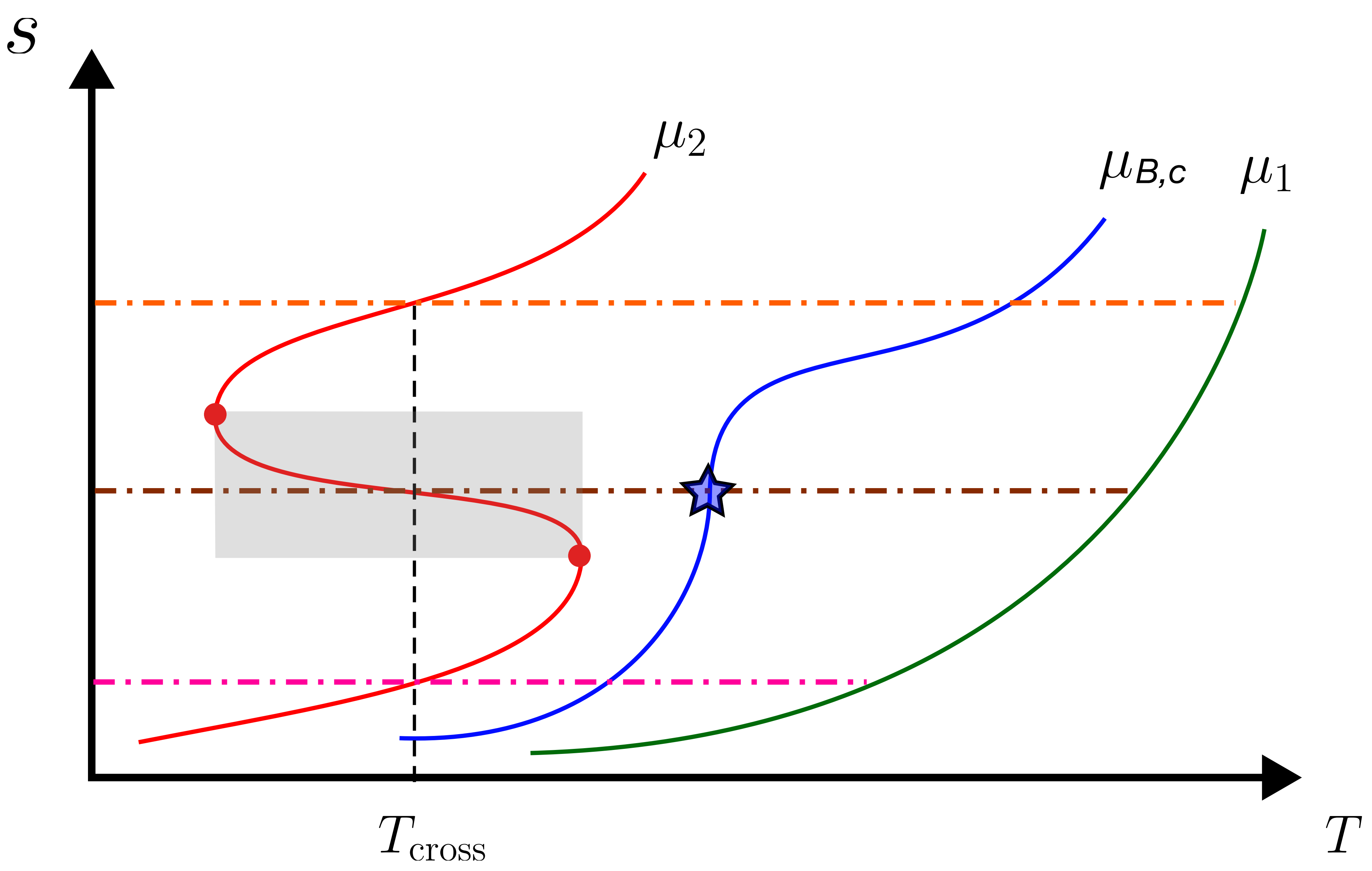


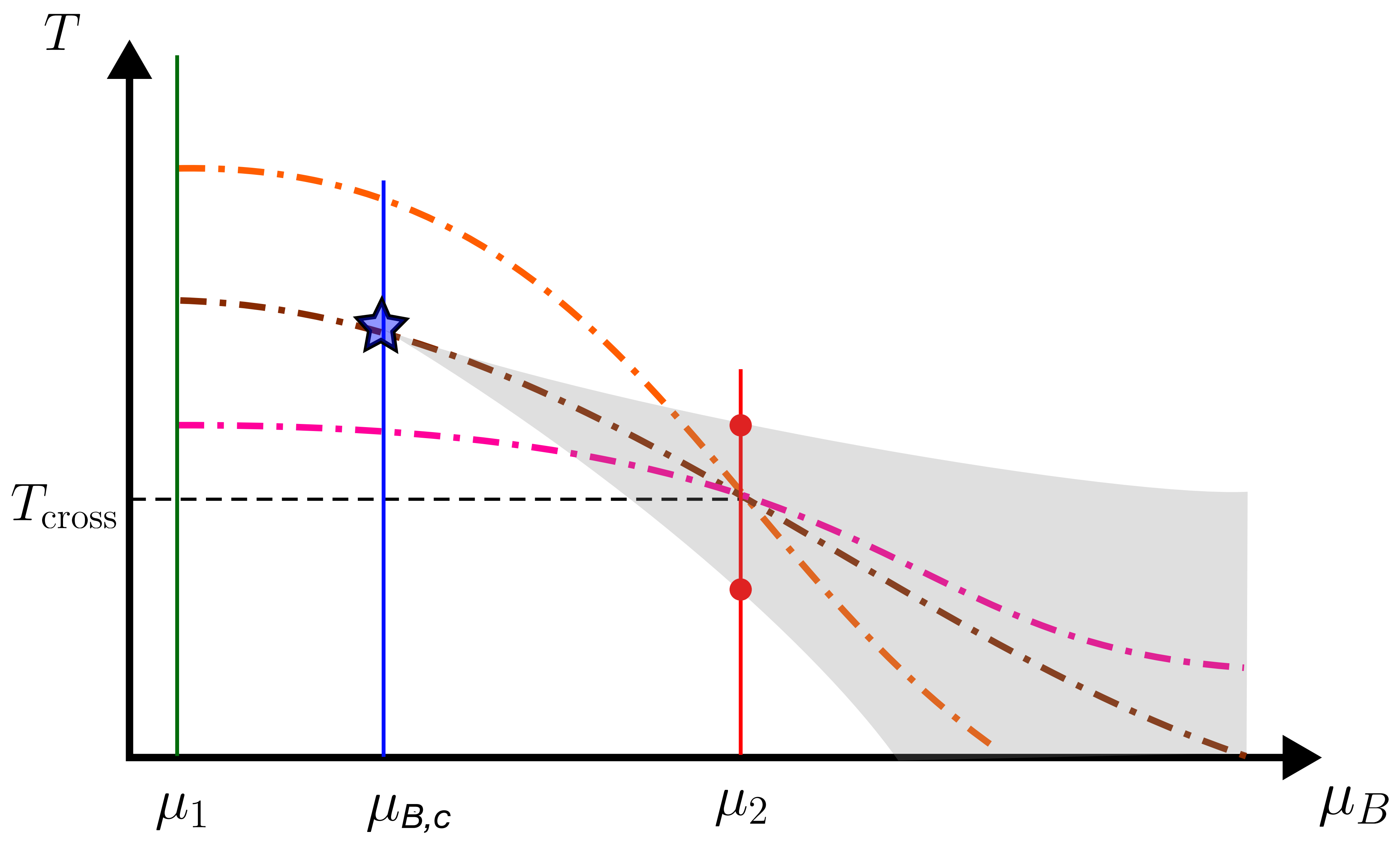


- $\dfrac{\chi_1^B(T,\mu_B)}{\hat{\mu}_B} = \chi_2^B(T',\,0); \quad T'(T,\mu_B) = T[1 + \kappa_2(T)\hat{\mu}_B^2 + \kappa_4(T)\hat{\mu}_B^4 + \ldots]$

Entropy based Contour

(PRC 113, L012201 (2026))





- $s(T_s, \mu_B) = s(T_0, 0); \quad T_s(T_0, \mu_B) = T_0 + \alpha_2(T_0)\dfrac{\mu_B^2}{2} + \ldots$

Different Schemes

(Extended to higher orders)

- **Entropy based scheme:** $\mathbf{s(T, \mu_B) = s(T_0, 0)}$
  - $T(T_0, \mu_B) = T_0(1 + \alpha_2(T_0)\hat{\mu}_B^2 + \alpha_4(T_0)\hat{\mu}_B^4 + \alpha_6(T_0)\hat{\mu}_B^6 + \alpha_8(T_0)\hat{\mu}_B^8 + \ldots)$

- **Baryon density based scheme:** $\mathbf{\dfrac{n_B(T, \mu_B)}{\hat{\mu}_B} = \chi_2^B(T_0, 0)}$
  - $T_0(T, \mu_B) = T(1 + \kappa_2(T)\hat{\mu}_B^2 + \kappa_4(T)\hat{\mu}_B^4 + \kappa_6(T)\hat{\mu}_B^6 + \ldots)$

- The coefficients $\kappa_6(T)$ and $\alpha_8(T_0)$ require the eighth order fluctuation $\chi_8^B(T)$.

Observable Matching

(Implicit vs Explicit)

$$\begin{tikzcd} \mathcal{O}(T, \mu_B) \arrow[r, shift right, "\text{Explicit } T_0(T, \mu_B)"'] & \mathcal{O}(T_0, 0) \arrow[l, shift right, "\text{Implicit } T(T_0, \mu_B)"'] \end{tikzcd}$$

Method

- For the n_B-based scheme at the leading order (LO):

$$\kappa_2(T) = \frac{1}{6T}\frac{\chi_4^B(T)}{\chi_2^{B'}(T)}$$

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- Similarly, for the s-based scheme at LO:

$$\alpha_2(T_0) = -\frac{2T_0\chi_2^B(T_0) + T_0^2\chi_2^{B'}(T_0)}{s'(T_0)}$$

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- The inverse relations allow us to express the lattice observables in terms of the expansion coefficients $\Rightarrow$ Jointly fitting $\{\chi_{2n}^B(T)\}$ by modeling $\{\kappa_{2n}(T)\}$ or $\{\alpha_{2n}(T)\}$
- Third order temperature derivatives of $\chi_{2n}^B(T)$ (or $s(T)$) are required for the highest coefficients, $\kappa_6(T)$(or $\alpha_8(T)$) $\Rightarrow$ Need higher degree splines.

Bayesian spline fitting to determine derivatives

(Green (1995), *Biometrika* 82(4), 711–732; Dimatteo et al. (2001), *Biometrika* 88(4), 1055–1071)

- We use Reversible Jump Markov Chain Monte Carlo to generate an ensemble of knot configurations, $\{(k_i, \vec{\xi}_i)\}$ with the following "action":

$$p(y\,|\,k, \vec{\xi}) = \exp\left(-\frac{1}{2}\mathrm{BIC}(k)\right); \quad \mathrm{BIC}(k) = (k + d + 1)\ln(n) - 2\ln(\hat{L})$$

where,

k: internal knots

$\vec{\xi}$: location of knots

p: degree of spline

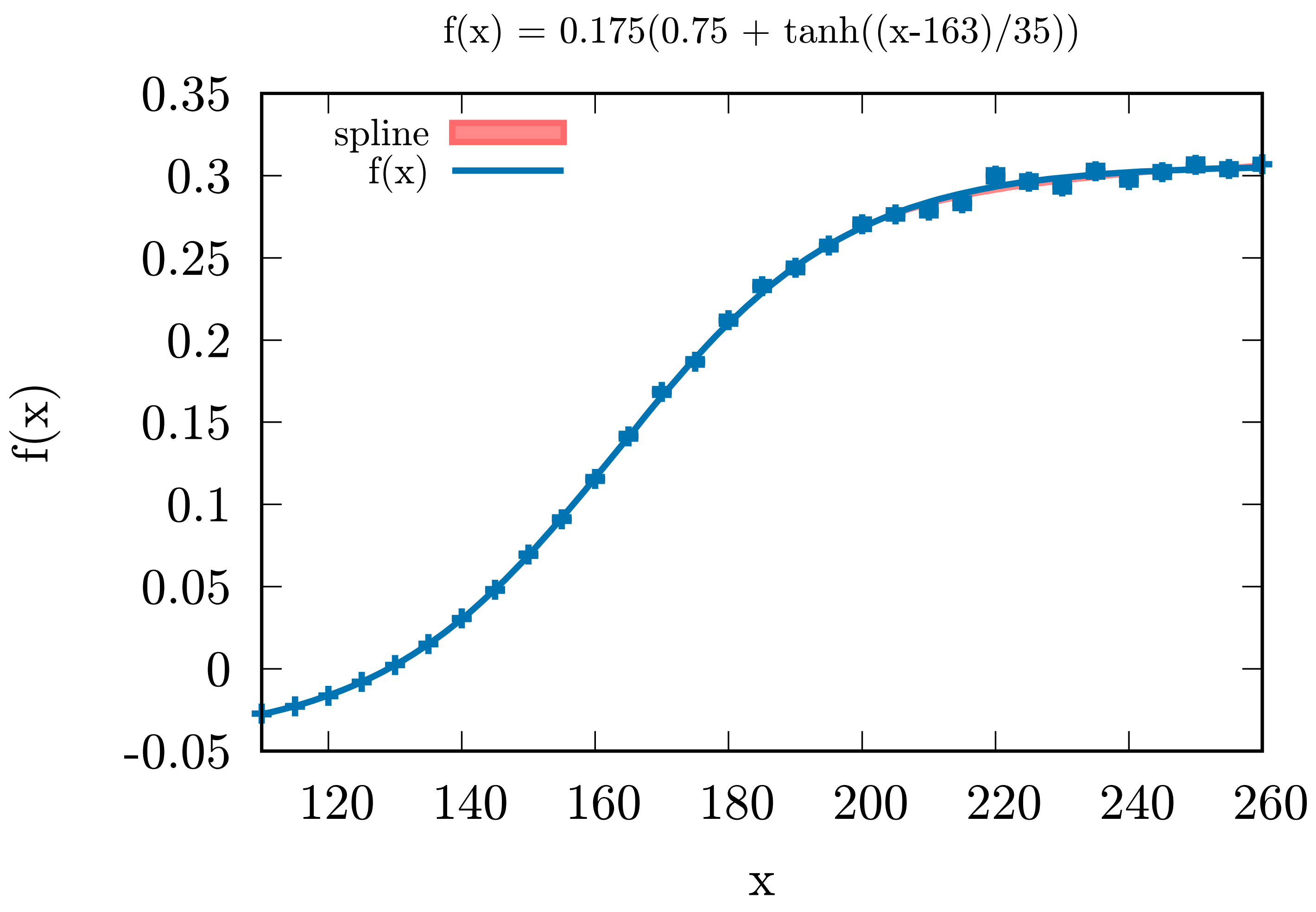
n: number of data points

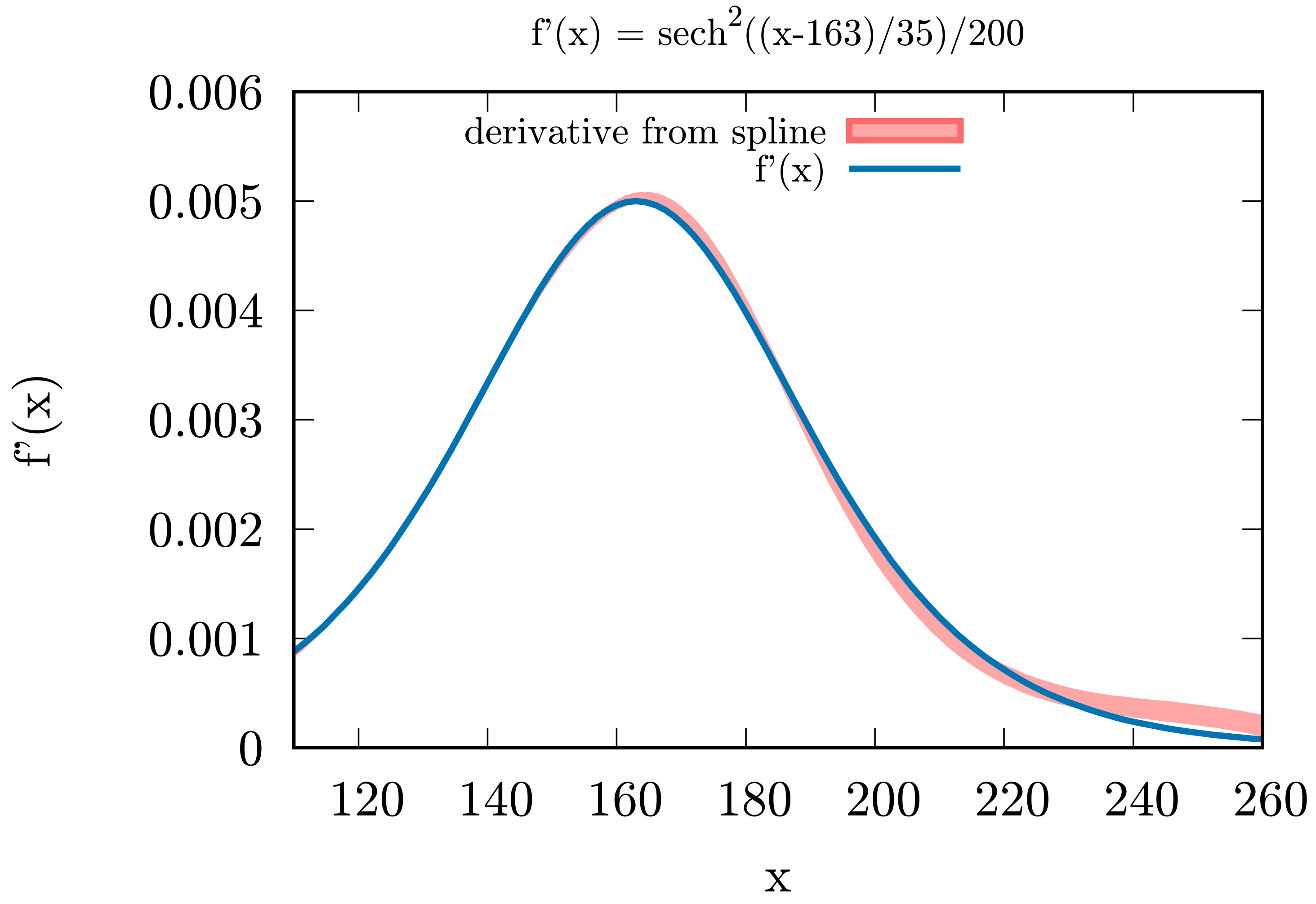
$\hat{L}$: maximized likelihood function

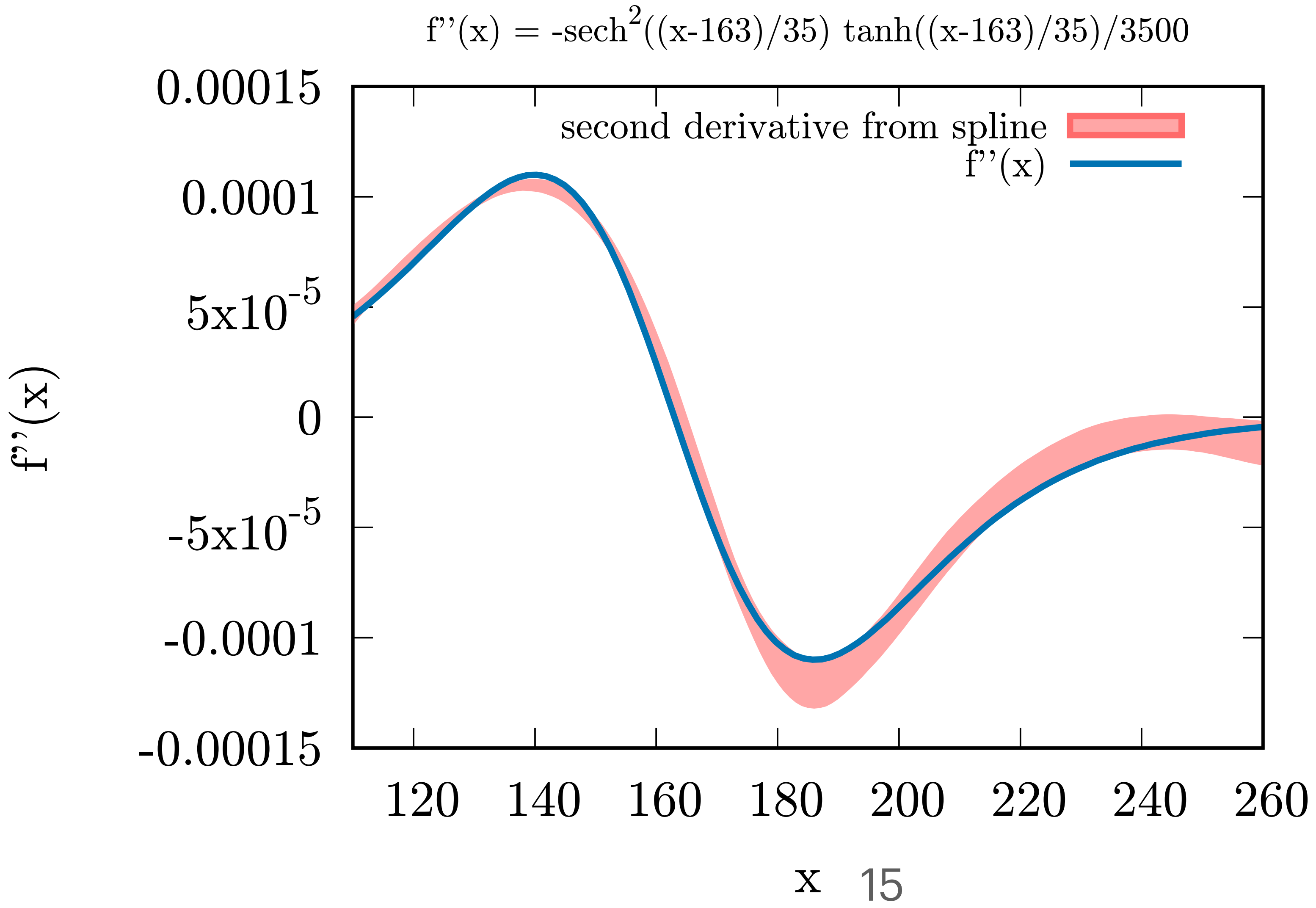
Bayesian spline fitting with free knots

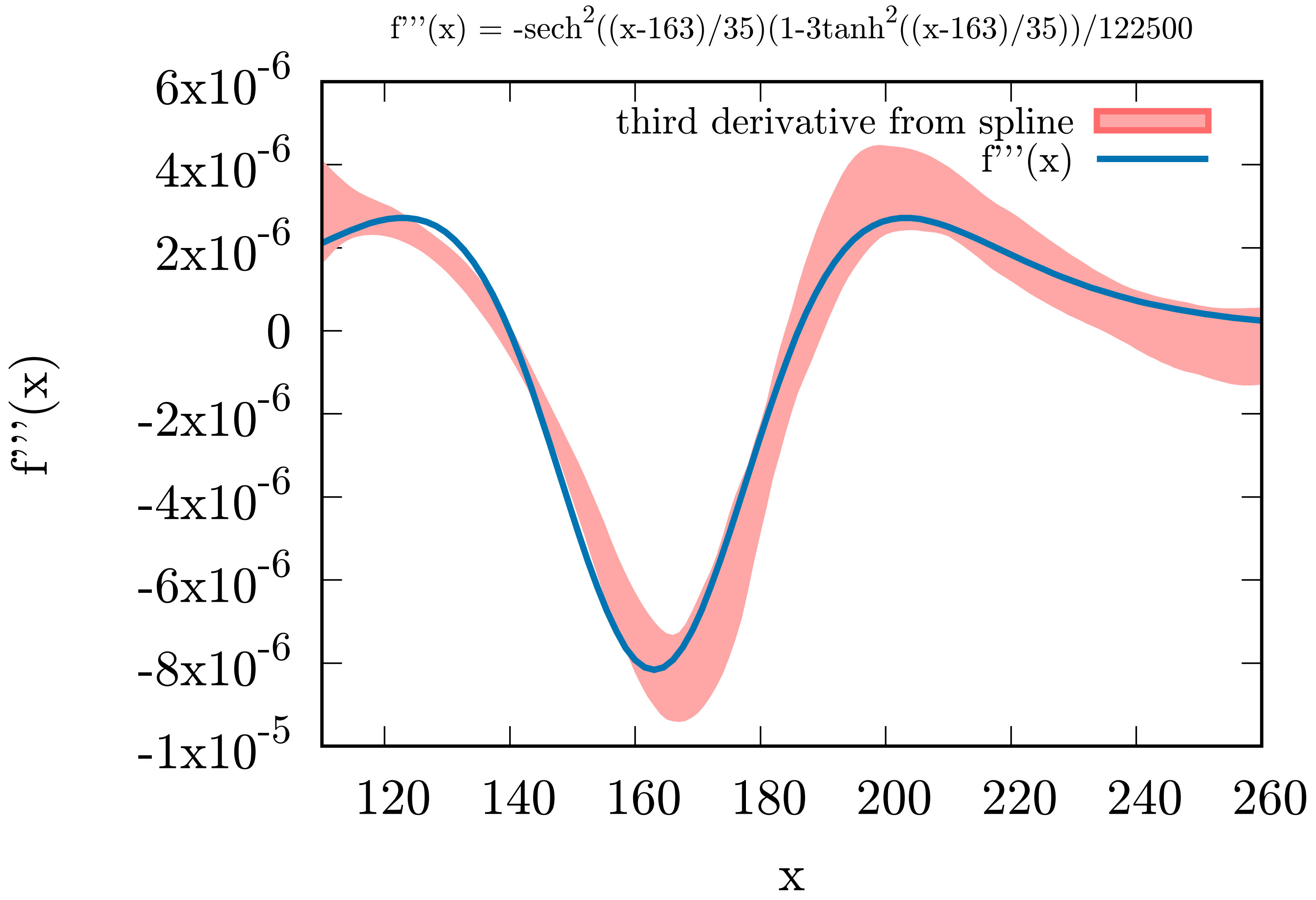
Test on mock data set

- We test the method on mock data sets for a sigmoid function.
- We use degree 5 spline to extract up to third order derivative and compare with the analytical results.







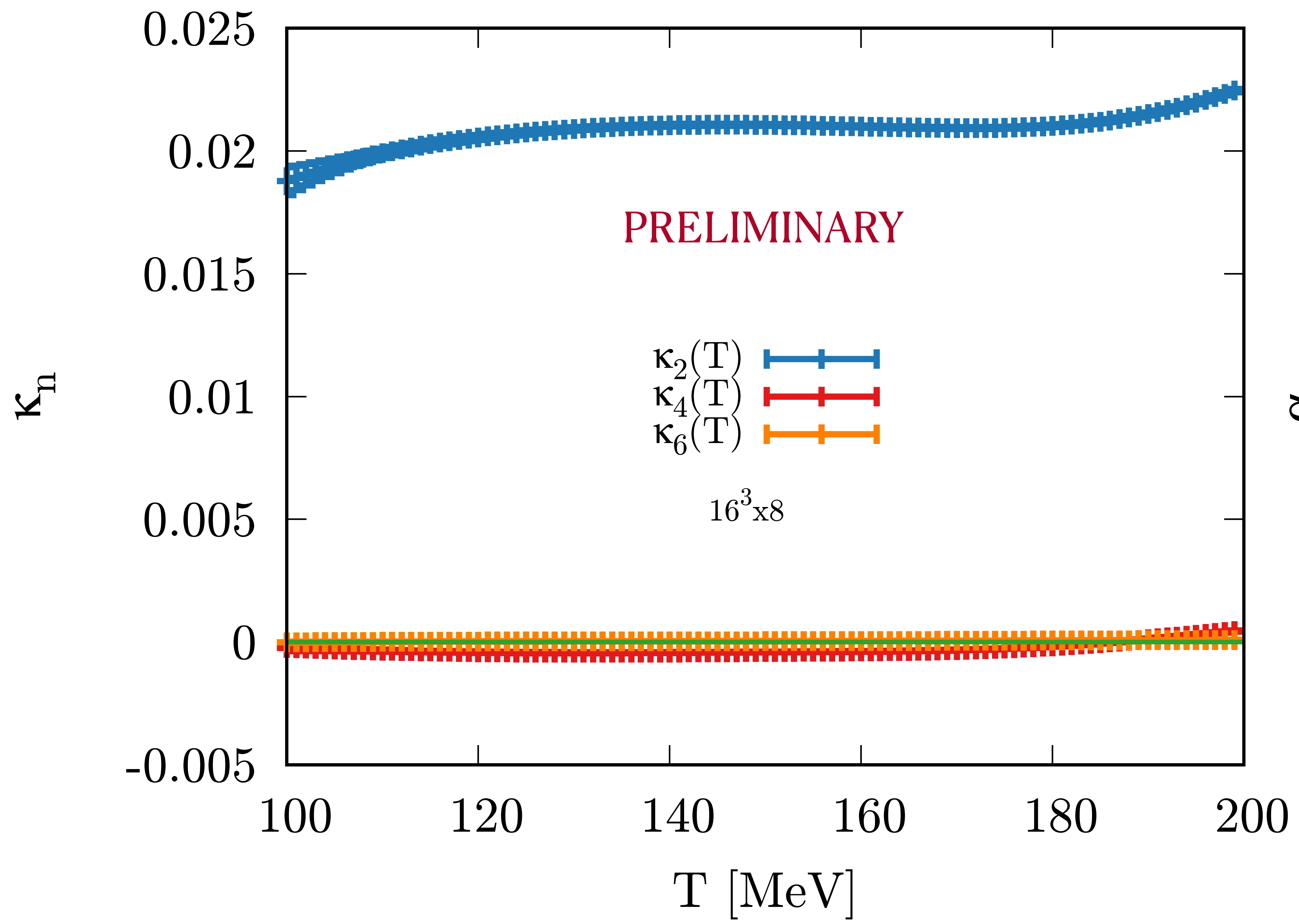


Results

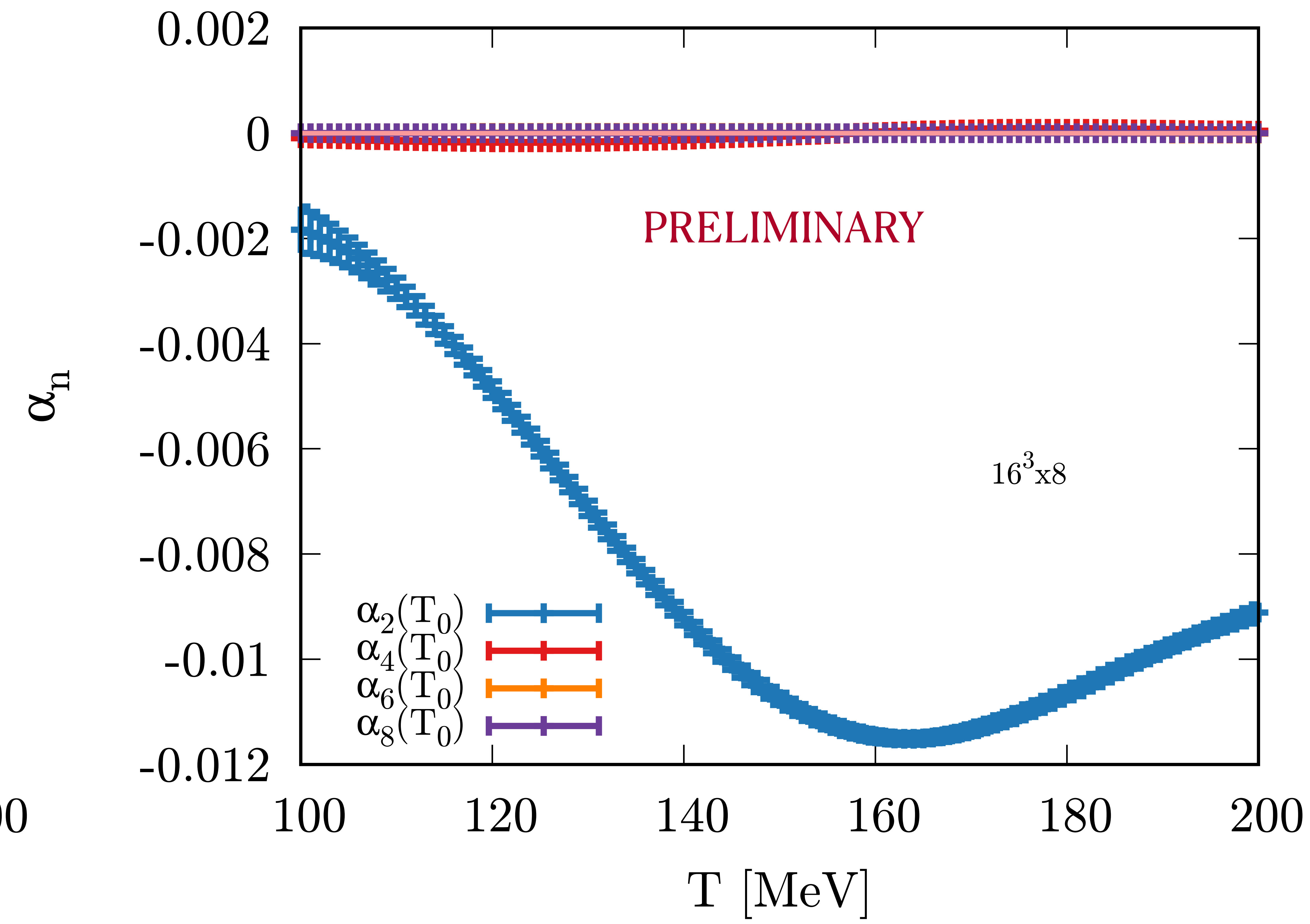
Results

(Expansion coefficients)

n_B-based Expansion

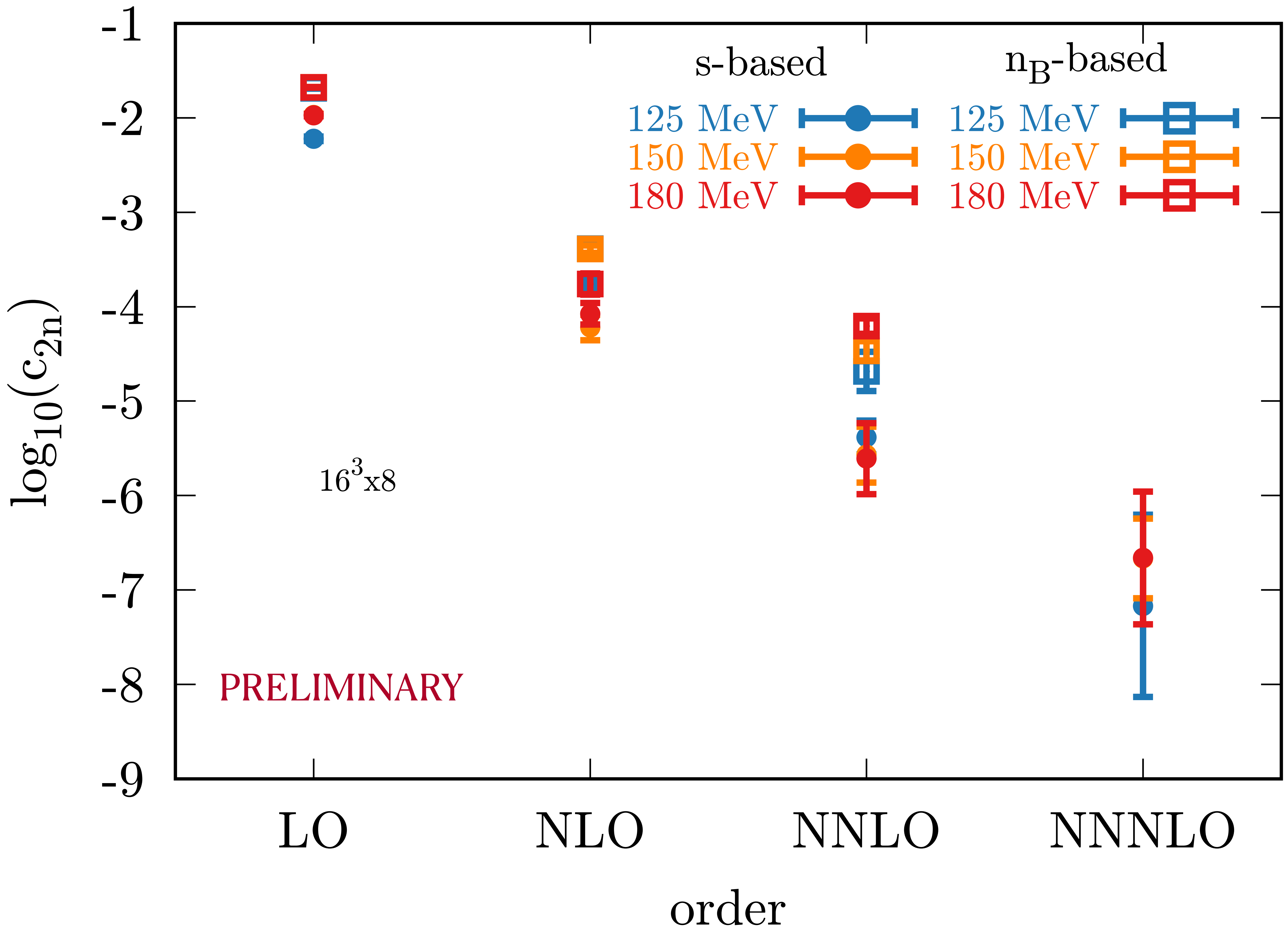


s-based Expansion



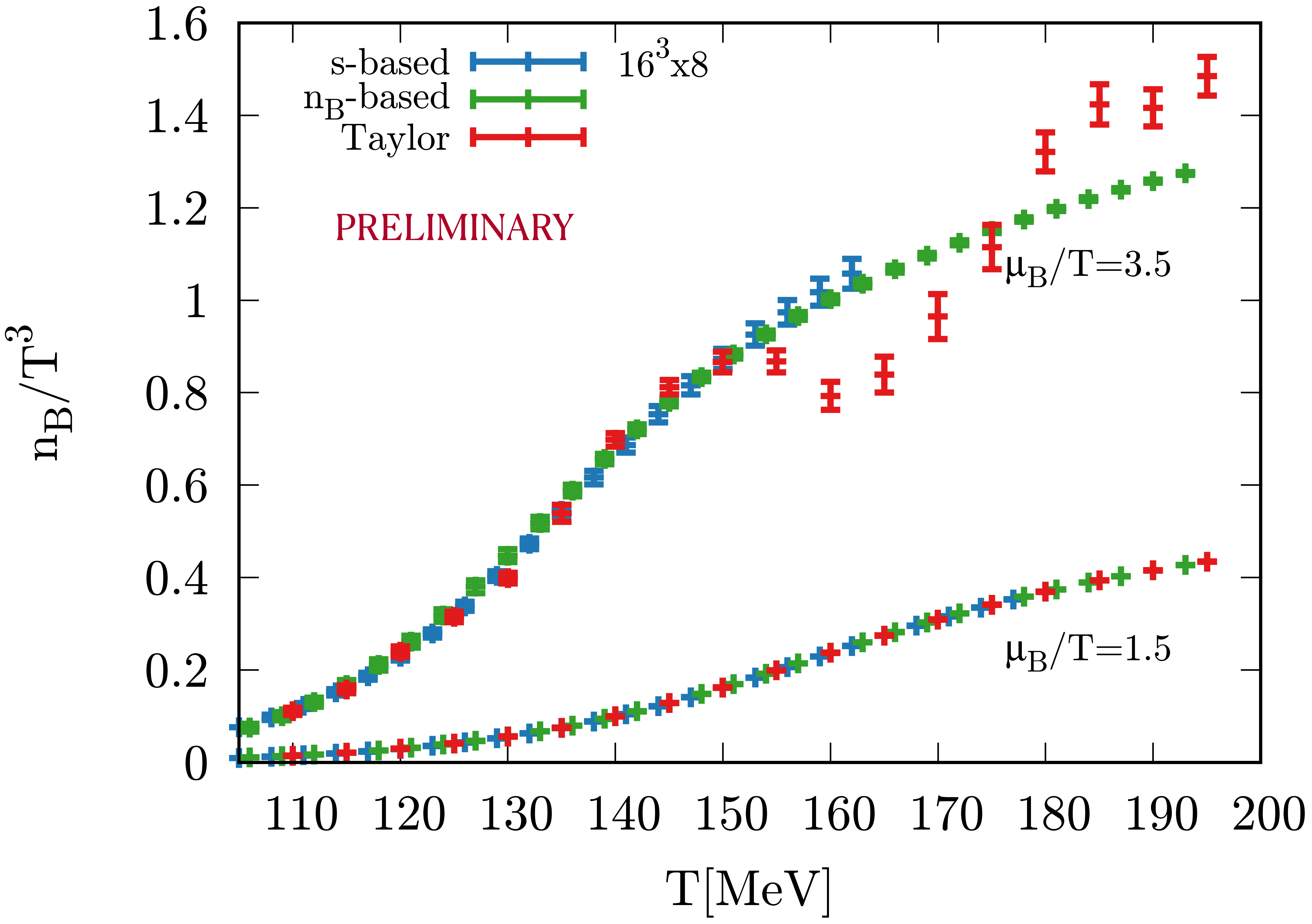
Results

(Magnitude of expansion coefficients across orders)



Results

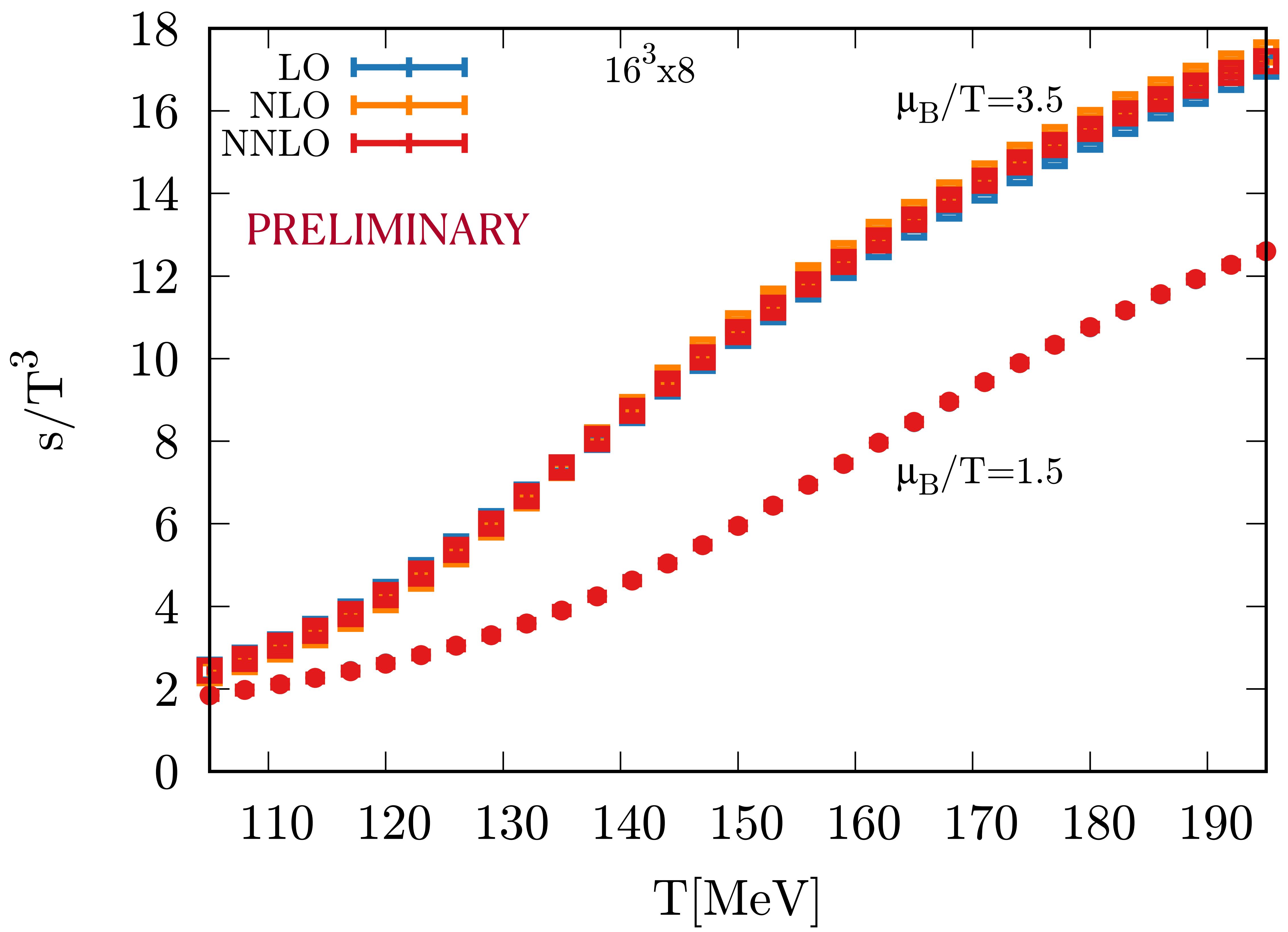
(Comparison of two schemes with Taylor at the highest order)



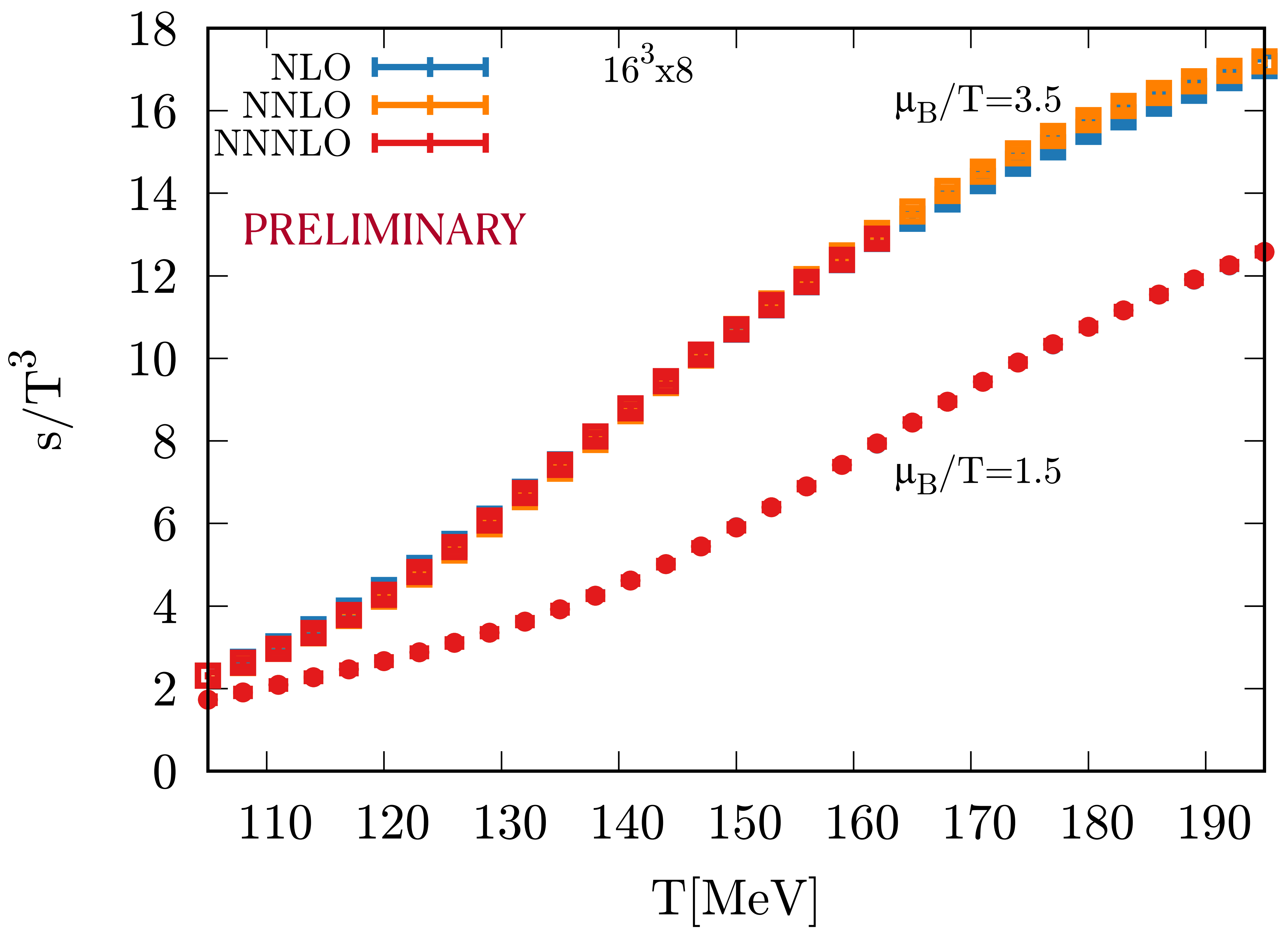
Results

(Entropy in two schemes)

n_B-based Expansion



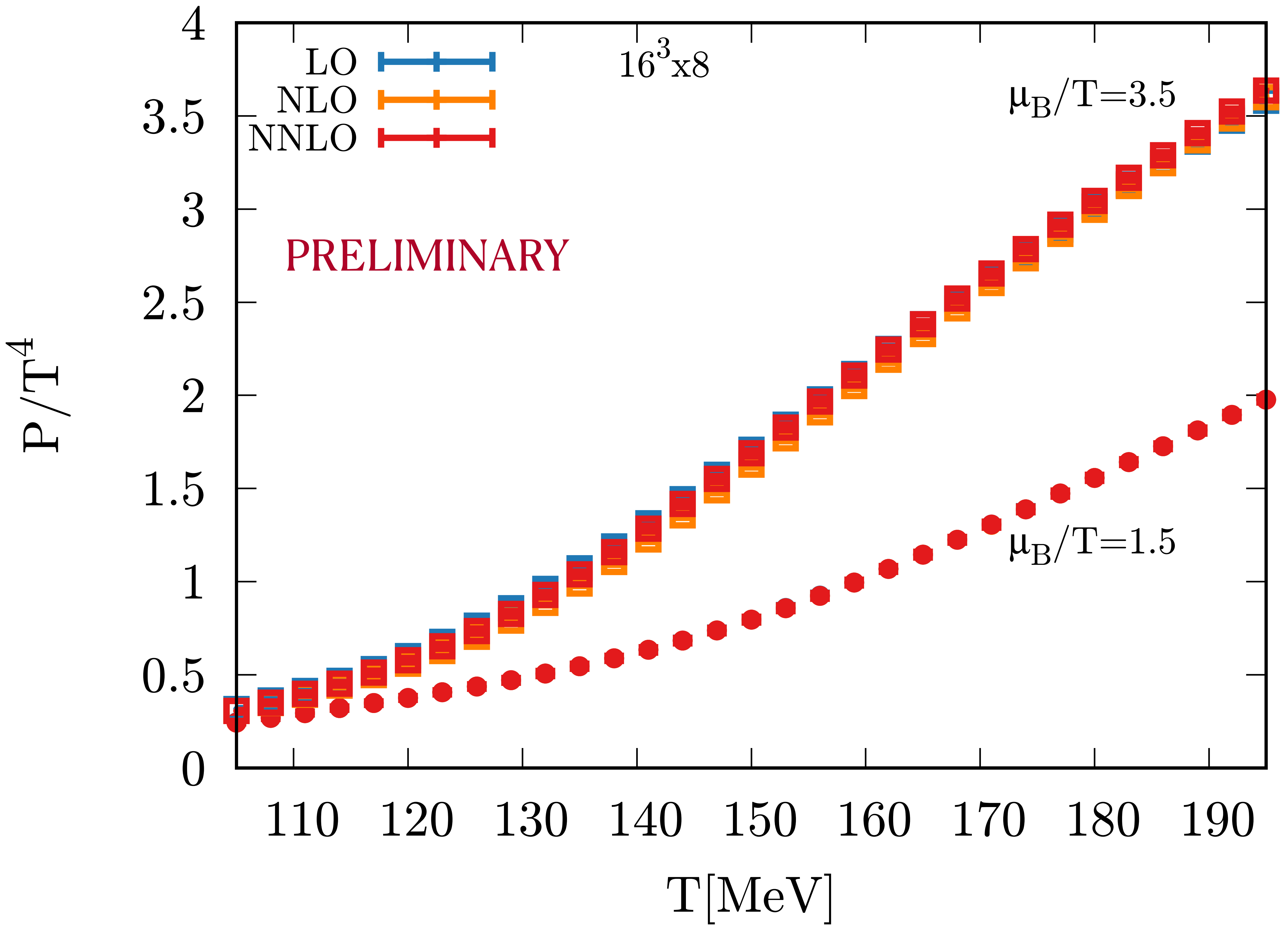
s-based Expansion



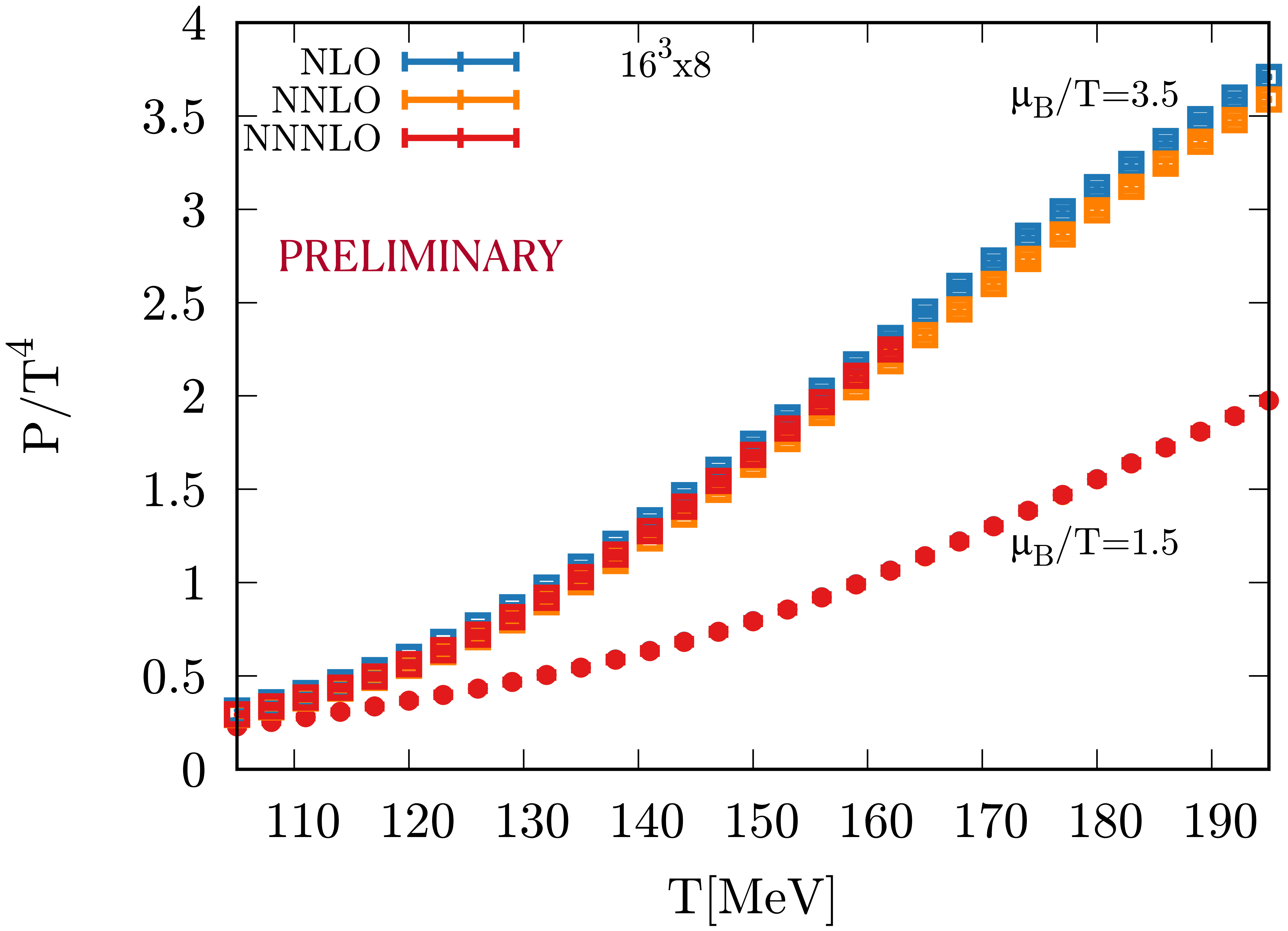
Results

(Pressure in two schemes)

n_B-based Expansion



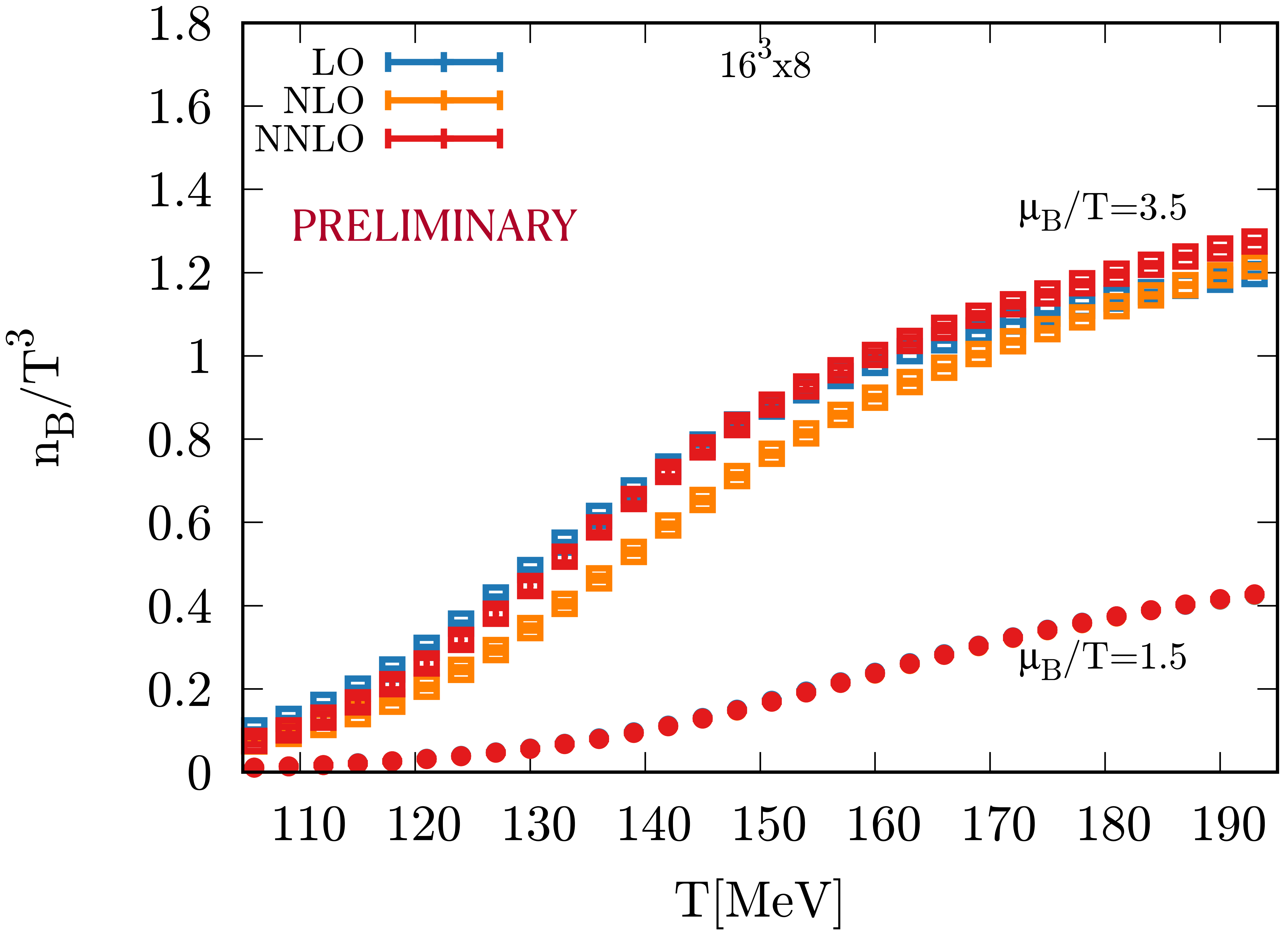
s-based Expansion



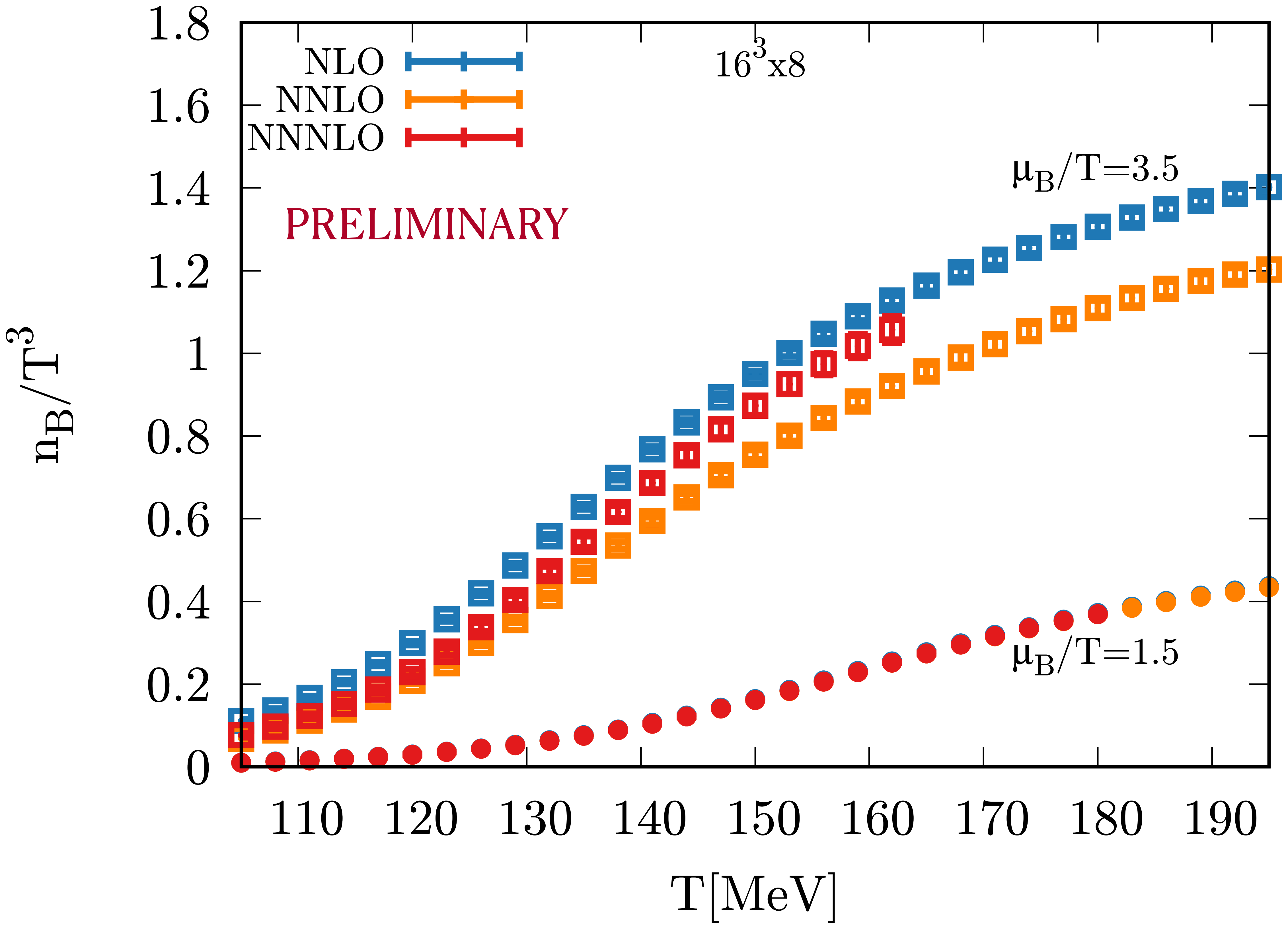
Results

(Baryon density in two schemes)

n_B-based Expansion

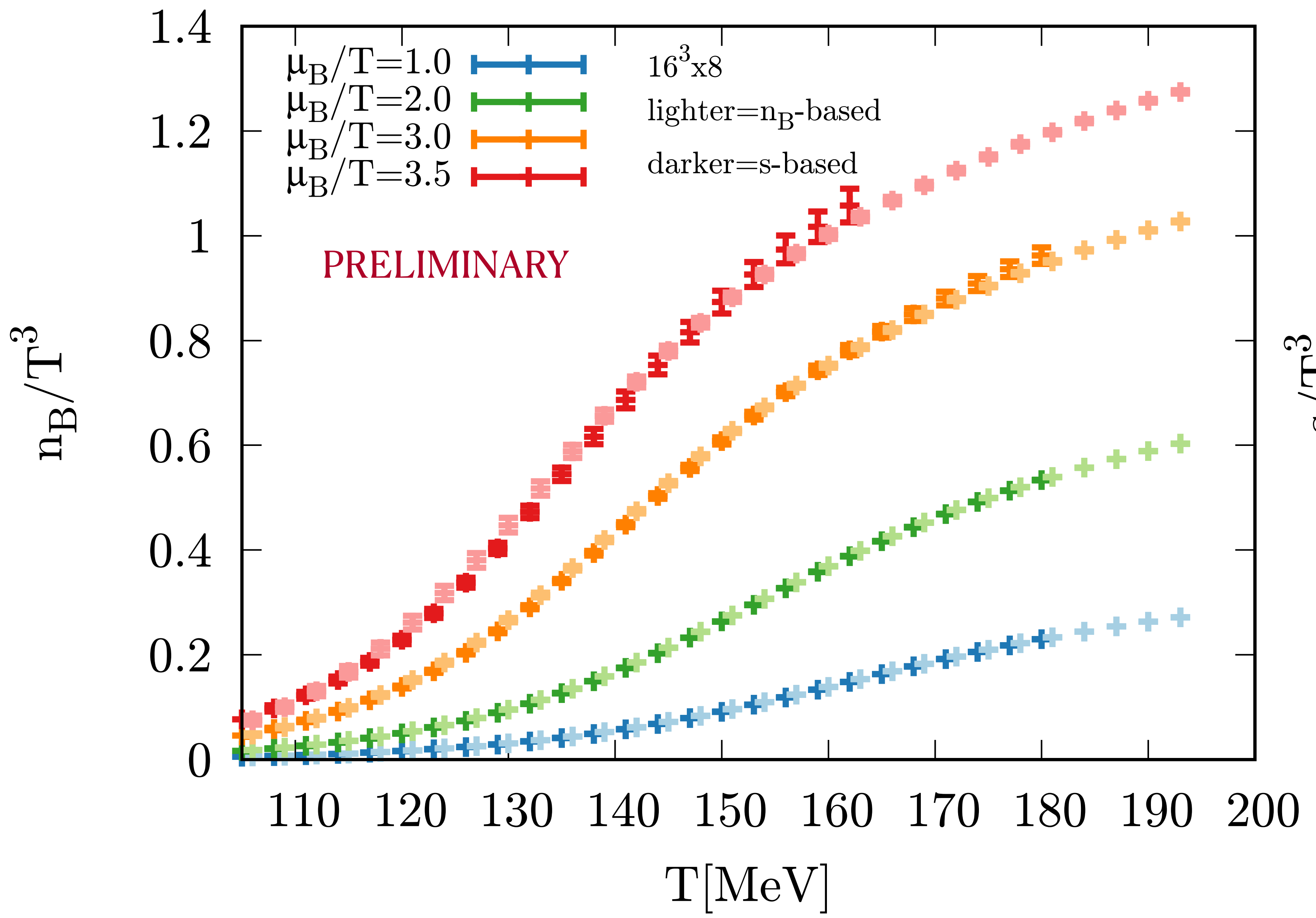


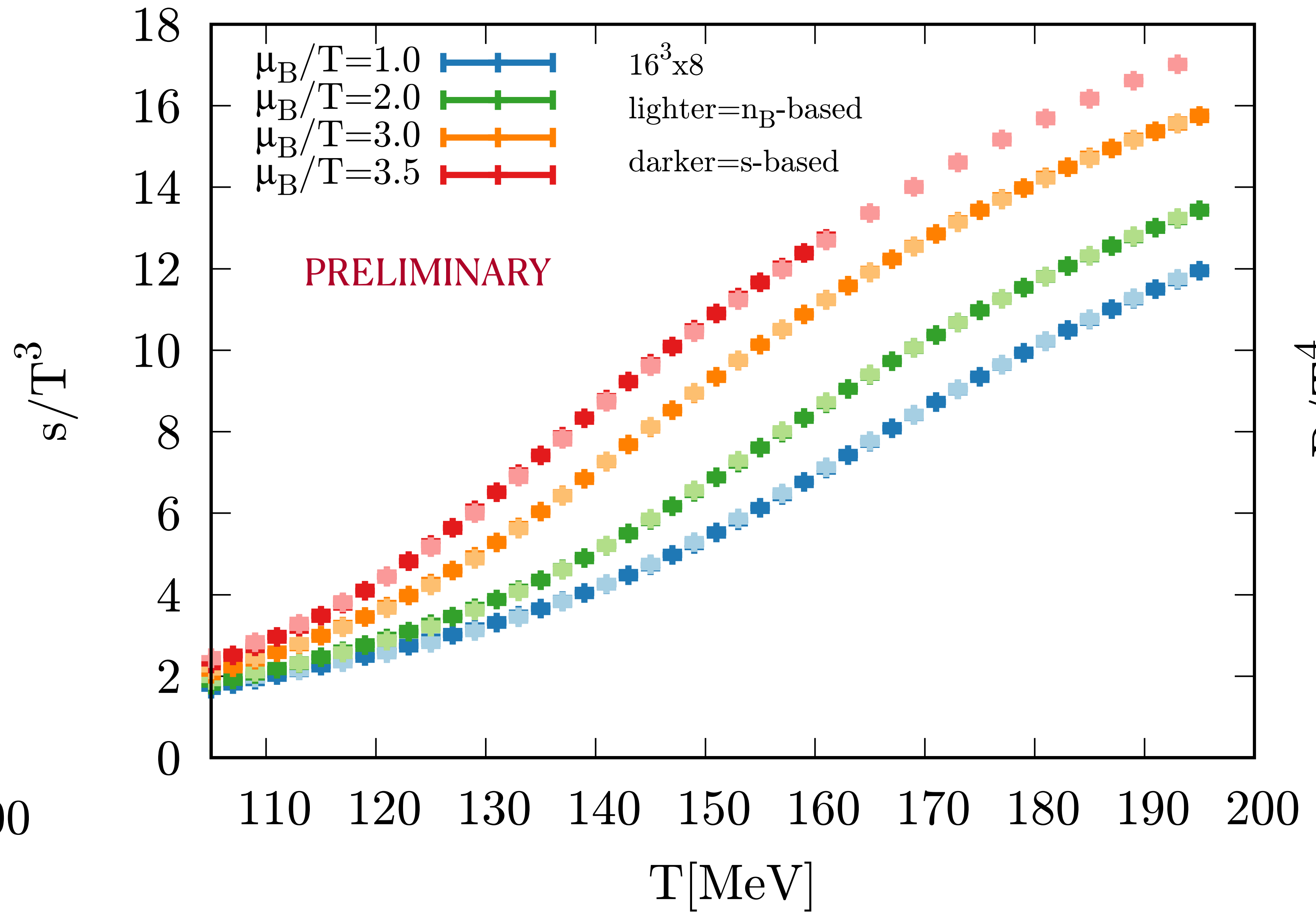
s-based Expansion

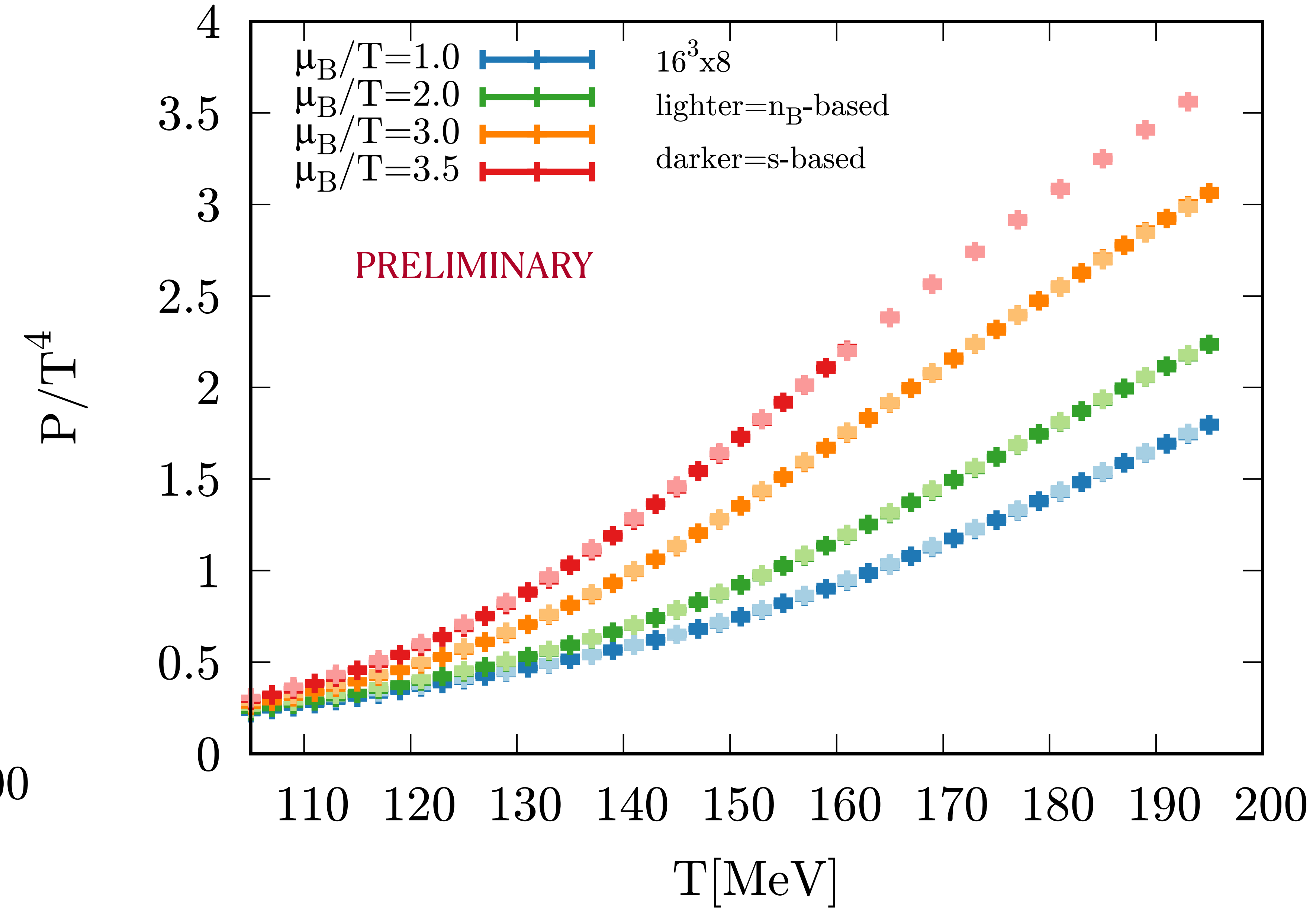


Results

(S, P, n_B at the highest order in two schemes)







Summary and Outlook

- Up to the highest available order (using χ_8^B), we observe that the EoS observables are consistent in the two schemes.
- The entropy based scheme requires a larger temperature coverage of the EoS observables at $\mu_B = 0 \rightarrow$ higher numerical cost driven by the renormalization.
- Work in Progress: Extending the comparison of the schemes in the 3D space (T, μ_B, μ_S).

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Thanks for your attention!

Backup slides

Relations between fluctuations and Temperature expansion coefficients

(T′ Explicit)

$$\chi_2^B(T) = \chi_2^B(T)$$

$$\chi_4^B(T) = 6T\kappa_2(T)\chi_2^{B'}(T)$$

$$\chi_6^B(T) = 120T\kappa_4(T)\chi_2^{B'}(T) + 60T^2\kappa_2(T)^2\chi_2^{B''}(T)$$

$$\chi_8^B(T) = 5040T\kappa_6(T)\chi_2^{B'}(T) + 5040T^2\kappa_2(T)\kappa_4(T)\chi_2^{B''}(T) + 840T^3\kappa_2(T)^3\chi_2^{B'''}(T)$$

Relations between fluctuations and Temperature expansion coefficients

(Entropy based implicit scheme)

$$\frac{T^2\chi_2^B}{2} = -\int\left(\frac{\alpha_2\mathcal{S}_0'}{2}\right)dT + C_2$$

$$\frac{\chi_4^B}{24} = \left(\frac{\alpha_2^2}{8}\mathcal{S}_0'\right) - \int\left(\frac{\alpha_4}{24}\mathcal{S}_0'\right)dT + C_4$$

$$\frac{\chi_6^B}{720T^2} = \left(\frac{\alpha_2\alpha_4}{48}\mathcal{S}_0' - \frac{\alpha_2^2\alpha_2'}{16}\mathcal{S}_0' - \frac{\alpha_2^3}{48}\mathcal{S}_0''\right) - \int\left(\frac{\alpha_6}{720}\mathcal{S}_0'\right)dT + C_6$$

$$\frac{\chi_8^B}{40320T^4} = \left(\frac{\alpha_2^4}{384}\mathcal{S}_0''' + \frac{\alpha_2^3\alpha_2''}{96}\mathcal{S}_0' + \frac{\alpha_2^3\alpha_2'}{48}\mathcal{S}_0'' - \frac{\alpha_2^2\alpha_4}{192}\mathcal{S}_0'' + \frac{\alpha_2^2(\alpha_2')^2}{32}\mathcal{S}_0'\right.$$
$$\left. - \frac{\alpha_2^2\alpha_4'}{192}\mathcal{S}_0' - \frac{\alpha_2\alpha_4\alpha_2'}{96}\mathcal{S}_0' + \frac{\alpha_2\alpha_6}{1440}\mathcal{S}_0' + \frac{\alpha_4^2}{1152}\mathcal{S}_0'\right) - \int\left(\frac{\alpha_8}{40320}\mathcal{S}_0'\right)dT + C_8$$

Thermodynamic observables from T-prime scheme

(Using Maxwell's relations)

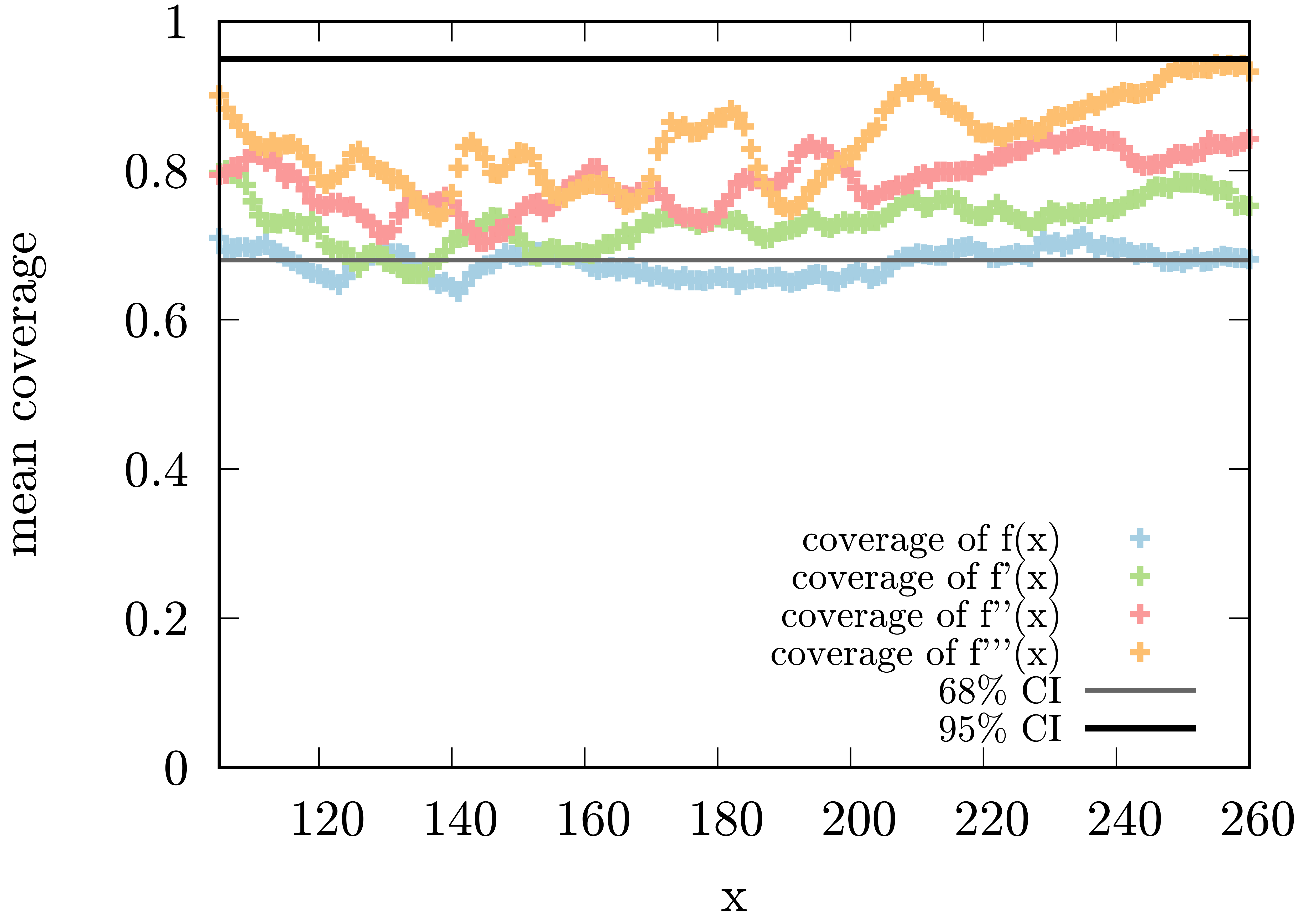
$$n_B(T,\mu_B) = T^2\mu_B\chi_2^B(T',0)$$

$$p(T,\mu_B) = p(T,0) + \int_0^{\mu_B} n_B(T,\mu_B)\ d\mu_B'$$

$$s(T,\mu_B) = s(T,0) + \int_0^{\mu_B} \left(\frac{\partial}{\partial T} n_B(T,\mu_B)\right) d\mu_B'$$

Bayesian spline fitting with free knots

(Point wise coverage)



Workflow

(Joint Fitting)

Start with the spline form for the expansion coefficients $\{\kappa_{2n}\}$ or $\{\alpha_{2n}\}$

Compute the fluctuations $\chi_2^B(T), \chi_4^B(T), \chi_6^B(T), \chi_8^B(T)$

Perform a least square minimization of the joint chi-square to estimate the optimal spline coefficients for the .