Finite-volume corrections to smeared spectral densities

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Outline

Motivation

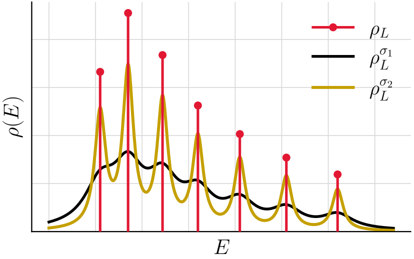
(Smeared) Spectral densities

- Probe several key phenomenological processes (e^+e^- hadron production, $\pi\pi$ scattering, inclusive τ decays) and observables (smeared R-ratio, timelike pion form factor, CKM matrix elements).
- LQCD: first principles, non-perturbative framework.
  Primary observables $\rightarrow$ Euclidean correlators
- Spectral densities are related to Euclidean, finite-volume correlators

$$C_L(t) = \int_0^\infty d\omega \rho_L(\omega) e^{-\omega|t|}$$

$\rho_L(\omega)$ from $C_L(t)$: Inverse Laplace

Inverse Laplace methods and finite-volume effects



- Smearing: $\widehat{\rho}_{L,\kappa}(\boldsymbol{\alpha}) = \int_0^\infty d\omega\ \widehat{\kappa}(\omega, \boldsymbol{\alpha})\ \rho_L(\omega)$

  $\widehat{\kappa}(\omega, \boldsymbol{\alpha}) \rightarrow$ formally required
  $\rightarrow$ physically motivated

- (Linear) Inverse Laplace methods: given $\{C_L(\tau_i)\}_{i=1}^N$,

  if $\{c_i(\boldsymbol{\alpha})\}_{i=1}^N$ s.t. $\widehat{\kappa}(\omega|\boldsymbol{c}(\boldsymbol{\alpha})) = \sum_{i=1}^N c_i(\boldsymbol{\alpha})\, e^{-\omega\tau_i} \rightarrow \widehat{\rho}_{L,\kappa}(\boldsymbol{c}(\boldsymbol{\alpha})) = \sum_{i=1}^N c_i(\boldsymbol{\alpha})\, C_L(\tau_i)$

  Vast literature: [G. Backus, J. Gilbert, '68] [M. Hansen, A. Lupo, N. Tantalo, '19]
  [M. Bruno, M. Saccardi, L. Giusti, '24] [G. Bailas, S. Hashimoto, T. Ishikawa, '20] ...

- **Focus of this work**: theoretical prediction of **finite-volume effects** on smeared spectral densities, starting from correlators

$$\widehat{\rho}_{L,\kappa}(\boldsymbol{\alpha}) = \widehat{\rho}_{\kappa}(\boldsymbol{\alpha}) + \delta\widehat{\rho}_{L,\kappa}(\boldsymbol{\alpha})$$

Setting the stage

Focus on

$$G_L(\tau) = -\frac{1}{3}\sum_{k=1}^{3}\int_0^L \mathrm{d}^3\boldsymbol{x}\,\langle j_k(x) j_k(0)\rangle_L, \qquad j_k : I = 1 \text{ Euclidean vector current}$$

$$\rho_L(\omega) = -\frac{L^3}{3\omega^2}\sum_{k=1}^{3}\langle 0| j_k(0)\delta_{\widehat{\boldsymbol{P}},\boldsymbol{0}}\delta(\omega - \widehat{H}) j_k(0)|0\rangle_L$$

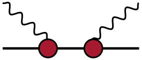
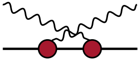
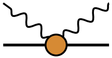
$$G_L(\tau) = \int_0^\infty \mathrm{d}\omega\, \omega^2\, e^{-\omega\tau}\,\rho_L(\omega) \quad \rightarrow \quad \text{Inverse Laplace}: \ \widehat{\kappa}(\omega|\boldsymbol{c}(\boldsymbol{\alpha})) = \omega^2 \sum_{i=1}^{N} c_i(\boldsymbol{\alpha})\, e^{-\omega\tau_i}$$

ω^2 : from Lorentz structure of j_k

Derivation 1

Contour deformation of $\delta G_L(\tau)$

$\delta G_L(\tau) = G_L(\tau) - G(\tau)$ driven by compton amplitude [M. T. Hansen, A. Patella, '20] [M. Lüscher, '85]

$\propto$  $+$  $+$ 

$$= \delta G_L^{\text{pole}} + \delta G_L^{\text{reg}} = (\delta G_L^A + \underbrace{\delta G_L^B}_{\mathcal{O}(e^{-m_\pi L}/L^2)}) + \delta G_L^{\text{reg}}$$

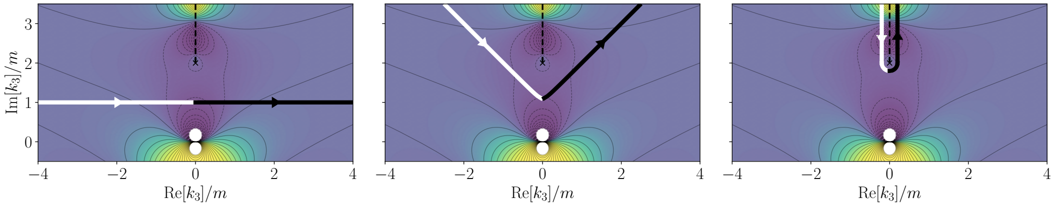
Leading term $\delta G_L^A(\tau) = \dfrac{1}{6}\displaystyle\sum_{\boldsymbol{n}\neq\boldsymbol{0}} \operatorname{Im}\int_{\mathbb{R}+i\mu} \frac{\mathrm{d}k_3}{2\pi}\, \frac{e^{-|\boldsymbol{n}|L\sqrt{m^2+k_3^2/4}}}{4\pi k_3|\boldsymbol{n}|L} e^{ik_3|\tau|}(4m^2+k_3^2)F(-k_3^2)^2$

Main intuition: contour deformation $e^{ik_3|\tau|} \to e^{-k_3|\tau|} \xrightarrow{\text{Linear Inverse Laplace}} \delta\widehat{\rho}_{L,\kappa}(\boldsymbol{\alpha})$

Contour deformation of $\delta G_L(\tau)$

$$\delta G_L^A(\tau) = \frac{1}{6}\sum_{\boldsymbol{n}\neq\boldsymbol{0}} \mathrm{Im}\int_{\mathbb{R}+i\mu}\frac{\mathrm{d}k_3}{2\pi}\,\frac{e^{-|\boldsymbol{n}|L\sqrt{m^2+k_3^2/4}}}{4\pi k_3|\boldsymbol{n}|L}e^{ik_3|\tau|}(4m^2+k_3^2)F(-k_3^2)^2$$

k_3 energy coordinate of two-pion state → first Riemann sheet, no resonance poles



Wick rotation: $F(-k_3^2) \to F(k_3^2)$, $\quad k_3 \sim s$

Watson's theorem: $F(s)^2 = |F(s)|^2 e^{2i\delta_{\pi\pi}(p)}$, $\quad p = \sqrt{\frac{s}{4}-m^2}$, $\quad s \sim k_3^2$

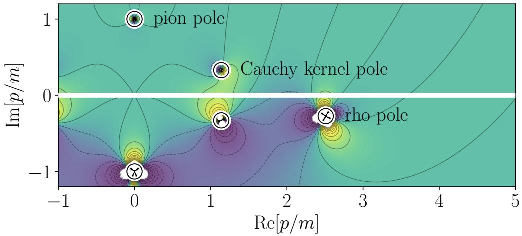
$\delta_{\pi\pi}(p)$ pion-pion scattering phase shift in the isospin-1 channel (elastic regime)

$$\delta G_L^A(\tau) = -\frac{1}{6}\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\frac{1}{\pi^2|\boldsymbol{n}|L}\mathrm{Im}\int_0^\infty \mathrm{d}p\,\frac{p^3 e^{-2\sqrt{p^2+m^2}|\tau|}}{p^2+m^2}\left|F\big(4(p^2+m^2)\big)\right|^2 e^{-2i\delta_{\pi\pi}(p)}e^{-i|\boldsymbol{n}|Lp}$$

$$\delta G_L^A(\tau) \xrightarrow{\text{Inverse Laplace}} \delta\widehat{\rho}_{L,\kappa}^A(\alpha)$$

Final result

$$\delta\widehat{\rho}^{A}_{L,\kappa}(\alpha) = \frac{1}{6}\sum_{\boldsymbol{n}\neq\boldsymbol{0}} \frac{1}{4\pi^2|\boldsymbol{n}|L}\,\mathrm{Im}\int_0^\infty \mathrm{d}p\, p^3 \underbrace{\frac{1}{(p^2+m^2)^2}}_{\text{pion pole}} \overbrace{\left|F\big(4(p^2+m^2)\big)\right|^2}^{\rho\text{ pole}} e^{i2\delta_{\pi\pi}(p)+i|\boldsymbol{n}|Lp}\ \underbrace{\widehat{\kappa}(2\sqrt{p^2+m^2},\boldsymbol{\alpha})}_{\text{kernel singularities}}$$



$$\widehat{\kappa}(2\sqrt{p^2+m^2};\boldsymbol{\alpha}) = \frac{1}{\pi}\frac{\sigma}{\left(2\sqrt{p^2+m^2}-\omega^*\right)^2+\sigma^2}$$

Exponential suppression:

$$p \to p + i\mu \quad \Rightarrow \quad e^{ipL} \to e^{-\mu L}, \quad \mu \propto \mathrm{Im}(p^*), \quad p^* = \text{first singularity hit by contour shift}$$

Derivation 2

Lellouch–Lüscher–Meyer formalism

$$G_L(\tau) = \frac{1}{3}\sum_n c_n(L)\, e^{-E_n(L)\tau}, \qquad c_n(L) = L^3 \sum_{k=1}^{3} |\langle n, \mathbf{0}|j_k(0)|0\rangle_L|^2$$

$$E_n(L) \longleftrightarrow \mathcal{Q}(E_n, L) = \pi n, \qquad c_n(L) \longleftrightarrow |F(E_n^2)|^2 \text{ by } \partial_{E_n}\mathcal{Q}(E_n, L)$$

[M. Lüscher, '86, '91] [L. Lellouch, M. Lüscher, '01] [H. B. Meyer, '11]

$$\widehat{\rho}_{L,\kappa}(\boldsymbol{\alpha}) = \frac{1}{3}\sum_{n=1}^{\infty} \frac{c_n(L)}{E_n(L)^2}\,\widehat{\kappa}(E_n(L), \boldsymbol{\alpha})$$

- $\sum_{n=-\infty}^{\infty} \delta\big(n\pi - \mathcal{Q}(E,L)\big) = \frac{1}{\pi}\sum_{l=-\infty}^{\infty} e^{2il\mathcal{Q}(E,L)}$ $\longrightarrow$ recover factors $e^{2i\delta_{\pi\pi}(p)}e^{i|\boldsymbol{n}|pL}$
- Large-L expansion Lüscher zeta function

Final expression: $\delta\widehat{\rho}^{A}_{L,\kappa}(\alpha) + \underbrace{\ldots}_{\text{subleading corrections}}$

Lellouch–Lüscher–Meyer formalism

$$G_L(\tau) = \frac{1}{3}\sum_n c_n(L)\, e^{-E_n(L)\tau}, \qquad c_n(L) = L^3 \sum_{k=1}^{3} |\langle n, \mathbf{0}| j_k(0)|0\rangle_L|^2$$

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[M. Lüscher, '86, '91] [L. Lellouch, M. Lüscher, '01] [H. B. Meyer, '11]

$$\widehat{\rho}_{L,\kappa}(\boldsymbol{\alpha}) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{c_n(L)}{E_n(L)^2}\,\widehat{\kappa}(E_n(L), \boldsymbol{\alpha}) = \frac{1}{3}\sum_{n=1}^{\infty}\int_{2m}^{\infty} dE\, \frac{\delta\big(E - E_n(L)\big)}{\mathcal{Q}^{(1,0)}(E, L)}\frac{p(E)^3}{2\pi E^3}\,|F(E^2)|^2\,\widehat{\kappa}(E, \boldsymbol{\alpha})$$

$$\delta(g(x)) = \frac{\delta(x - x_0)}{g'(x_0)} \longrightarrow \sum_{n=-\infty}^{\infty}\delta\big(n\pi - \mathcal{Q}(E, L)\big) = \frac{1}{\pi}\sum_{l=-\infty}^{\infty} e^{2il\mathcal{Q}(E,L)} \qquad \text{(PSF)}$$

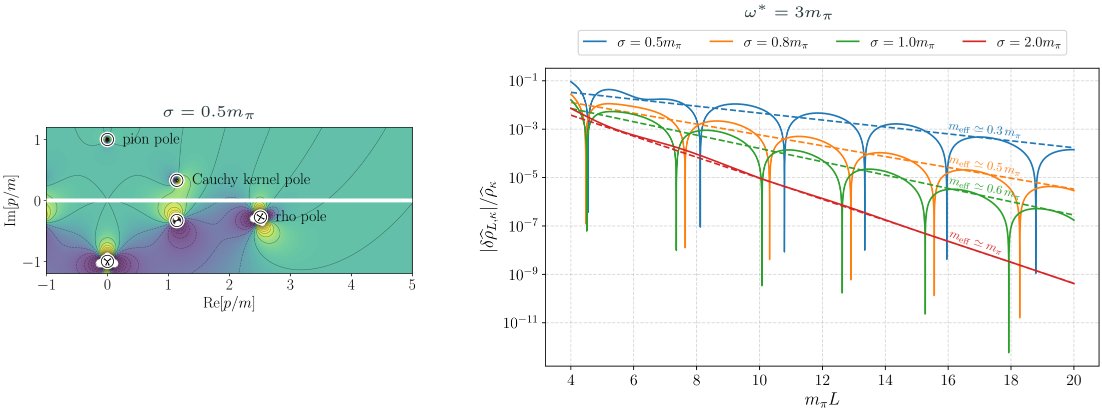
$$\text{(PSF) } l = 1 \longrightarrow e^{2i\delta_{\pi\pi}(p)}e^{2i\phi(q(E,L))} \xrightarrow{\text{large } L\text{-expansion}} \frac{e^{2i\delta_{\pi\pi}(p)}}{2ipL}\sum_{\boldsymbol{n}\neq\mathbf{0}}\frac{e^{i|\boldsymbol{n}|pL}}{|\boldsymbol{n}|} + \underbrace{\mathcal{O}(e^{2ipL}/(Lp)^2)}_{\text{subleading corrections}}$$

$$2\,\mathrm{Re}\,\overbrace{\widehat{\rho}^{(1)}_{L,\kappa}(\boldsymbol{\alpha})}^{l=1} = \delta\widehat{\rho}^{A}_{L,\kappa}(\alpha) + \underbrace{\cdots}_{\text{subleading corrections}}$$

Numerical explorations

Gounaris-Sakurai model

$F(s)$ Gounaris-Sakurai, $\widehat{\kappa}(\sqrt{s}, \boldsymbol{\alpha})$ Cauchy, $\boldsymbol{\alpha} = (\omega^*, \sigma)$



$$|F(4(p^2+m^2))|^2 e^{2i\delta_{\pi\pi}(p)} = 1 \rightarrow \quad \delta\widehat{\rho}^A_{L,\kappa}(\alpha)\Big|_{\text{Cauchy poles}} \propto \frac{e^{-m_{\mathrm{eff}}L}}{L}, \; m_{\mathrm{eff}} = \frac{1}{2}\,\mathrm{Im}\sqrt{(\omega^*+i\sigma)^2 - 4m^2}$$

Conclusions & Outlooks

Wrapping up

Summary

- Universal expression for $\delta\widehat{\rho}_{L,\kappa}(\boldsymbol{\alpha})$. Naive expectation $e^{-\sigma L}$ made precise.
- Two complementary approaches to describe FVE

Spacelike FF		Timelike FF
Hansen–Patella	$\xleftrightarrow{\text{Wick rotation}}$	Lellouch–Lüscher–Meyer
correlator FVE		matrix elements FVE

- Sufficiently reliable model of F: FVE of smeared spectral densities quantifiable.

Outlooks

- Compare estimate of FVE of smeared densities from real data with our formula. Expected validity at low energies.
- Extension to more general spectral densities (e.g. three-particle states)
- Main result can be expressed as FVE $[\rho_\kappa] = \int d\omega\ \kappa \times \underbrace{\Delta(L)}_{\text{FVE kernel}} \times \underbrace{[\rho]}_{\text{Spec. dens. itself}}$

Thank you!

Backup

Backup: derivation (I)

$$\delta G_L(\tau) = -\frac{1}{6}\sum_{n\neq 0}\int \frac{dp_3}{2\pi}\frac{e^{-|\boldsymbol{n}|L\sqrt{m^2+p_3^2}}}{4\pi|\boldsymbol{n}|L}\int\frac{dk_3}{2\pi}\cos(k_3\tau)\,\mathrm{Re}\,T(-k_3^2,-p_3k_3), \quad T = T_{\text{pole}} + T_{\text{reg}}$$

$$T_{\text{pole}}(-k_3^2,-p_3k_3) = \sum_{x=\pm}\frac{2(4m^2+k_3^2)F(-k_3^2)^2}{k_3^2+2xk_3p_3-i\epsilon} \quad \Rightarrow \quad \delta G_L(\tau) = \delta G_L^{\text{pole}}(\tau) + \delta G_L^{\text{reg}}(\tau),\ \delta G_L^{\text{pole}}(\tau) = \delta G_L^A(\tau) + \delta G_L^B(\tau)$$

$$\delta G_L^A(\tau) = \frac{1}{6}\sum_{n\neq 0}\mathrm{Im}\int_{\mathbb{R}+i\mu}\frac{dk_3}{2\pi}\frac{e^{-|\boldsymbol{n}|L\sqrt{m^2+k_3^2/4}}}{4\pi k_3|\boldsymbol{n}|L}e^{ik_3|\tau|}(4m^2+k_3^2)F(-k_3^2)^2, \quad \delta G_L^A(\tau) = \delta G_L^{A,+}(\tau) + \delta G_L^{A,-}(\tau)$$

$$\delta G_L^{A,-}(\tau) = \delta G_L^{A,+}(\tau), \quad k_3 = xe^{i\theta} + i\mu$$

$$\delta G_L^{A,+}(\tau) = \frac{1}{6}\sum_{\boldsymbol{n}\neq \boldsymbol{0}}\mathrm{Im}\left[e^{i\theta}\int_0^\infty\frac{\mathrm{d}x}{2\pi}\frac{e^{i(xe^{i\theta}+i\mu)|\tau|-|\boldsymbol{n}|L\sqrt{m^2+(xe^{i\theta}+i\mu)^2/4}}}{4\pi(xe^{i\theta}+i\mu)|\boldsymbol{n}|L}\left(4m^2+(xe^{i\theta}+i\mu)^2\right)F\left[-(xe^{i\theta}+i\mu)^2\right]^2\right]$$

$$\overset{\theta\to\frac{\pi}{2}}{=}\frac{1}{6}\sum_{\boldsymbol{n}\neq \boldsymbol{0}}\mathrm{Im}\left[\int_{2m}^\infty\frac{\mathrm{d}x}{2\pi}\frac{e^{-x|\tau|-i|\boldsymbol{n}|L\sqrt{x^2/4-m^2}}}{4\pi x|\boldsymbol{n}|L}\left(4m^2-x^2\right)F((x-i0^+)^2)^2\right]$$

$$\delta G_L(\tau) = -\frac{1}{6}\sum_{\boldsymbol{n}\neq \boldsymbol{0}}\frac{1}{\pi^2|\boldsymbol{n}|L}\mathrm{Im}\int_0^\infty \mathrm{d}p\,\frac{p^3e^{-2\sqrt{p^2+m^2}|\tau|}}{p^2+m^2}\left|F\left(4(p^2+m^2)\right)\right|^2 e^{-2i\delta_{\pi\pi}(p)}e^{-i|\boldsymbol{n}|Lp} + \varepsilon_L(\tau)$$

Backup: derivation (I)

$$4(p^2+m^2)\sum_{i=1}^{N} c_i(\boldsymbol{\alpha})\, e^{-2\sqrt{p^2+m^2}\tau_i} = \widehat{\kappa}(2\sqrt{p^2+m^2},\boldsymbol{\alpha}), \quad \epsilon_{L,\kappa}(\boldsymbol{\alpha}) = \sum_{i=1}^{N} c_i(\boldsymbol{\alpha})\, \varepsilon_L(\tau_i)$$

$$\delta\widehat{\rho}^{A}_{L,\kappa}(\boldsymbol{\alpha}) = \frac{1}{6}\sum_{\boldsymbol{n}\neq\boldsymbol{0}} \frac{1}{4\pi^2|\boldsymbol{n}|L}\mathrm{Im}\int_0^\infty \mathrm{d}p\frac{p^3}{(p^2+m^2)^2}\left|F\big(4(p^2+m^2)\big)\right|^2 e^{i2\delta_{\pi\pi}(p)+i|\boldsymbol{n}|Lp}\,\widehat{\kappa}(2\sqrt{p^2+m^2},\boldsymbol{\alpha})$$

$$\delta\widehat{\rho}^{A}_{L,\kappa}(\boldsymbol{\alpha}) = \int \mathrm{d}\omega\, \rho_{\pi\pi}(\omega)\,\widehat{\kappa}(\omega,\boldsymbol{\alpha})\Delta(\omega,L)$$

$$\rho_{\pi\pi}(\omega) = \tfrac{p^3}{6\pi^2\omega^3}\left|F(\omega^2)\right|^2\Big|_{p=\sqrt{\omega^2/4-m^2}}, \quad \Delta(\omega,L) = \textstyle\sum_{\boldsymbol{n}\neq\boldsymbol{0}} \tfrac{\sin(2\delta_{\pi\pi}(p)+|\boldsymbol{n}|Lp)}{|\boldsymbol{n}|Lp}\Big|_{p=\sqrt{\omega^2/4-m^2}}$$

Backup: derivation (II) - Large-L expansion details

$$\mathcal{Q}(E,L) = \delta_{\pi\pi}(p(E)) + \phi(q(E,L)), \quad \cot\phi(q) = -\frac{\mathcal{Z}_{00}(1,q^2)}{q\pi^{3/2}}, \quad p(E) = \sqrt{E^2/4 - m^2}, \quad q(E,L) = \frac{p(E)L}{2\pi}$$

$$\cot\phi(q) = -\frac{1}{\pi p L}\lim_{\Lambda\to\infty}\sum_{\boldsymbol{n}}^{\Lambda}\left[\frac{1}{\boldsymbol{n}^2 - (pL/2\pi)^2} - 4\pi\Lambda\right] \overset{\boldsymbol{k}=\frac{2\pi\boldsymbol{n}}{L}}{=} -\frac{4\pi}{p}\lim_{\overline{\Lambda}\to\infty}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}}^{\overline{\Lambda}}\frac{1}{\boldsymbol{k}^2 - p^2} - \frac{\overline{\Lambda}}{2\pi^2}\right], \ \overline{\Lambda} = 2\pi\Lambda/L$$

$$(\text{PSF}) \longrightarrow \ \cot\phi(q) = -\frac{4\pi}{p}\int\frac{d^3x}{(2\pi)^3}\left(\frac{1}{x^2-p^2} - \frac{1}{x^2}\right) - \frac{4\pi}{p}\sum_{n\neq 0}\int\frac{d^3x}{(2\pi)^3}\frac{e^{iL\boldsymbol{n}\cdot\boldsymbol{x}}}{x^2-p^2}$$

$$\frac{4\pi}{p}\int\frac{d^3x}{(2\pi)^3}\left(\frac{1}{x^2-p^2} - \frac{1}{x^2}\right) = i, \qquad \frac{4\pi}{p}\sum_{n\neq 0}\int\frac{d^3x}{(2\pi)^3}\frac{e^{iL\boldsymbol{n}\cdot\boldsymbol{x}}}{x^2-p^2} = \frac{1}{Lp}\sum_{n\neq 0}\frac{1}{|n|}e^{iL|n|p}$$

$$\cot\phi(q) = -i - \frac{\zeta(pL)}{pL}, \qquad \zeta(z) = \sum_{n\neq 0}\frac{e^{iz|n|}}{|n|}, \quad e^{2iz} = \frac{\cot z + i}{\cot z - i}$$

$$e^{2i\phi(q)} = \frac{1}{2ipL}\frac{\zeta(pL)}{1+\zeta(pL)/(2ipL)} \ \Rightarrow \ e^{2i\phi(q)} = \frac{1}{2ipL}\sum_{n\neq 0}\frac{e^{i|n|pL}}{|n|} + \mathcal{O}\Big(e^{2ipL}/(Lp)^2\Big)$$

Backup: $\delta G_L^B(\tau)$ contribution

$$\delta G_L^B(\tau) = -\frac{1}{4\pi^3 L}\mathrm{Re}\left[\int \mathrm{d}p_z e^{-L\sqrt{m^2+p_z^2}} \int \mathrm{d}k_z \frac{e^{i(k_z+i\mu)|\tau|}(4m^2+(k_z+i\mu)^2)}{(k_z+i\mu)^2-4p_z^2} F\left[-(k_z+i\mu)^2\right]^2\right]$$

- Non-interacting limit $(F=1) \to \delta G_L^B(\tau) = 0$ for $\tau \neq 0$
- Interacting case $\to$ Large-L expansion analysis

$$\delta G_L^A(\tau) = -\frac{m^2}{2\pi L}\, e^{-mL} + \cdots$$

$$\delta G_L^B(\tau) = \frac{me^{-mL}}{4\pi^2 L^2}\, \mathcal{I}_L^B(\tau) + \cdots \qquad \mathcal{I}_L^B(\tau) \text{ bounded by } L\text{-independent constant}$$

$m\tau = 1$
$F(s) = [1 - s/M^2]^{-1}$
$M/m = 727/137$

mL	δG_L^A	δG_L^B	$\delta G_L^B/\delta G_L^A$
4	-1.51×10^{-4}	-1.01×10^{-5}	0.0667
5	-5.99×10^{-5}	-2.64×10^{-6}	0.0442
6	-2.21×10^{-5}	-7.38×10^{-7}	0.0333
7	-7.96×10^{-6}	-2.15×10^{-7}	0.0270
8	-2.82×10^{-6}	-6.46×10^{-8}	0.0229
9	-9.94×10^{-7}	-1.99×10^{-8}	0.0200
10	-3.49×10^{-7}	-6.24×10^{-9}	0.0179