Kernel Transformations
for smeared spectral functions

William I. Jay

Colorado State University

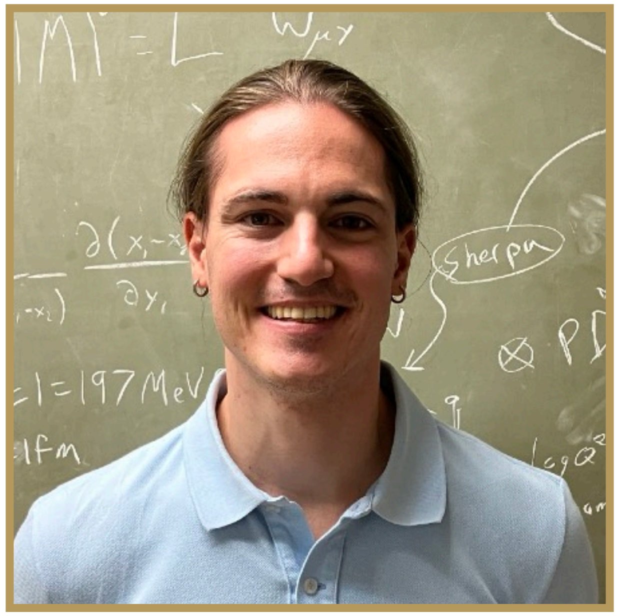
43rd International Symposium on
Lattice Field Theory (Lattice 2026)

26 July - 1 Aug 2026
University of Maryland

Outline

- Spectral reconstruction and meaning of “kernel transformation"
- Spectral positivity $\Longrightarrow$ bounds
- Certified bounds for linear kernel approximations
- Numerical example

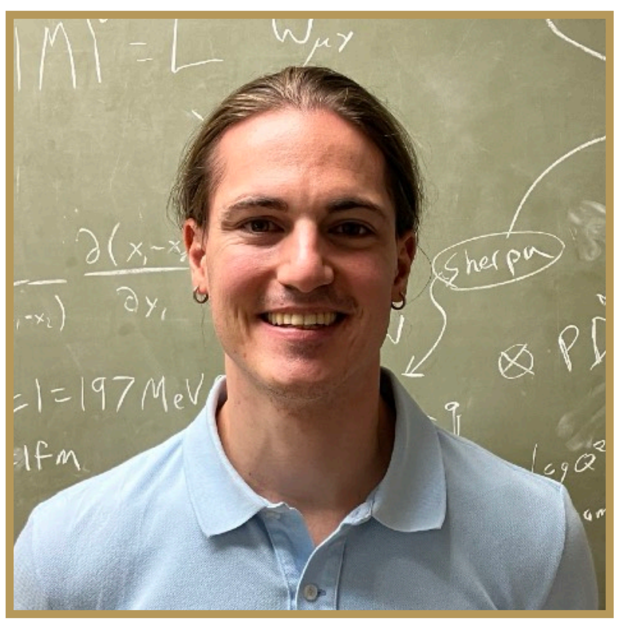
WJ and M. Saccardi
arXiv:2606.19503
submitted to PRD



Matteo Saccardi
Postdoc @ CSU

Connections to other talks

- *The Causal Bootstrap: Bounding Smeared Spectral Functions from Non-Perturbative Euclidean Data*, R. Abbott, S. Fields, WJ, P. Oare, M. Saccardi, arXiv:2605.20509, submitted to PRD
- Matteo: Monday 2:20 pm - Energy-smeared *R*-ratio with these and other methods
- Sarah: Monday 2:40 pm - Nevanlinna—Pick interpolation with noisy data
- Ryan: Monday 3:00 pm - Causal bootstrap



Matteo Saccardi
Postdoc @ CSU

Ryan Abbott
Postdoc @ Columbia

Patrick Oare
Postdoc @ BNL

Sarah Fields
PhD @ Columbia

Smeared Spectral Functions

Via Euclidean-time correlators

- Numerical lattice QCD gives access to finite-volume, Euclidean-time correlation functions, e.g.,

$$C(t) = \langle \mathcal{O}(t)\mathcal{O}^\dagger(0)\rangle = \int d\omega\, \rho(\omega) e^{-\omega t}.$$

- As for any QM system in a box, the finite-volume spectral function is discrete:

$$\rho(\omega) = \sum_n |\langle n | \mathcal{O} | \Omega \rangle|^2 \delta(\omega - E_n)$$

- Inverting the Laplace transform to obtain the spectral function is ill-defined for a finite number inputs $C(t)$.

Smeared Spectral Functions

Via Euclidean-time correlators

- Spectral function is a singular distribution. Study by smearing.
- LQCD gives:

$$C(t) = \int d\omega\, \rho(\omega) e^{-\omega t} \iff \text{“linear functional } \rho[\,\cdot\,] \text{ acting on } e^{-\omega t}\text{”}$$

- Meanwhile, phenomenological applications often require:

$$\rho[\kappa] \equiv \int d\omega\, \rho(\omega) \kappa(\omega) \iff \text{“linear functional } \rho[\,\cdot\,] \text{ acting on } \kappa(\omega)\text{”}$$

- 2606.19503: generic $\kappa_1(\omega) \to \kappa_2(\omega)$
- This talk: focus on $e^{-\omega t} \to \kappa(\omega)$
- Guiding idea: express kernel as linear combination of “basis functions” $e^{-\omega t}$, constructed using available $C(t)$.

Linear reconstruction methods

- Begin with linear kernel approximation — g_t coefficients

$$\kappa(\omega) \approx \kappa_T(\omega) \equiv \sum_t g_t e^{-\omega t}$$

Hansen, Meyer, Robaina
PRD 96 (2017) 9, 094513
arXiv:1704.08993

Hansen, Lupo, Tantalo
PRD 99 (2019) 9, 094508
arXiv:1903.06476

- Smear with spectral function:

$$\rho[\kappa] \equiv \int d\omega\, \rho(\omega)\, \kappa(\omega)$$

$$\approx \rho[\kappa_T] = \int d\omega\, \rho(\omega) \left(\sum_t g_t e^{-\omega t} \right) = \sum_t g_t C(t)$$

Bailas, Hashimoto, Ishikawa
PTEP 2020 (2020) 4, 043B07
arXiv:2001.11779

Bruno, Giusti, Saccardi
PRD 111 (2025) 9, 094515
arXiv:2407.04141

Del Debbio, Lupo, Panero, Tantalo
Eur.Phys.J.C 85 (2025) 2, 185
arXiv:2409.04413

✓ Smeared spectral function is a linear combination of Euclidean input data

◉ Underlying spectral positivity not guaranteed / leveraged

Lupo and Tantalo
arXiv:2605.14652

◉ Exponential growth in $|g_t|$, delicate numerical cancellations

Tsuji and Hashimoto
arXiv:2605.15674

✴ Upshot: $\kappa(\omega) \neq \kappa_T(\omega)$

$\Longrightarrow$ Must control for systematic bias $\rho[\kappa - \kappa_T]$

Giusti, Saccardi, Toniolo
arXiv:2606.28167

Smeared Spectral Functions

Spectral positivity $\implies$ Bounds

- For generic $\kappa(\omega)$, even $\rho[e^{-\omega t}] \to \rho[\kappa]$ remains ill-posed
- Fortunately: smearing “softens” the problem
- Enforcing $\rho \geq 0$ gives rigorous, tight pointwise bounds

Nevanlinna—Pick interpolation

Bergamaschi, WJ, P. Oare
PRD 108 (2023) 7, 074516
arXiv:2305.16190

N. Christ and S. Fields
PRD 113 (2026) 5, 5
arXiv:2510.12136

Moment problems

R. Abbott, WJ, P. Oare
arXiv:2508.01377

SDP “Causal Bootstrap”

S. Lawrence
arXiv:2408.11766

R. Abbot et al. [WJ]
arXiv:2605.20509

S. Mutzel and A. Tilloy
arXiv:2606.09791

Smeared Spectral Functions

Spectral positivity $\Longrightarrow$ Bounds

- Assume a non-negative spectral function.
- Consider non-negative smearing kernels $\kappa(\omega) \geq 0$.
- Smearing defines a positive, monotone operator.

$$\kappa_1(\omega) \geq \kappa_2(\omega) \implies \rho[\kappa_1] \geq \rho[\kappa_2]$$

- That is, *smearing preserves order* $\Longrightarrow$ design kernels to bound $\kappa, \rho[\kappa]$
- If one has $\kappa_T(\omega) \equiv \sum_t g_t e^{-\omega t}$ and $\kappa(\omega) \geq \sum_t g_t e^{-\omega t}$ for all energies ω
  - $\Longrightarrow$ Rigorous lower bound: $\rho[\kappa] \geq \rho[\kappa_T]$
  - An analogous argument applies for upper bounds

Smeared Spectral Functions

Toward certified bounds for $\rho[\kappa - \kappa_T]$

- Linear methods usually approximate the kernel $\kappa(\omega) \approx \kappa_T(\omega)$
- Output kernels $\kappa_T(\omega)$ usually oscillate around the target kernel and therefore do *not* satisfy the desired bounding property.

  - Q: Can these results be used to construct rigorous bounds?
  - A: Yes, via a *certification* step.

Hölder's Inequality

Toward certified bounds for $\rho[\kappa - \kappa_T]$

- Hölder's inequality generalizes the Cauchy-Schwarz inequality:
  - Cauchy-Schwarz: $\|fg\|_1 \leq \|f\|_2 \, \|g\|_2$
  - Hölder: $\|fg\|_1 \leq \|f\|_p \, \|g\|_q$ with $1/p + 1/q = 1$
- Useful special case: $p = \infty$, $q = 1$
  - $\|fg\|_1 \leq \|f\|_\infty \, \|g\|_1$, with $\|f\|_\infty = \max_\omega f(\omega)$

Hölder's Inequality

Toward certified bounds for $\rho[\kappa - \kappa_T]$

$$\text{bias} = \int d\omega \, \left(\kappa(\omega) - \kappa_T(\omega)\right) \rho(\omega) = \, ?$$

- **Insight:** Hölder's inequality can isolate bounds from "kinematics" ($\kappa - \kappa_T$) and "dynamics" $\rho(\omega)$

- Step 1: Multiply + divide by positive function (say, $e^{-\omega t_{\text{ref}}}$)

- $$\rho[\kappa - \kappa_T] = \int d\omega \left(\frac{\kappa(\omega) - \kappa_T(\omega)}{e^{-\omega t_{\text{ref}}}} \right) \rho(\omega) e^{-\omega t_{\text{ref}}}$$

- Step 2: Invoke Hölder

$$|\rho[\kappa - \kappa_T]| \leq \left\| \frac{\kappa(\omega) - \kappa_T(\omega)}{e^{-\omega t_{\text{ref}}}} \right\|_\infty \cdot C(t_{\text{ref}})$$

Hölder's Inequality

Toward certified bounds for $\rho[\kappa - \kappa_T]$

$$\text{bias} = \int d\omega \, \big(\kappa(\omega) - \kappa_T(\omega)\big)\, \rho(\omega) = \, ?$$

- **Insight:** Hölder's inequality can isolate bounds from "kinematics" $(\kappa - \kappa_T)$ and "dynamics" $\rho(\omega)$

- Step 1: Multiply + divide by positive fur

- $$\rho[\kappa - \kappa_T] = \int d\omega \left(\frac{\kappa(\omega) - \kappa_T(\omega)}{e^{-\omega t_{\text{ref}}}} \right.$$

Calculable bounds in terms of input data $C(t)$ only (!)

- Step 2: Invoke Hölder

$$|\rho[\kappa - \kappa_T]| \leq \left\| \frac{\kappa(\omega) - \kappa_T(\omega)}{e^{-\omega t_{\text{ref}}}} \right\|_\infty \cdot C(t_{\text{ref}})$$

Hölder's Inequality

Toward certified bounds for $\rho[\kappa - \kappa_T]$

$$\text{bias} = \int d\omega\ \big(\kappa(\omega) - \kappa_T(\omega)\big)\,\rho(\omega) = ?$$

- **Insight:** Hölder's inequality can isolate bounds from "kinematics" $(\kappa - \kappa_T)$ and "dynamics" $\rho(\omega)$

- Step 1: Multiply + divide by positive fur

Calculable bounds in terms of input data $C(t)$ only (!)

Of course, $||\cdot||_\infty$ must be finite for a non-trivial bound. See our paper (arXiv:2606.19503) for discussion.

$$|\rho[\kappa - \kappa_T]| \le \left\| \frac{\kappa(\omega) - \kappa_T(\omega)}{e^{-\omega t_{\text{ref}}}} \right\|_\infty \cdot C(t_{\text{ref}})$$

Kernel Transformations

Analysis pipeline for certified bounds

- Step 1: Choose a regulated target kernel $\kappa_T(\omega) \approx \kappa(\omega)$
- Step 2: Propagate uncertainties from data to obtain $\delta\rho[\kappa_T]_{\mathrm{stat}}$
- Step 3: Bound bias

$$\delta\rho[\kappa - \kappa_T; t_{\mathrm{ref}}]_{\mathrm{syst}} \equiv \left\| \frac{\kappa(\omega) - \kappa_T(\omega)}{e^{-\omega t_{\mathrm{ref}}}} \right\|_\infty \cdot C(t_{\mathrm{ref}}) \geq |\rho[\kappa - \kappa_T]|$$

- Step 4: Optimize total error over reference time t_{ref}, regulator T

$$\longrightarrow \delta\rho[\kappa]_{\mathrm{total}} = \delta\rho[\kappa_T]_{\mathrm{stat}} \oplus \delta\rho[\kappa - \kappa_T; t_{\mathrm{ref}}]_{\mathrm{syst}}$$

Kernel Transformations

Formal guarantees

- **Recall:** Finite # of inputs $\implies \rho[\kappa]$ not uniquely defined, only pointwise bounds accessible $\rho[\kappa]_\pm$
- **Limiting case:** finite Euclidean data, noise $\to 0$
  - Tight bounds $\iff$ *least* upper bound / *greatest* lower bound.
  - Tight bounds $\rho[\kappa]_\pm$ accessible via SDPs (“causal bootstrap”)
  - Hölder’s bounds are rigorous but not necessarily tight.
- **Practical reality:** finite Euclidean data, finite noise
  - “Noisy bounds” $\iff$ confidence intervals for $\rho[\kappa]_\pm$ (*not* for $\rho[\kappa]$)
- Both bounds leverage positivity, but use data in different ways

Kernel Transformations

Noisy reality: certified linear methods vs SDP

- **Case 1:** SDP/ causal bootstrap
  - Greatest lower bound via $\kappa(\omega) \geq \Sigma_t \, g_t e^{-\omega t}$
  - Coefficients g_t optimized to give least upper bound / greatest lower bound
  - Roughly: coefficients leverage implicit information about spectral support
- **Case 2:** Certified linear methods:
  - $\kappa(\omega) \approx \kappa_T(\omega) = \Sigma_t \, g_t e^{-\omega t}$,
  - Coefficients $g_t \approx$ independent of input data
  - HLT: data only influence g_t via covariance matrix in regulator
  - Certification step uses $C(t_{\text{ref}})$ at a single point to shift a single g_t

Kernel Transformations

Noisy reality: synthesis

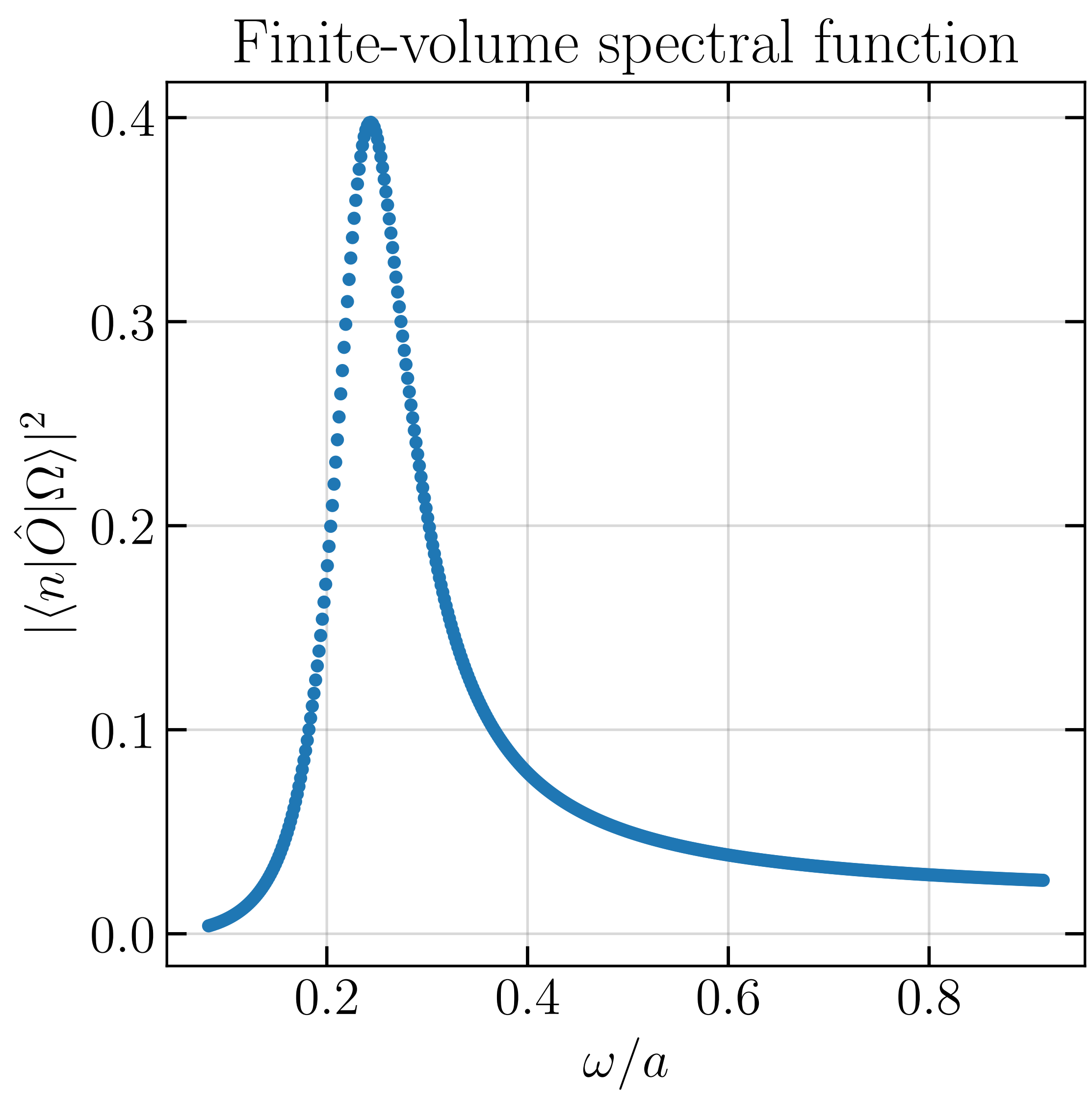
$\Longrightarrow$ Together, the methods…

- provide complementary information about spectral bounds
- couple in different ways to statistical uncertainties
- reduce to a known hierarchy in limit of exact data
- augment existing techniques to estimate bias (e.g., stability of results as regulator varied)

Example of Certified Bounds

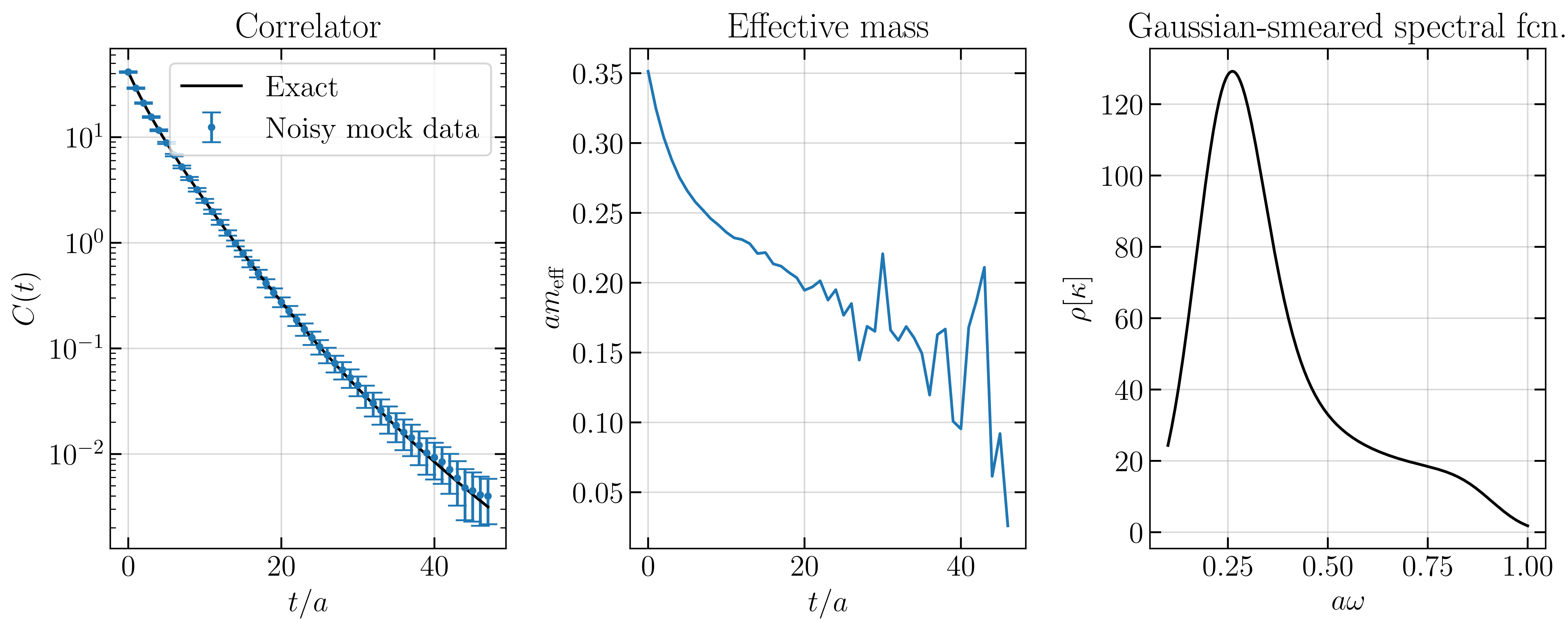
Toy model for a resonance

- Consider a toy spectral function for a resonance
- $$\rho(\omega) = \frac{\omega^2\,(\Gamma/\pi)}{(\omega - M)^2 + \Gamma^2}\Theta(\omega - 2M_\pi)$$
    - $M_\pi = 135\ \mathrm{MeV}$
    - $M = 770\ \mathrm{MeV}$
    - $\Gamma = 150\ \mathrm{MeV}$
- Take lattice units $a = 0.06\ \mathrm{fm}$
- Suppose dense spectrum ("large V")



Example of Certified Bounds

Toy model for a resonance

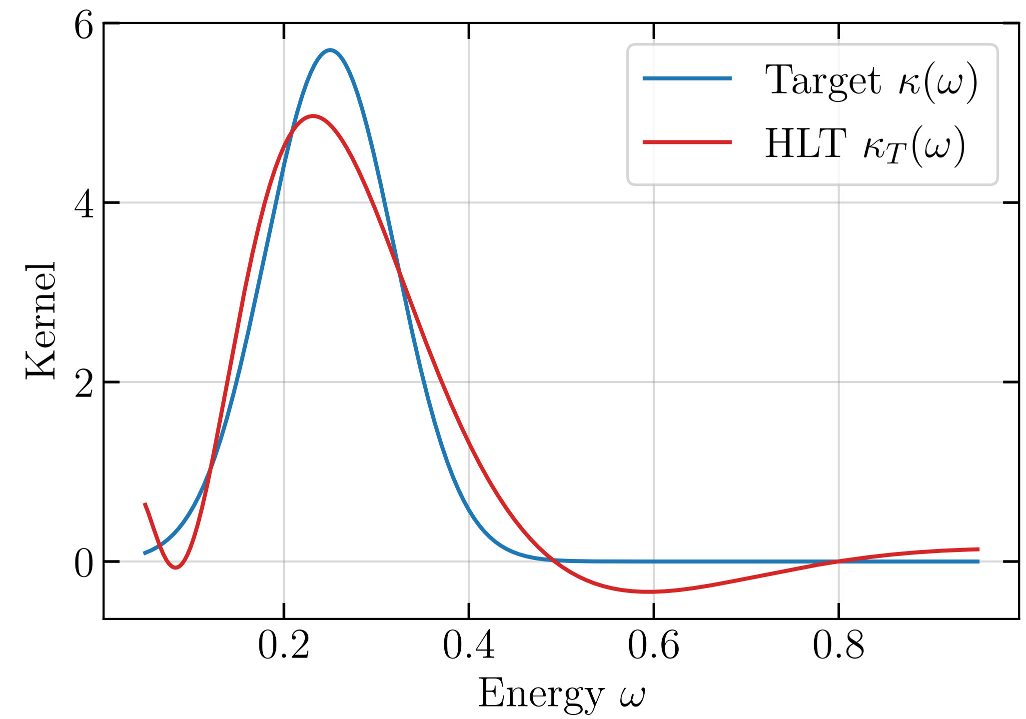


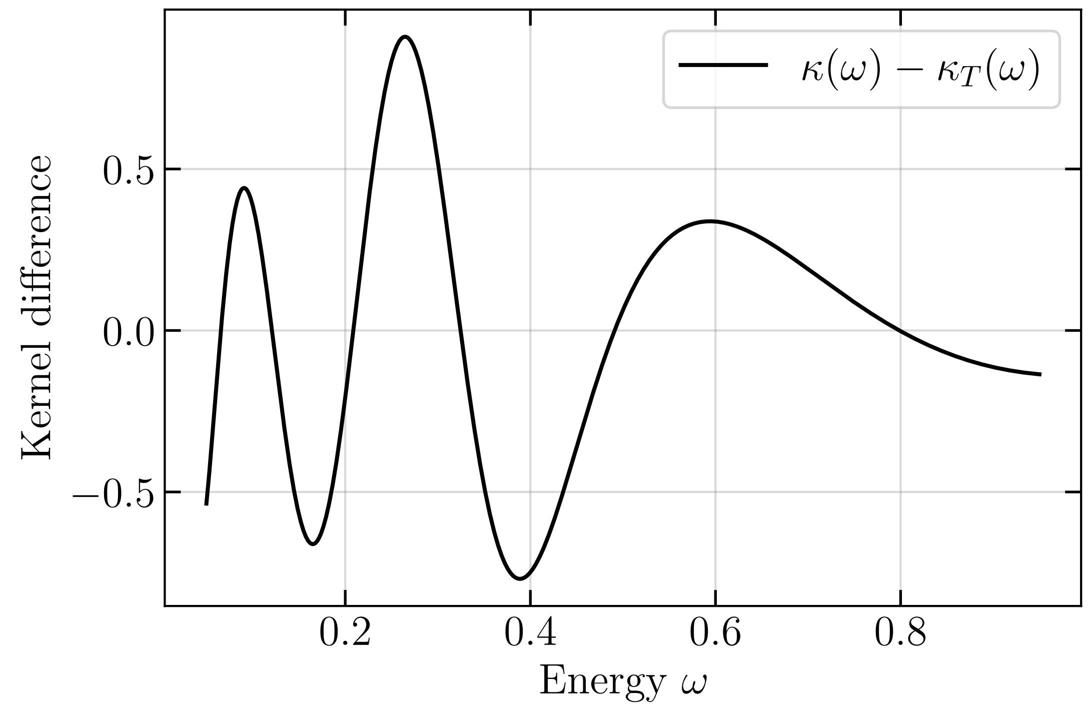
Gaussian smearing width $a\sigma = 0.07$

Example of Certified Bounds

Toy model for a resonance

- Consider HLT reconstruction using input data on $N_t = 24$ Euclidean times
- Consider gaussian smearing at $a\omega_0 = 0.25$ with $a\sigma = 0.07$
- Reconstruction approximates kernel: $\kappa_T(\omega) \approx \kappa(\omega)$
- $\kappa_T(\omega)$ oscillates around $\kappa(\omega)$

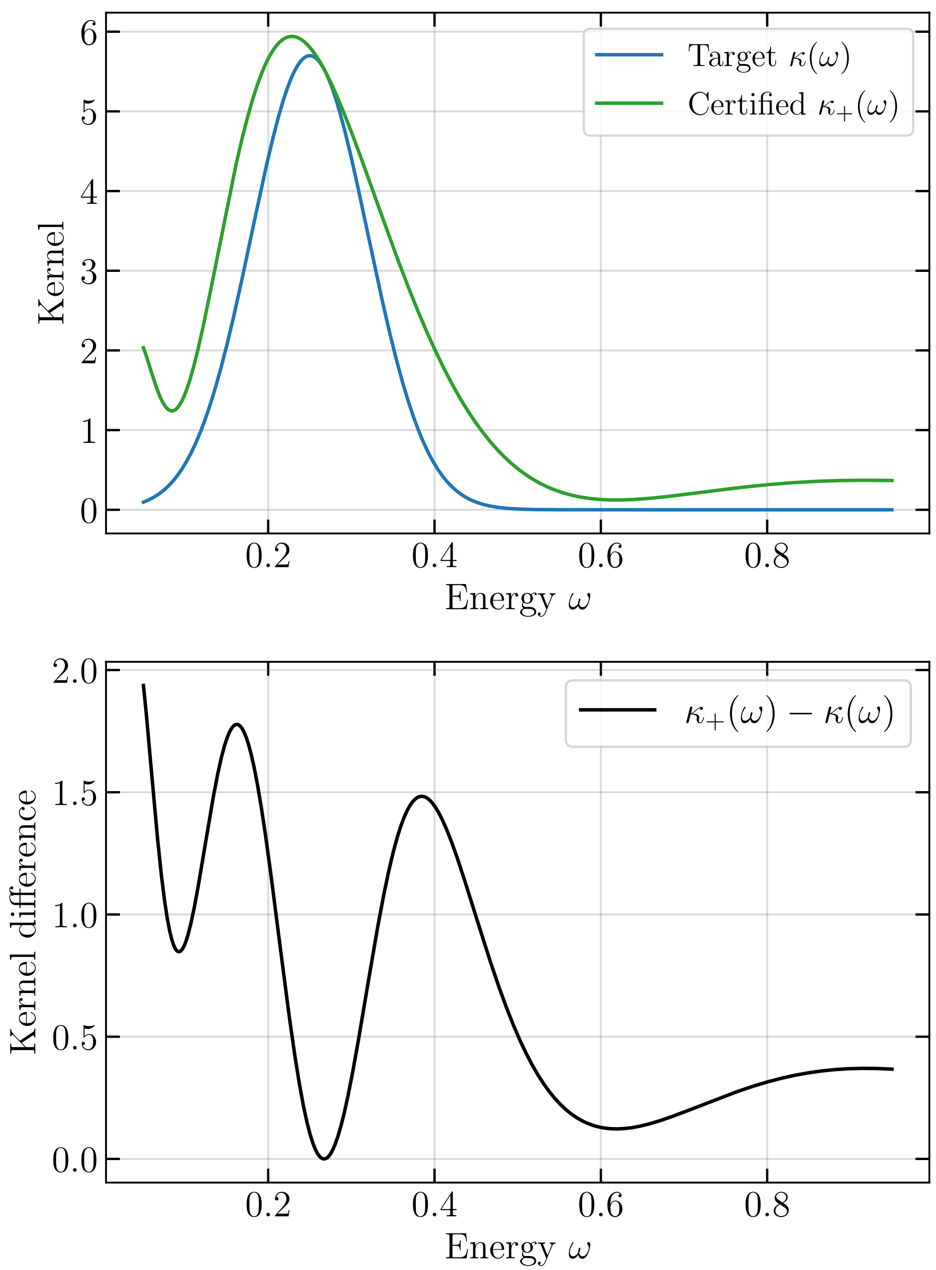




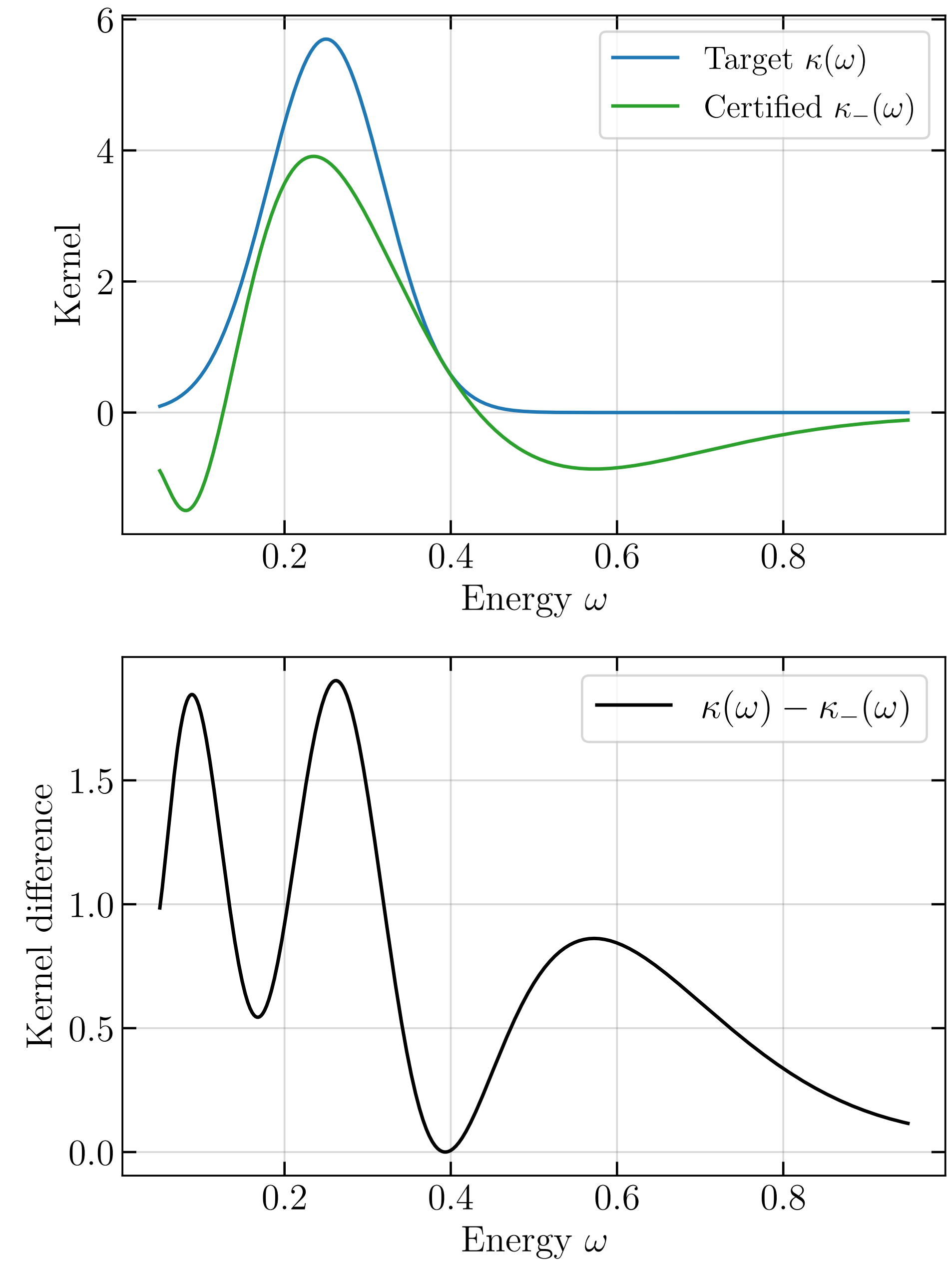
Example of Certified Bounds

Invoke Hölder to construct bounding kernels

Certified bounding kernel $\kappa_+(\omega)$

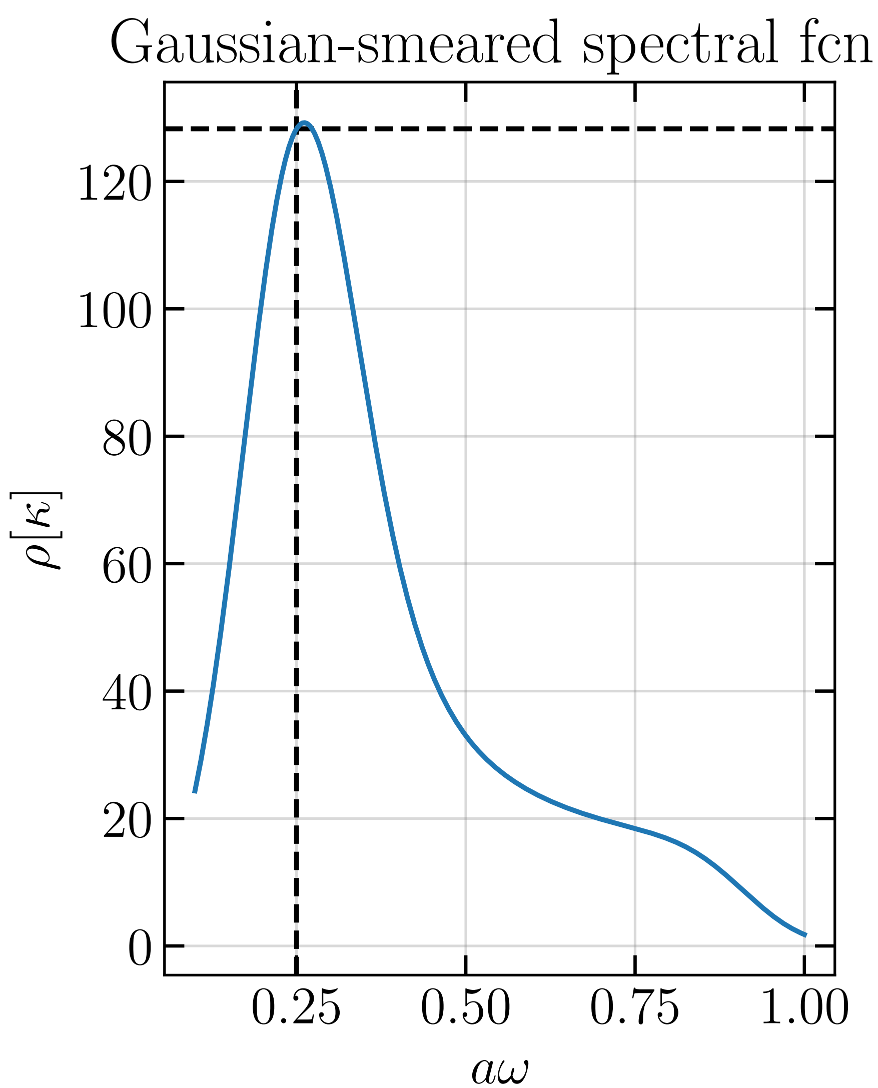


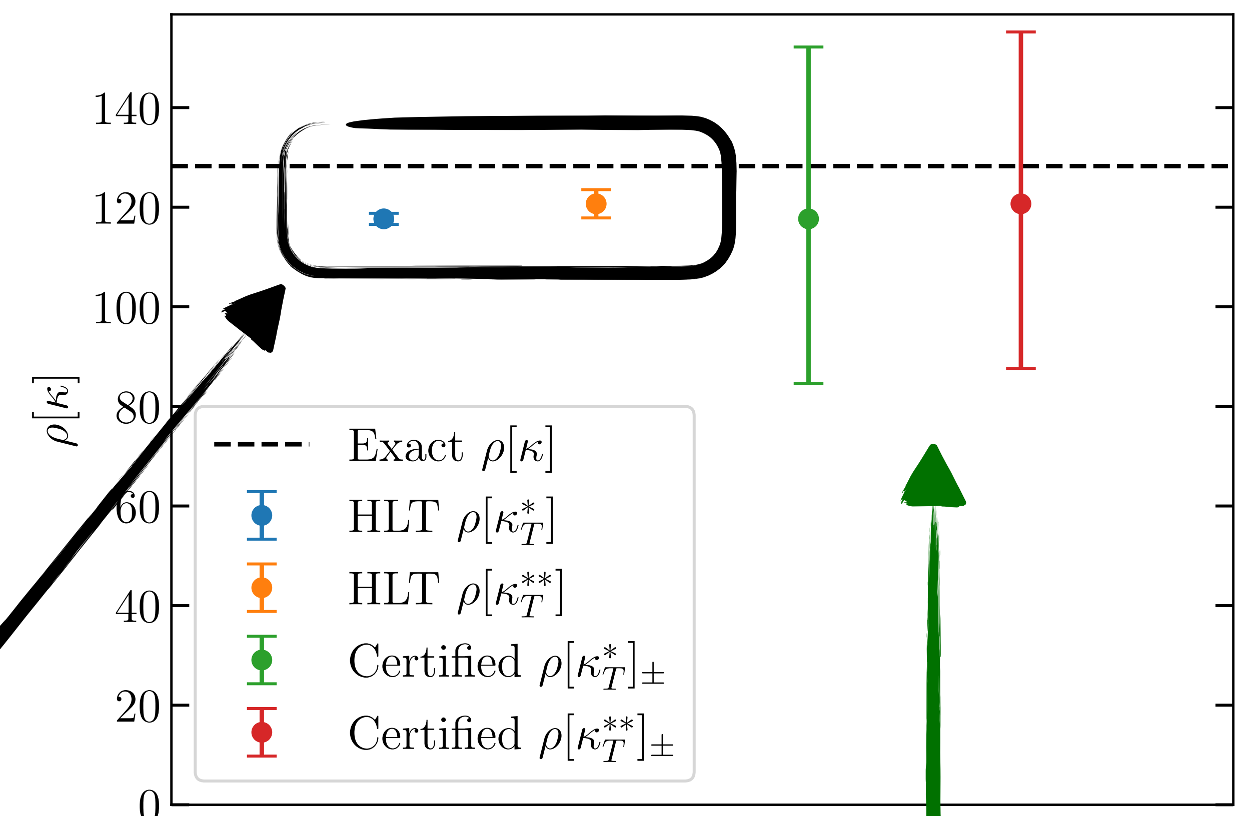
Certified bounding kernel $\kappa\_(\omega)$



Example of Certified Bounds

Comparison to exact result $a\omega_0 = 0.25$, $a\sigma = 0.07$





Results of stability analysis

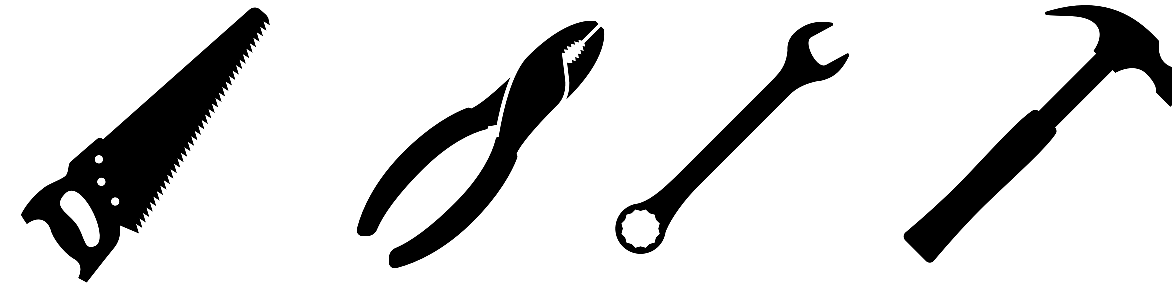
Certified bounds

(Consistent with exact result by construction)

(Manifestly *not* tight)

Not shown: Optimization of bounds over regulator/t_{ref}. See 2606.19503.

Conclusions



- Spectral reconstructions are already being used in phenomenologically relevant calculations

- My hope: these methods will join our field's toolkit for robust uncertainty quantification and (eventually) full systematic control

- Spectral positivity is a powerful nonlinear constraint and leads to kernel bounds

- Certification can be applied to existing linear methods to give (statistical) bounds of the kernel-mismatch bias $\rho[\kappa - \kappa_T]$

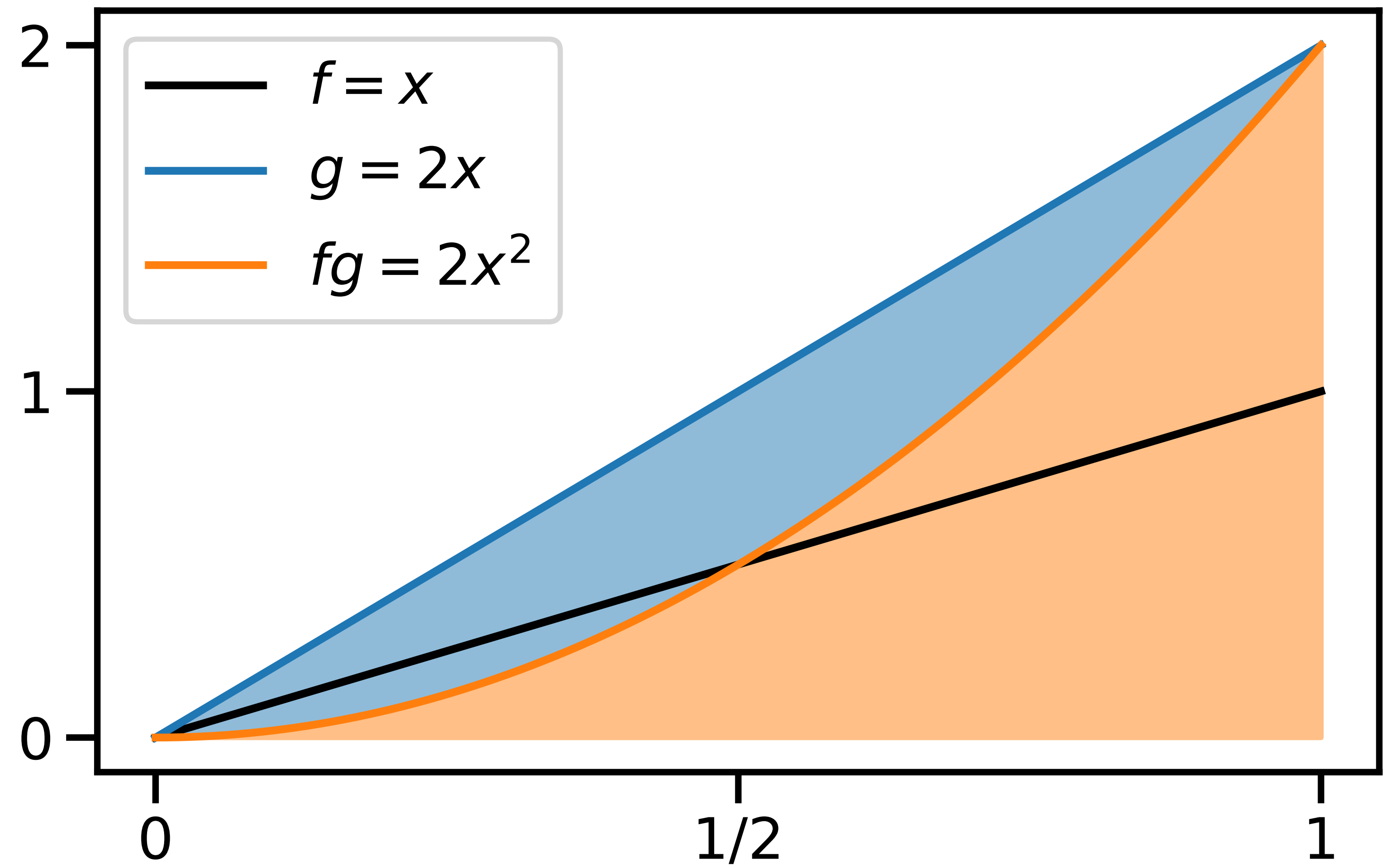
- With finite noise, certified bounds from linear methods complement existing stability analyses and alternative bounds from SDPs/"causal bootstrap"

Backup

Hölder's Inequality

Illustrative example

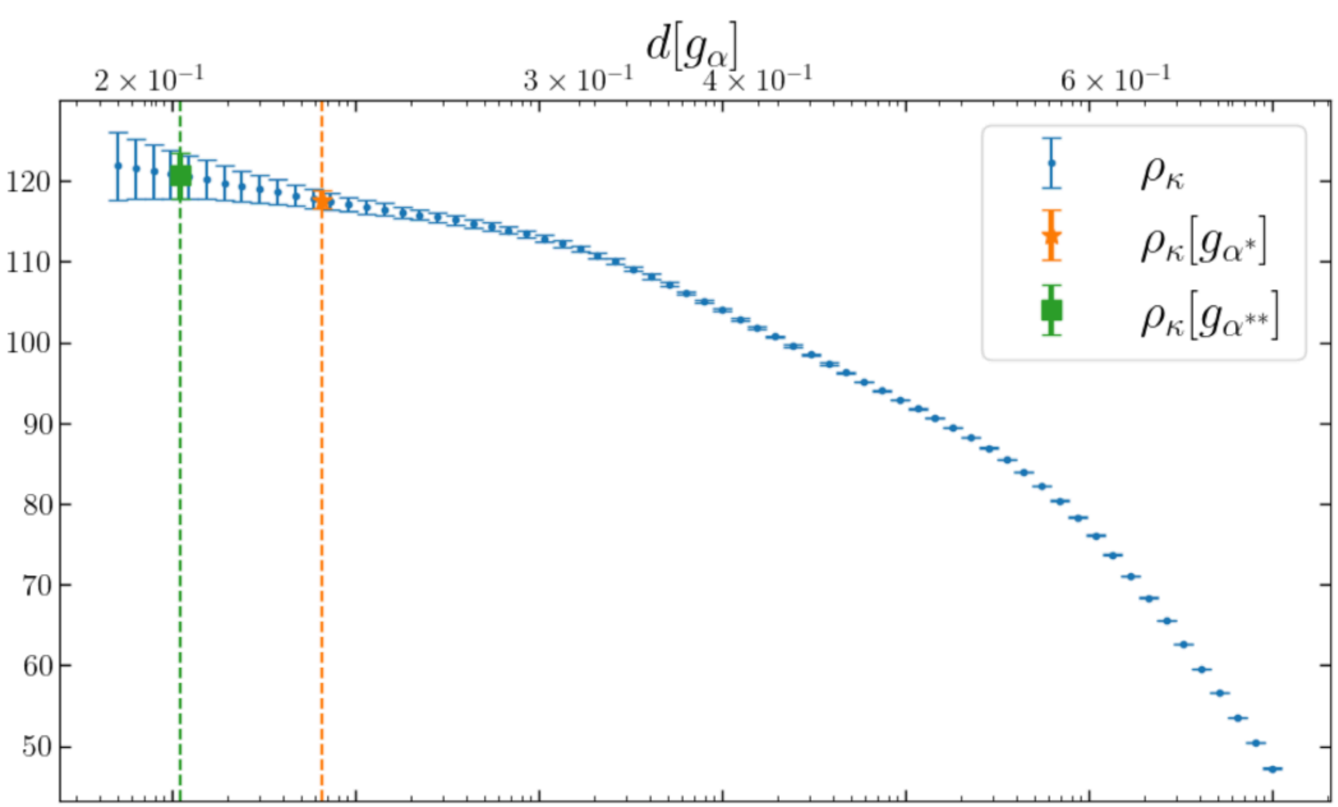
- $f(x) = x$ and $g(x) = 2x$
  - $\int_0^1 dx\, f(x) g(x) = \frac{2}{3}$
  - $\int_0^1 dx\, g(x) = 1$
  - $\max_{x \in [0,1]} f = 1$
  - Hölder: $\frac{2}{3} \leq 1 \cdot 1$

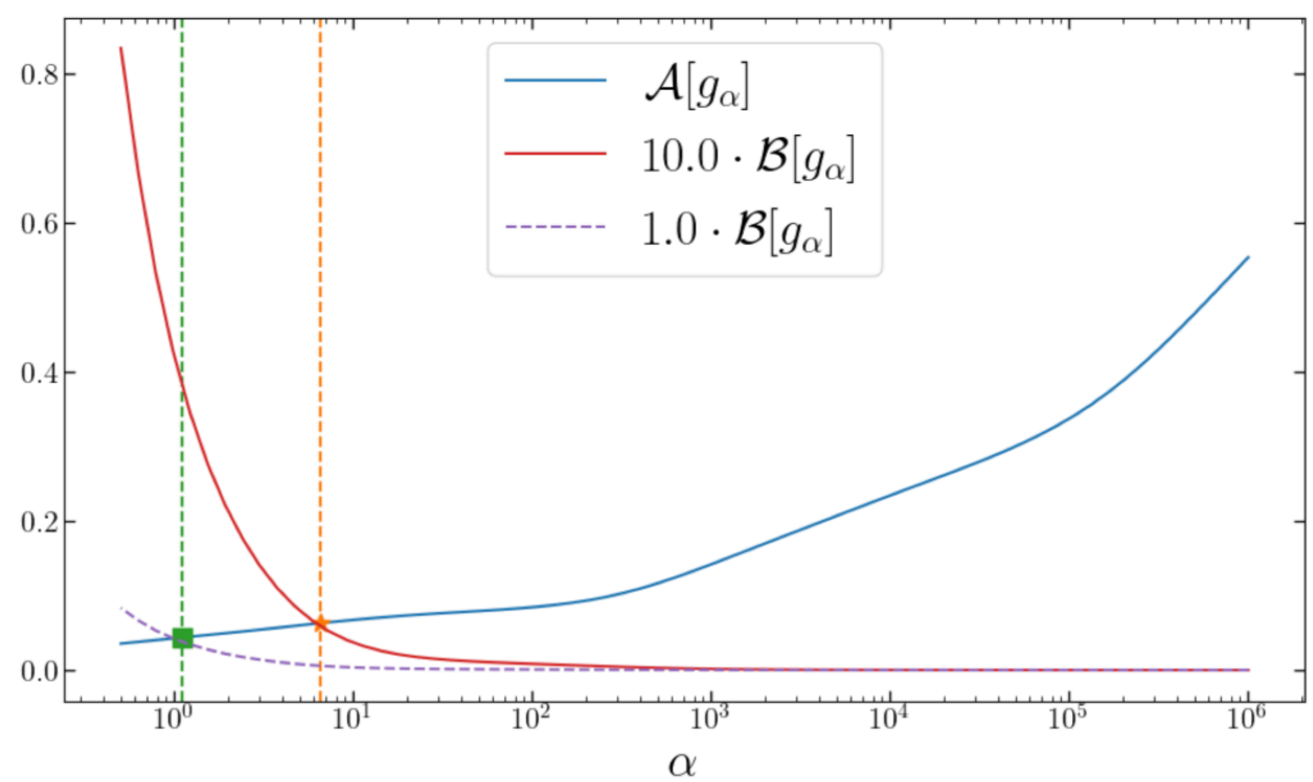


Numerical Example - Details

HLT Stability Analysis

- HLT minimizes convex functional $W[\alpha, \mathbf{g}] = (1-\alpha)A[\mathbf{g}] + \alpha B[\mathbf{g}]$
- $A[g]$: L^2 distance to target kernel
- $B[g]$: statistical regulator, $\propto \mathbf{g}^T \Sigma \mathbf{g}$
- $d[\mathbf{g}_\alpha]$: conventional measure of regulator size (cf. 2212.08467)





william.jay@colostate.edu

Numerical Example - Details

Covariance / correlation matrix

- Errors decay exponentially, but slower than C(t)
- Correlations decay smoothly away from the diagonal