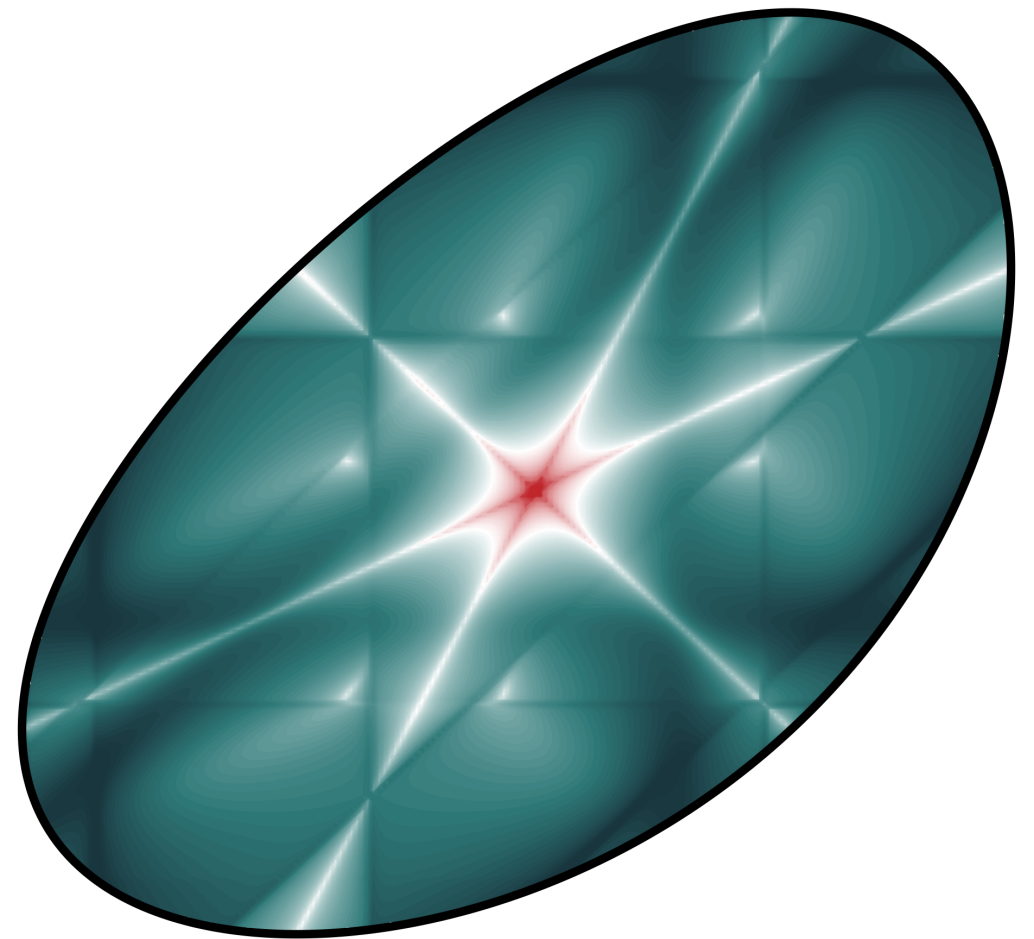


Symmetrizing Three-Body Partial Wave Amplitudes

Nicholas Chambers

with Andrew Jackura, Raúl Briceño

Lattice 2026, UMD College Park
July 30, 2026



WILLIAM & MARY

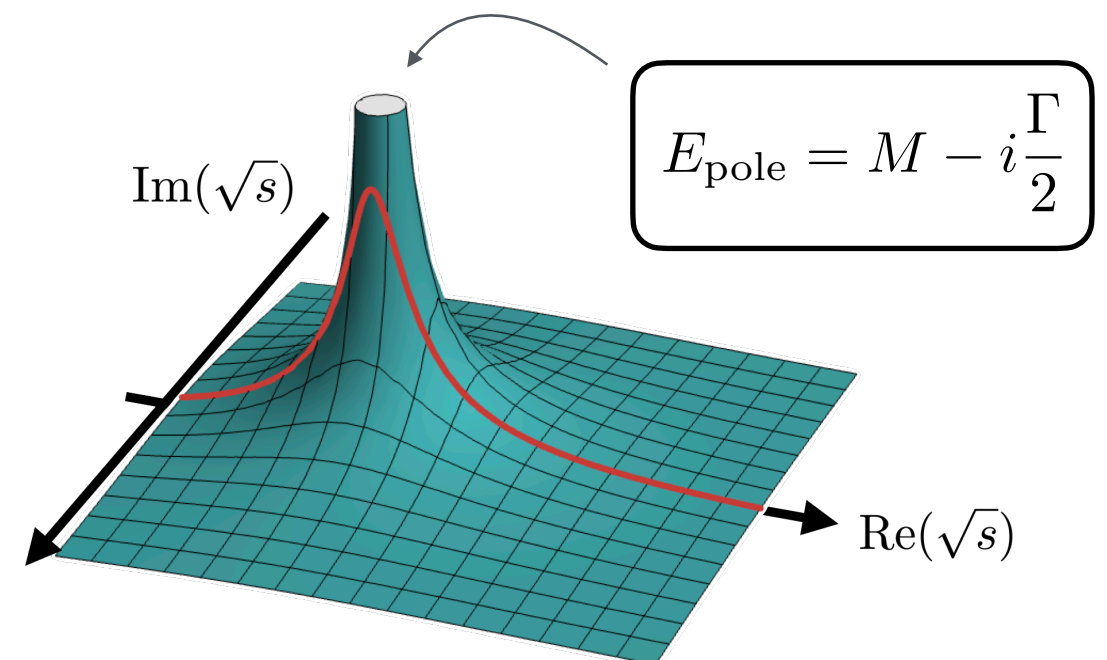
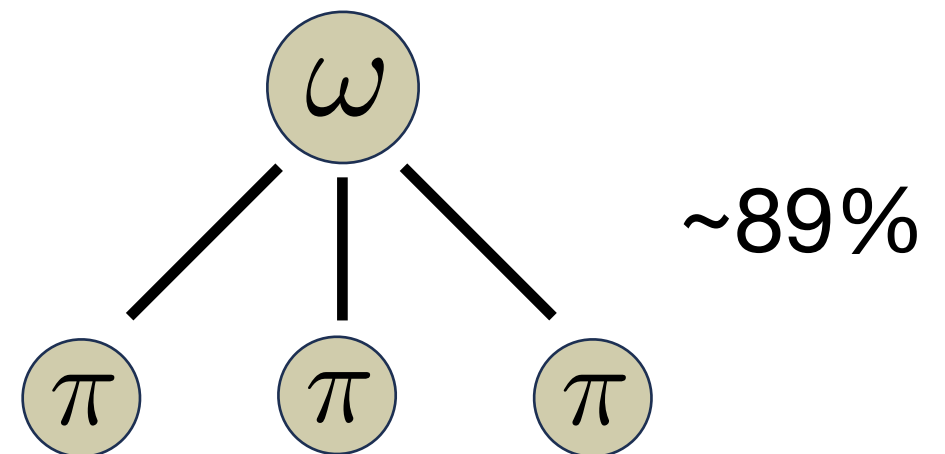
Three particles from QCD

3B = "Three-Body"

Many hadrons couple to
three-particle channels

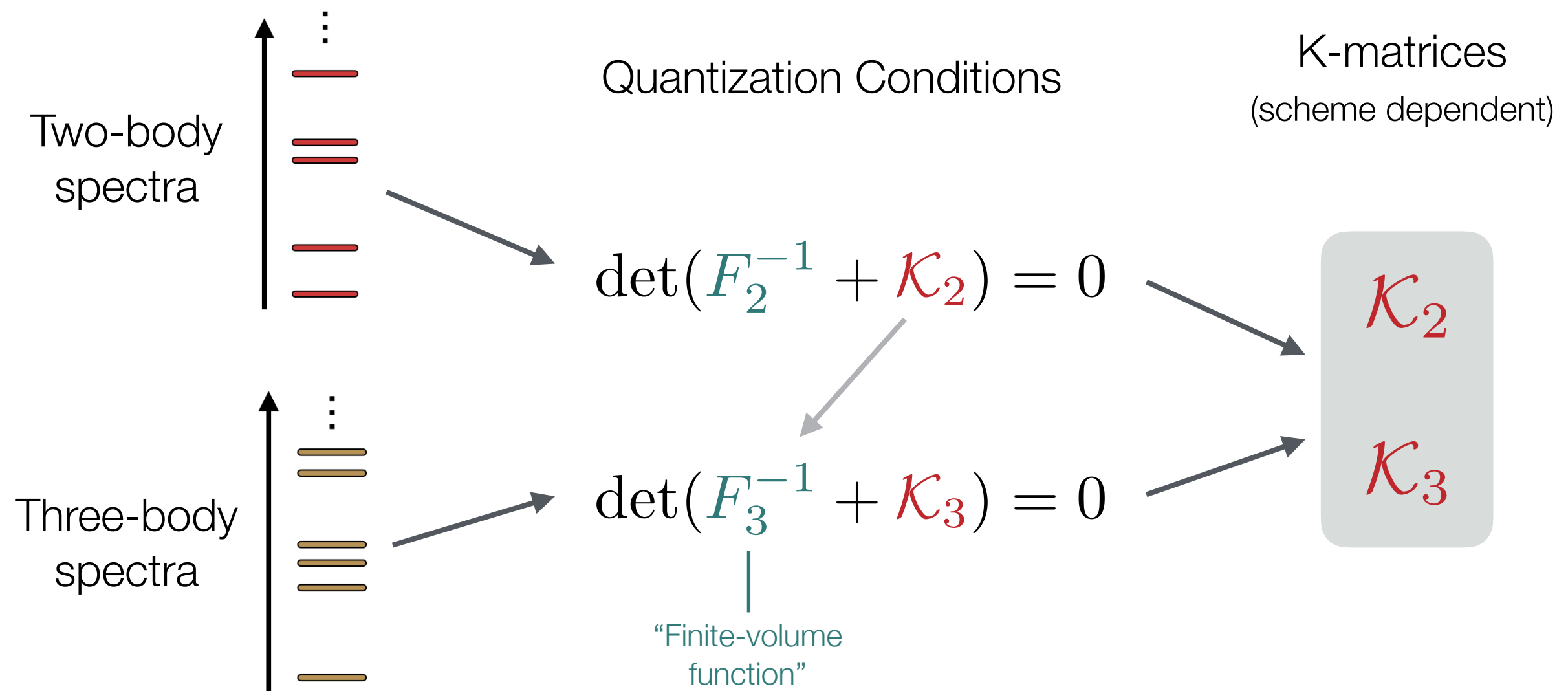
Amplitude analysis is essential

- Not just for resonance extraction
- Unique kinematic effects
- 2- and 3-body forces
- Lattice spectra and K-matrices are only a **piece** of the puzzle...



Lattice-to-Amplitude Workflow

Finite-Volume



Lattice-to-Amplitude Workflow

PW = "Partial Wave"

Infinite-Volume

QCD Input

Integral
Equations

$$i\mathcal{M}_3 = \text{[Diagram: 3 external lines meeting at a black vertex]} = \text{[Diagram: 3 external lines meeting at a white vertex]} + \text{[Diagram: 3 external lines, one white vertex, one black vertex]} + \text{[Diagram: 3 external lines, two white vertices, one loop]} + \text{[Diagram: 3 external lines, two black vertices, one loop]} + \text{[Diagram: 3 external lines, one white vertex, one black vertex, one loop]}$$

$\mathcal{K}_3$ and $\mathcal{K}_2$ are indicated by red lines pointing to the second and third diagrams respectively.

Post-analysis

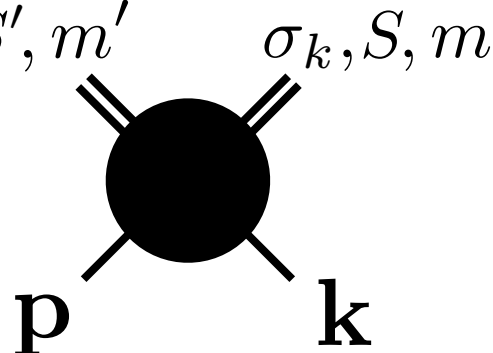
- PW Projection
- Analytic Continuation
- **Symmetrization**
- $\vdots$

A concluding step in the formalism...

Need for Symmetrization

- Integral equations use **pair-spectator** basis
 - Singles out initial/final momenta as **spectators**
 - Remaining two form a **pair**
- Final amplitude has unphysical dependence on spectator

Asymmetric Pair-Spectator Amplitude

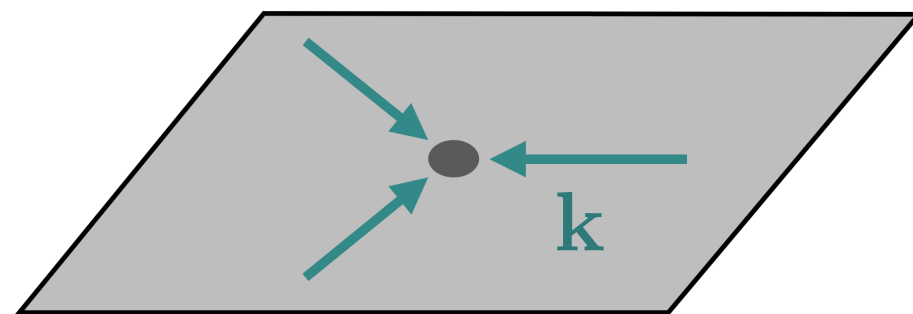
$$\mathcal{M}_{S'm';Sm}(\mathbf{p}, \mathbf{k}) =$$


The diagram shows a central black circle with four external lines. Two lines enter from the bottom, labeled $\mathbf{p}$ and $\mathbf{k}$. Two lines exit from the top, labeled σ_p, S', m' and σ_k, S, m . The top lines are double-lined, while the bottom lines are single-lined.

Partial Wave Formalism

Partial-Wave Projection

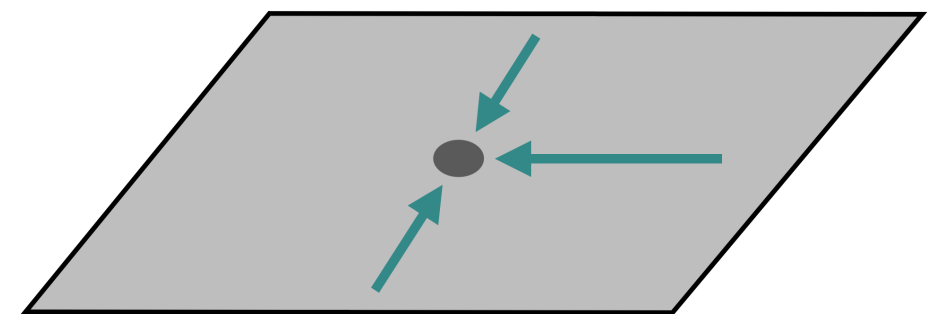
considering spinless particles...



3B rest frame (initial state)



boost



Pair rest frame

Pair-spectator Kinematics

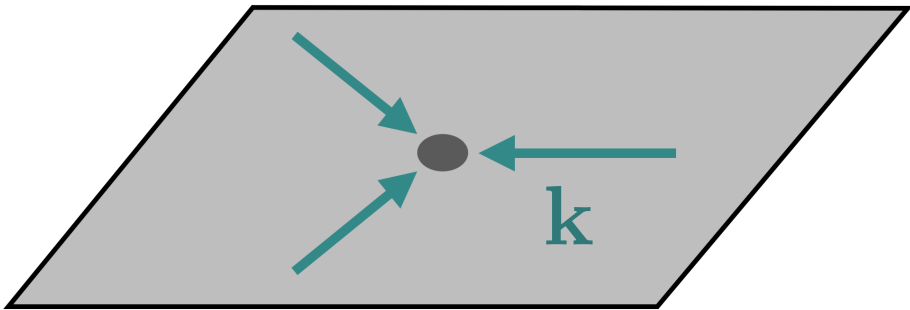
Spectator Momenta
Pair Invariant Masses
Spin of pairs
Pair-spectator orbital ang. mom

p **k**

Partial-Wave Projection

PW = "Partial Wave"

considering spinless particles...

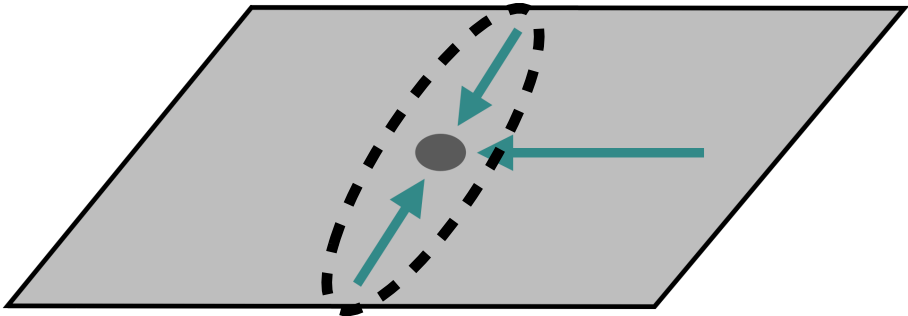


3B rest frame (initial state)



boost

PW project **pair** to
definite S (and isospin I_2)



Pair rest frame

Pair-spectator Kinematics

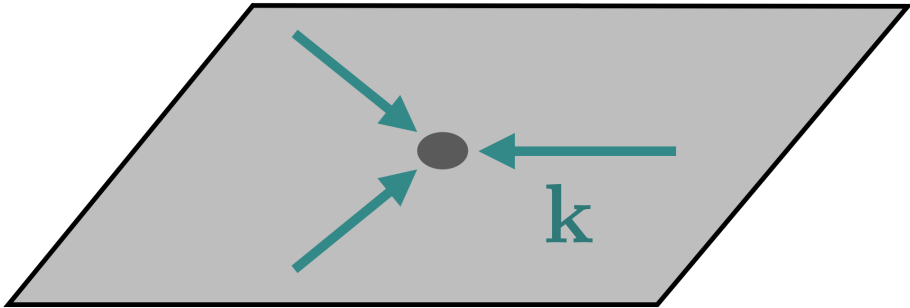
- Spectator Momenta
- Pair Invariant Masses
- Spin of pairs
- Pair-spectator orbital ang. mom

$\mathbf{p}$	$\mathbf{k}$
σ'_p	σ_k
S'	S

Partial-Wave Projection

PW = "Partial Wave"

considering spinless particles...

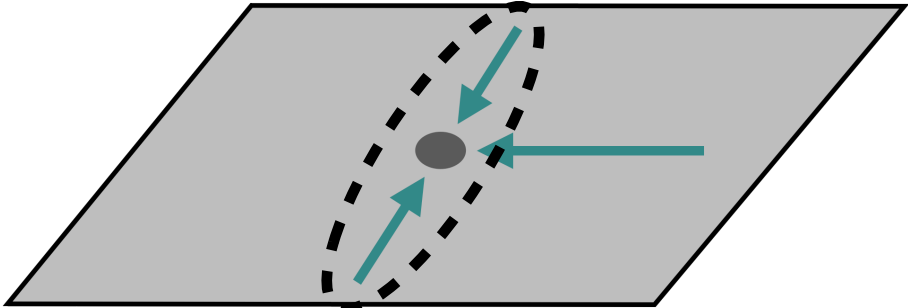


3B rest frame (initial state)



boost

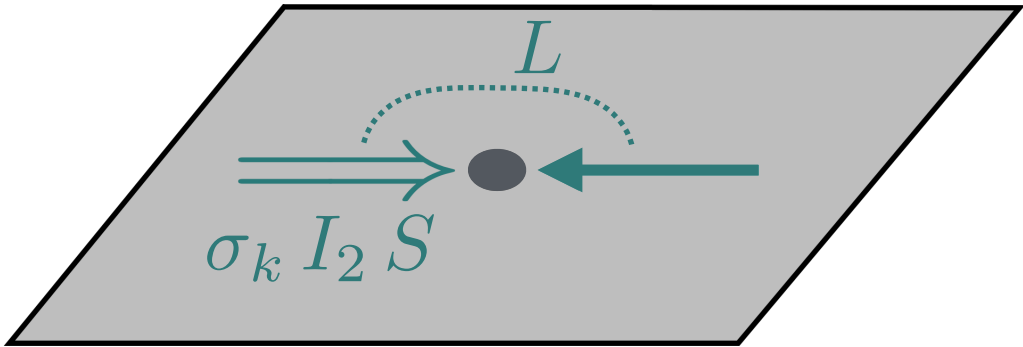
PW project **pair** to
definite S (and isospin I_2)



Pair rest frame



undo boost



3B rest frame
(spectator + quasi-particle)

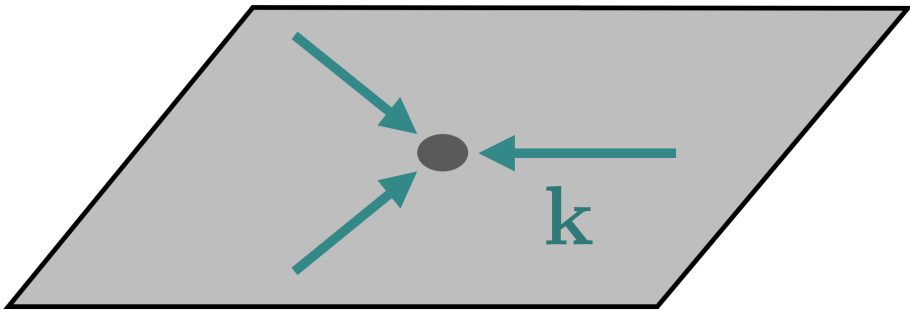
Pair-spectator Kinematics

Spectator Momenta	$\mathbf{p}$	$\mathbf{k}$
Pair Invariant Masses	σ'_p	σ_k
Spin of pairs	S'	S
Pair-spectator orbital ang. mom	L'	L

Partial-Wave Projection

PW = "Partial Wave"

considering spinless particles...

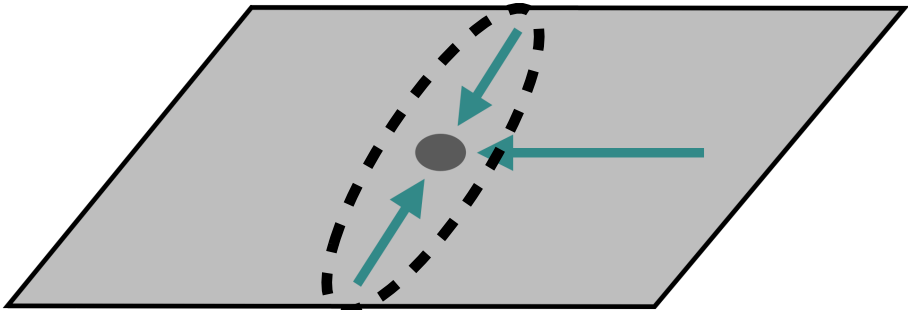


3B rest frame (initial state)



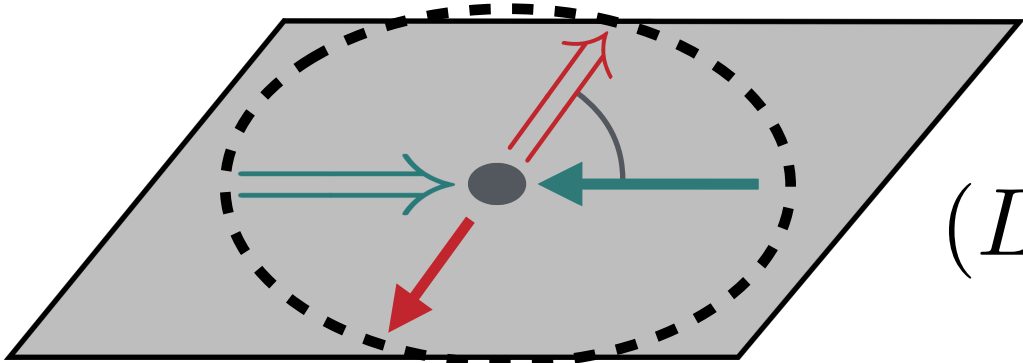
boost

PW project **pair** to
definite S (and isospin I_2)



Pair rest frame

undo boost



3B rest frame
(spectator + quasi-particle)

$$(L, S) \rightarrow J$$

Pair-spectator Kinematics

Spectator Momenta	$\mathbf{p}$	$\mathbf{k}$
Pair Invariant Masses	σ'_p	σ_k
Spin of pairs	S'	S
Pair-spectator orbital ang. mom	L'	L
Total ang. mom	J	

We need to symmetrize...

The pair-spectator basis creates an unphysical dependence on spectator choice

$$\begin{array}{ccc}
 \text{Symmetrized} & & \text{Asymmetric Pair-Spectator Amplitude} \\
 \mathcal{M} & & \mathcal{M}^{(j',j)}
 \end{array}$$

$$\text{Diagram } \mathcal{M}_3 = \sum_{j'j} \left\{ \text{Diagram } \mathcal{M}_3^{(j',j)} \right\}$$

The diagram on the left is a teal circle with three external lines. The diagram on the right is a red circle with three external lines, labeled with red indices: j' and j at the bottom, and $j'+1$ and $j+1$ at the top, with $j'+2$ and $j+2$ above them. The entire expression is enclosed in large curly braces.

Notation

Spectator Index

$$j, j' = 1, 2, 3$$

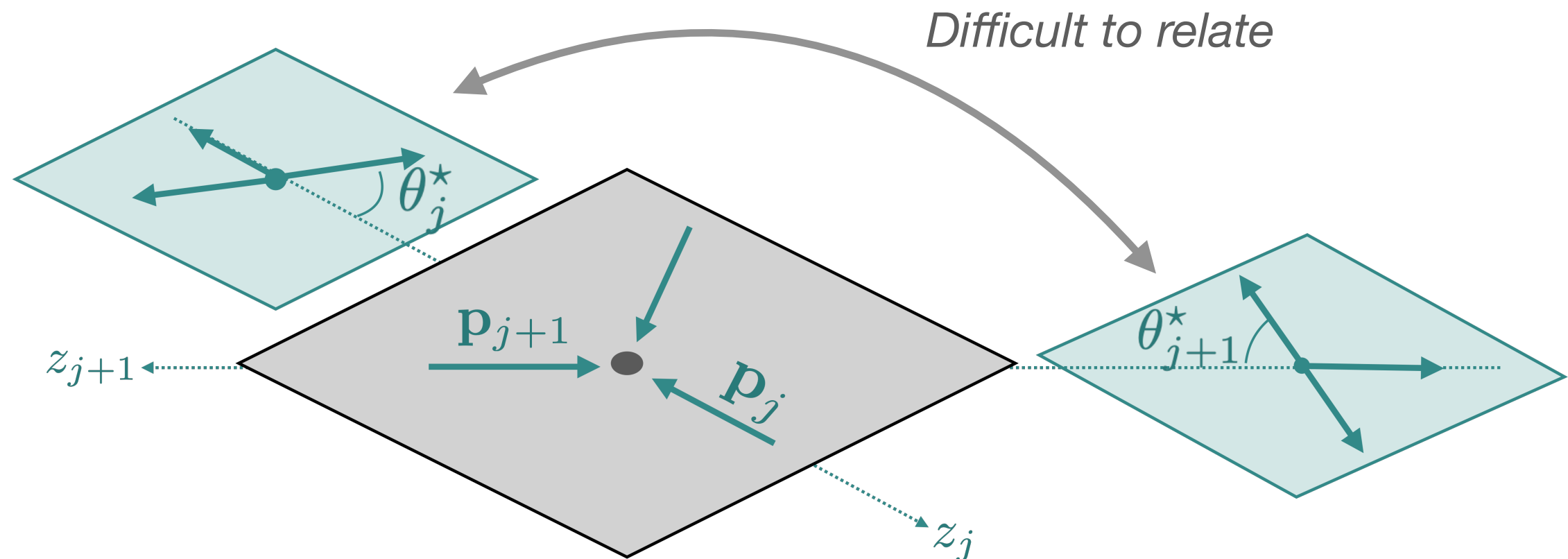
$$\mathbf{p}_j \rightarrow \mathbf{p}_{j+1} \rightarrow \mathbf{p}_{j+2}$$

We need to symmetrize...

Key Concept

Difficult to symmetrize in pair-spectator basis

- Requires relating angular kinematics in different Lorentz frames

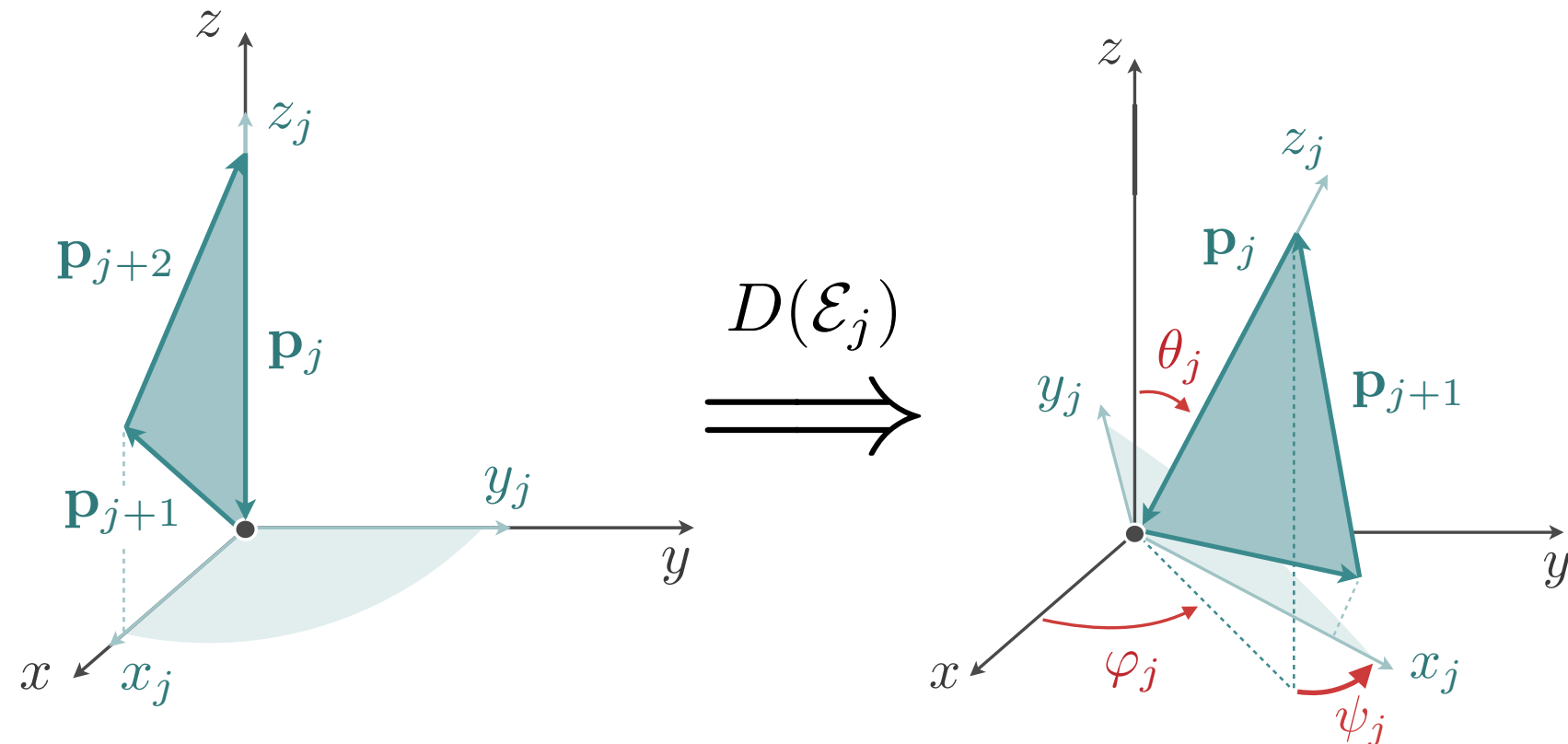


...so let's try a new basis

Dalitz Basis

Separate angular kinematics in
3B rest frame

No more boosting

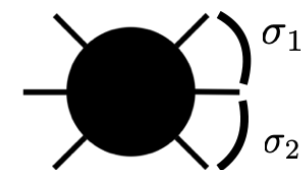


Dalitz PW Projection

$$\mathcal{M}^{(j',j)}(\mathbf{p}, \mathbf{k}) = \sum_J \sum_{\lambda'_{j'}, \lambda_j} \mathcal{M}^J_{\lambda'_{j'}, \lambda_j}(\{\sigma'\}, \{\sigma\}) D^{(J)*}_{\lambda_j \lambda'_{j'}}(\mathcal{E}_{j'j})$$

$\uparrow$ 5 variables $\uparrow$ 3 variables

Pair invariant masses

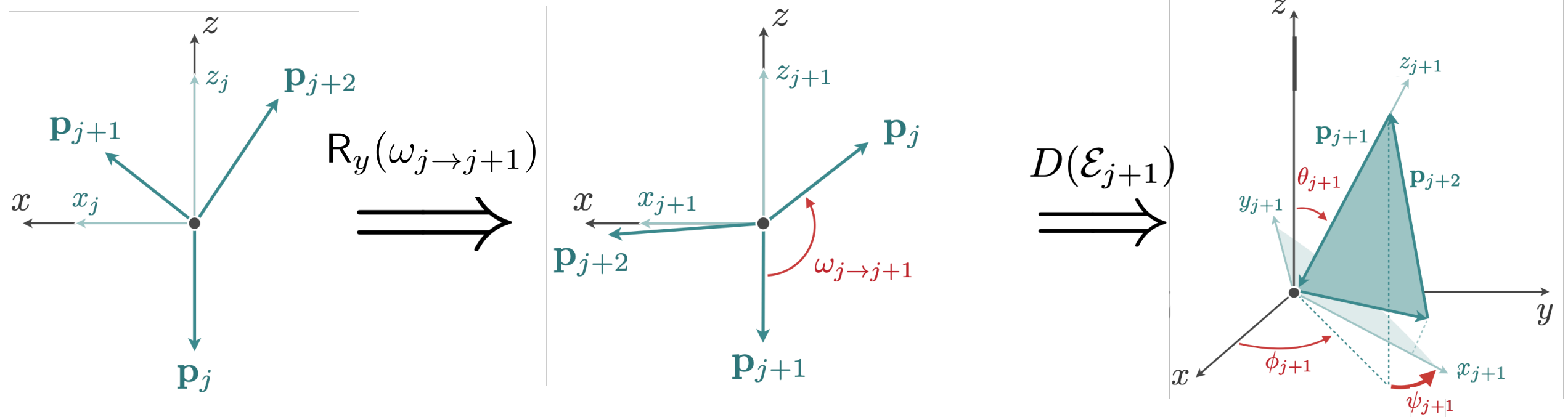


$$\sigma = P_{\text{pair}}^2 = (\sqrt{s} - E_j)^2 - \mathbf{p}_j^2$$

Symmetrizing Dalitz PWs

Now, it's easy to move between spectator choices

- Rotate, then re-orient reaction plane



Symmetrizing Dalitz PWs

[Jackura, **Chambers**, Briceño. PRD **113**, 014019 (2026)]

New strategy:

1. Construct amplitudes from K-matrices in **pair-spectator** basis
2. Complete symmetrization step in **Dalitz** basis

Symmetrized PW Amplitude

$$\mathcal{M}_{\lambda'\lambda}^{\mathfrak{q}} = \sum_{\substack{j'j \\ \{L, S, I_2\}}} \left(\mathcal{R}_{\lambda'}^{(j')} \right)^{\top} \cdot \mathcal{M}^{\mathfrak{q}}(j', j) \cdot \mathcal{R}_{\lambda}^{(j)}$$

I(J^P) points to $\mathfrak{q}$ in $\mathcal{M}_{\lambda'\lambda}^{\mathfrak{q}}$

$\{L, S, I_2\}$ points to the summation index

A red vertical line points from $\mathcal{R}_{\lambda}^{(j)}$ to the **Recoupling Coefficients** box

Jargon

$\mathfrak{q}$

“Target Quantum Numbers” (e.g. omega)

Recoupling Coefficients

- Clebsch-Gordans
- Wigner-d matrix elements
- Wigner 6-j symbols (isospin)

Toy Model Application

Model input: Three Pions

One-particle exchange (OPE)

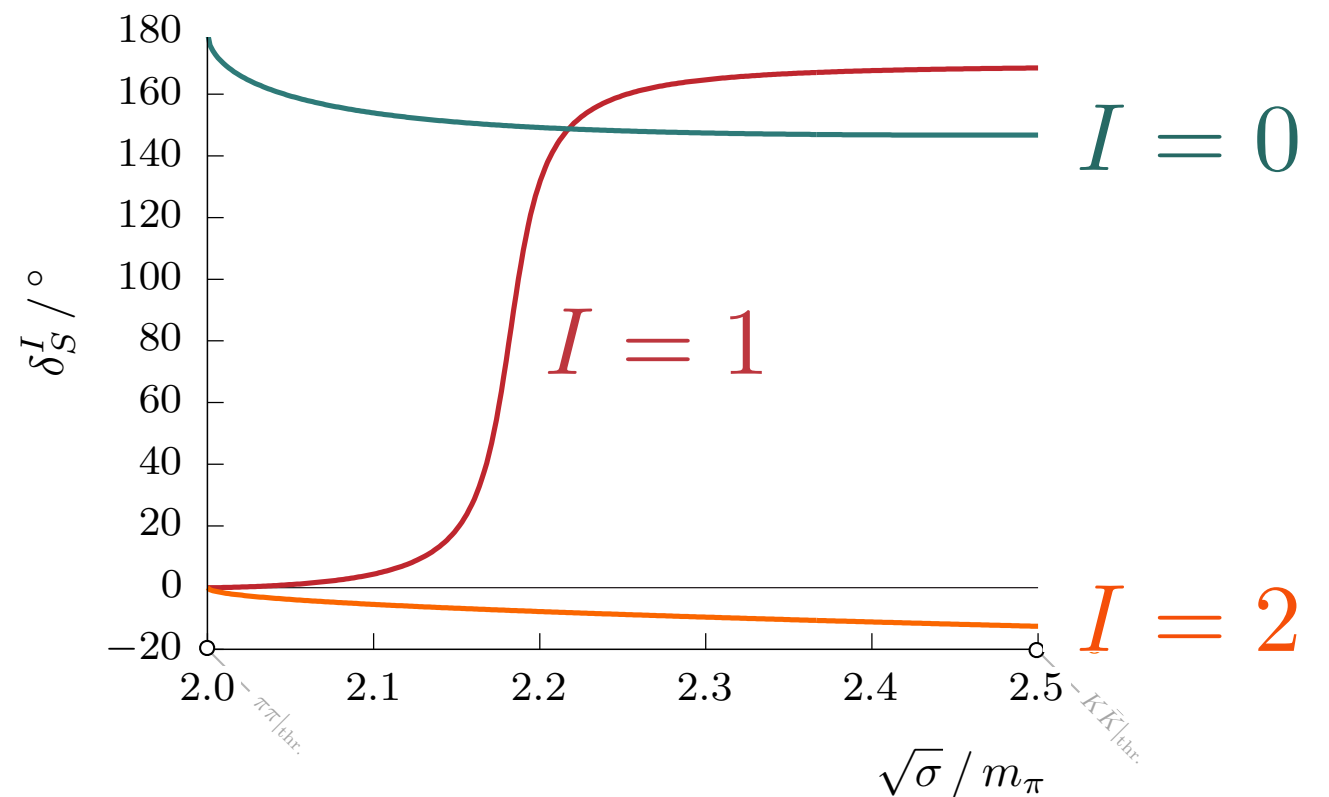
- $\mathcal{K}_3 = 0$
- No integral equations

$$i\mathcal{M}_3 = \text{Diagram} \quad \mathcal{G} \sim \frac{1}{(P - p - k)^2 - m^2}$$

“OPE function”

2π amplitude input

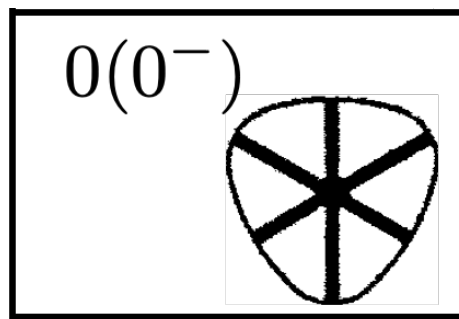
- Exact isospin
- $S \leq 1$
- $m_\pi \approx 400 \text{ MeV}$
- $m_\rho \approx 2.2m_\pi$



Dalitz Distributions

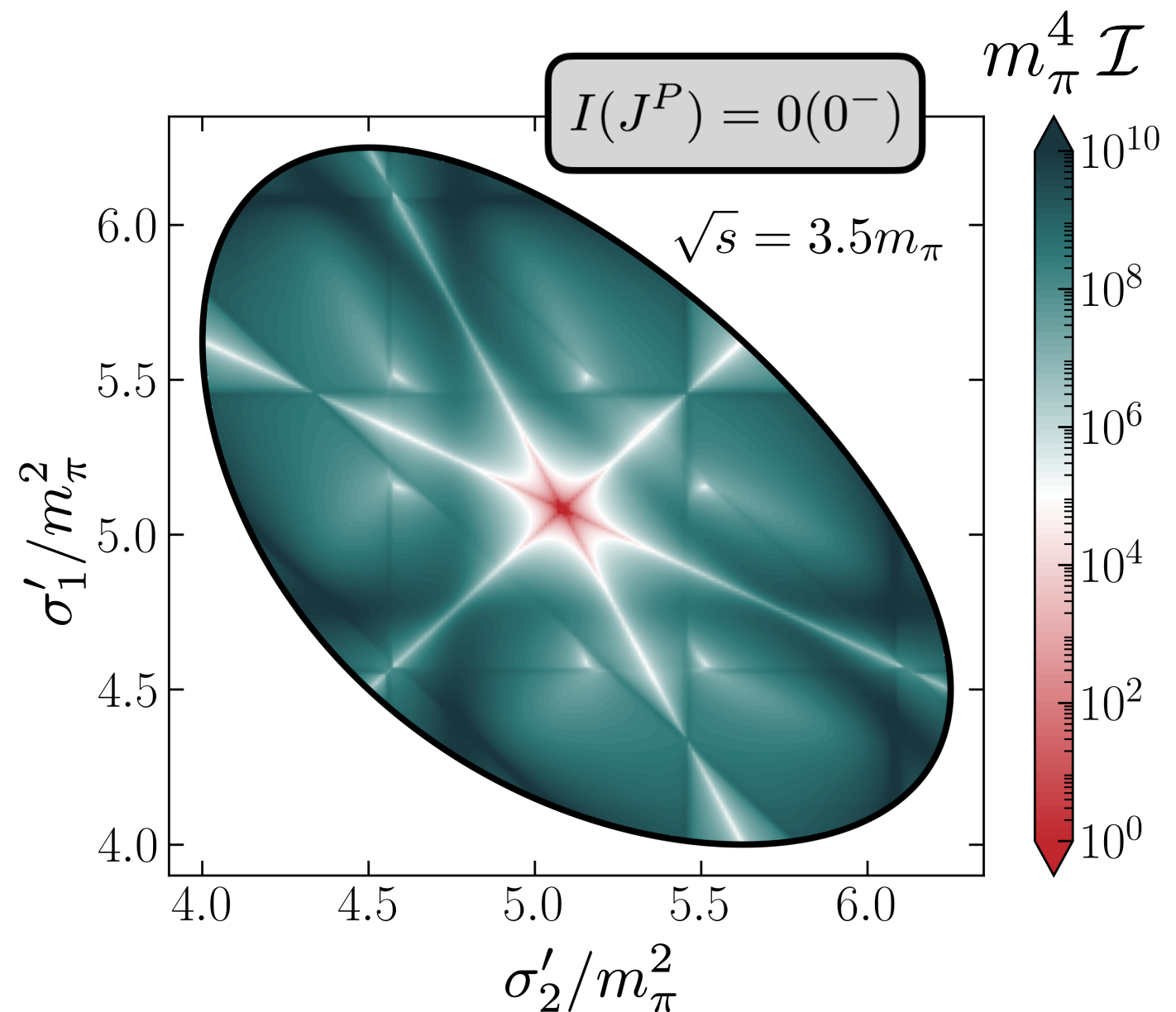
Differential 3B “Reaction Rate”

$$\mathcal{I}^q \propto \sum_{\lambda' \lambda} |\mathcal{M}_{3,\lambda',\lambda}^q|^2$$



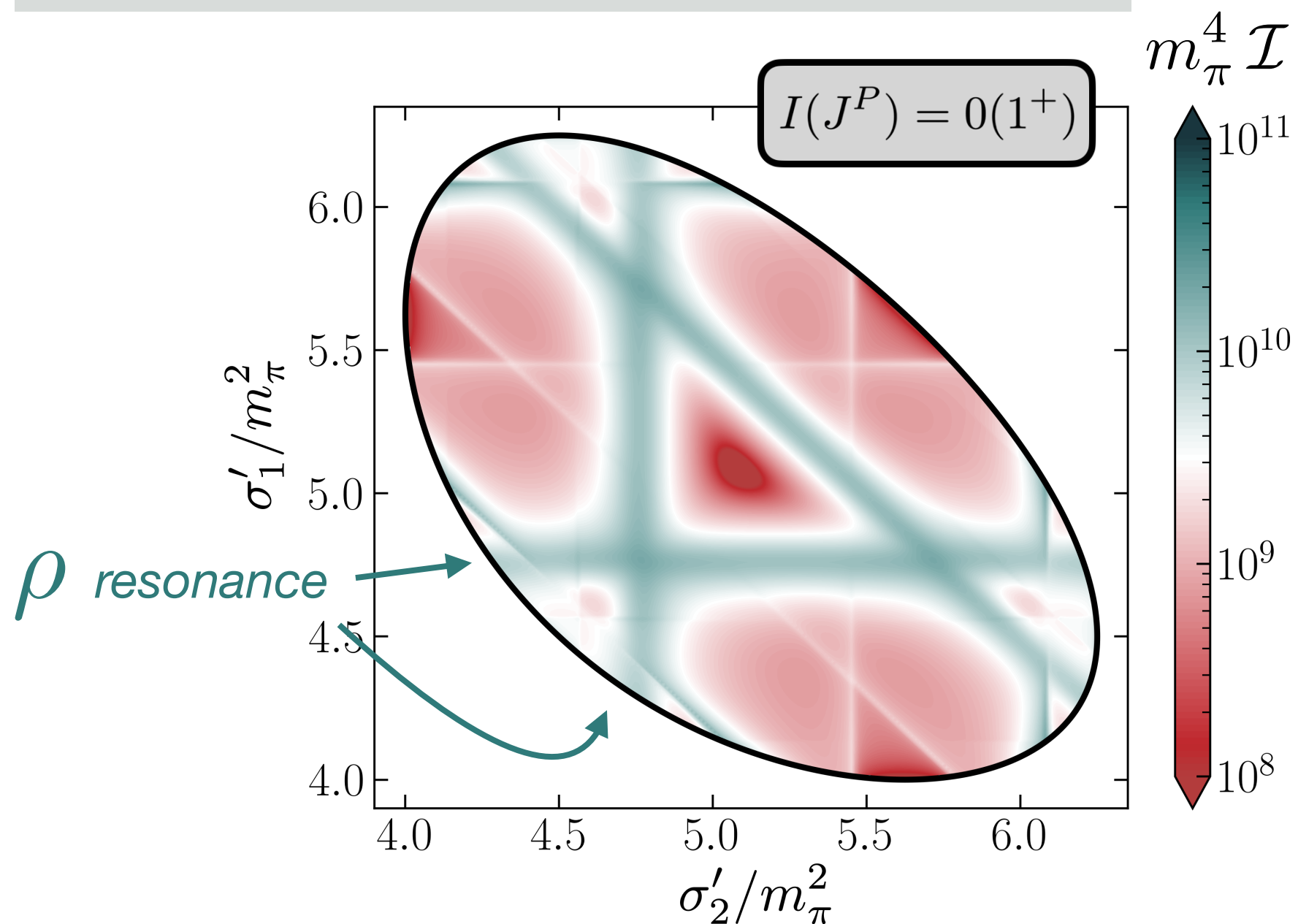
*Exhibits symmetry
predicted by Zemach*

C. Zemach, Phys. Rev. 133, B1201 (1964)



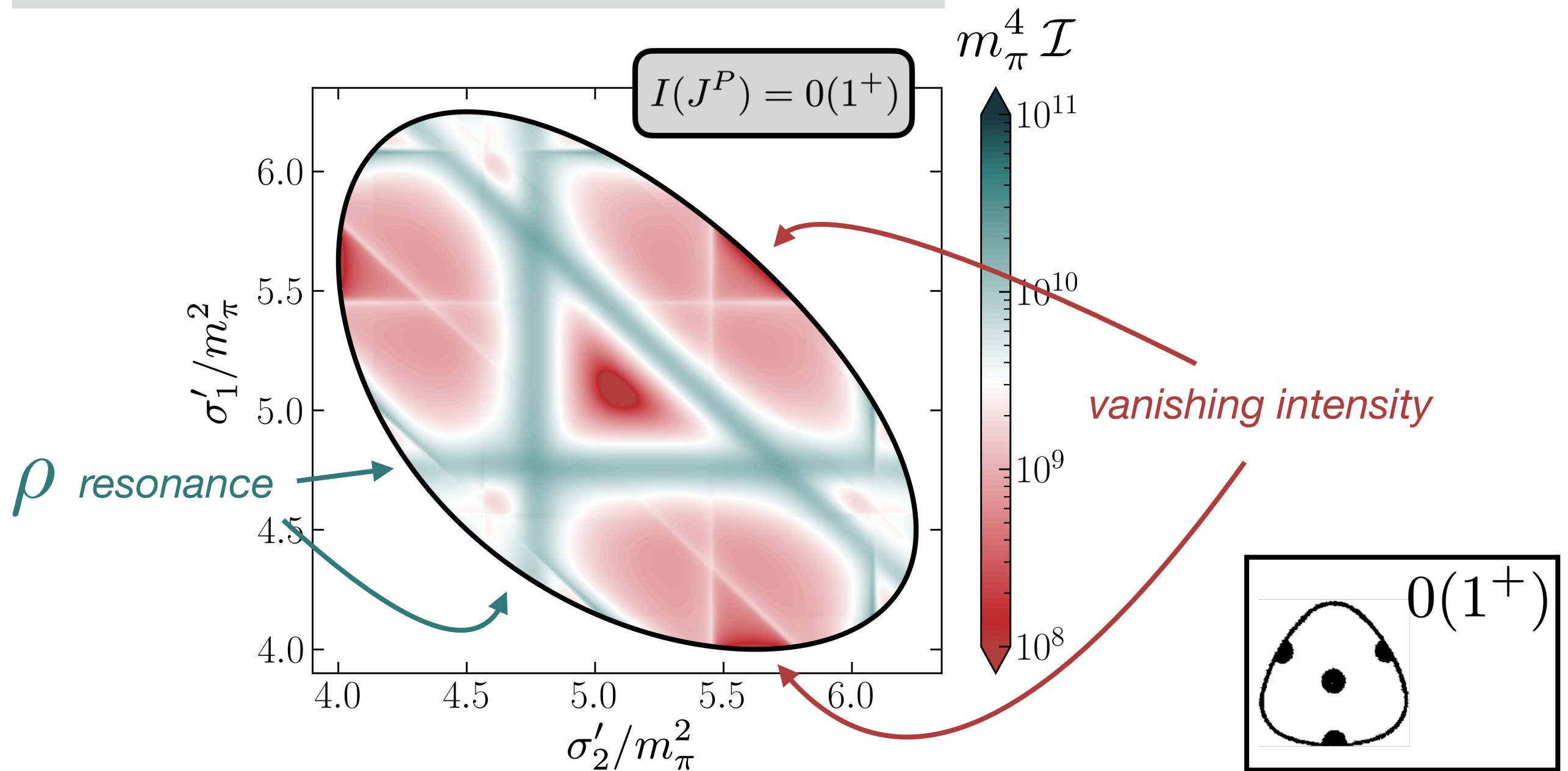
Dalitz Distributions

Dalitz distributions exhibit rich **symmetry features** and two-body subchannel physics

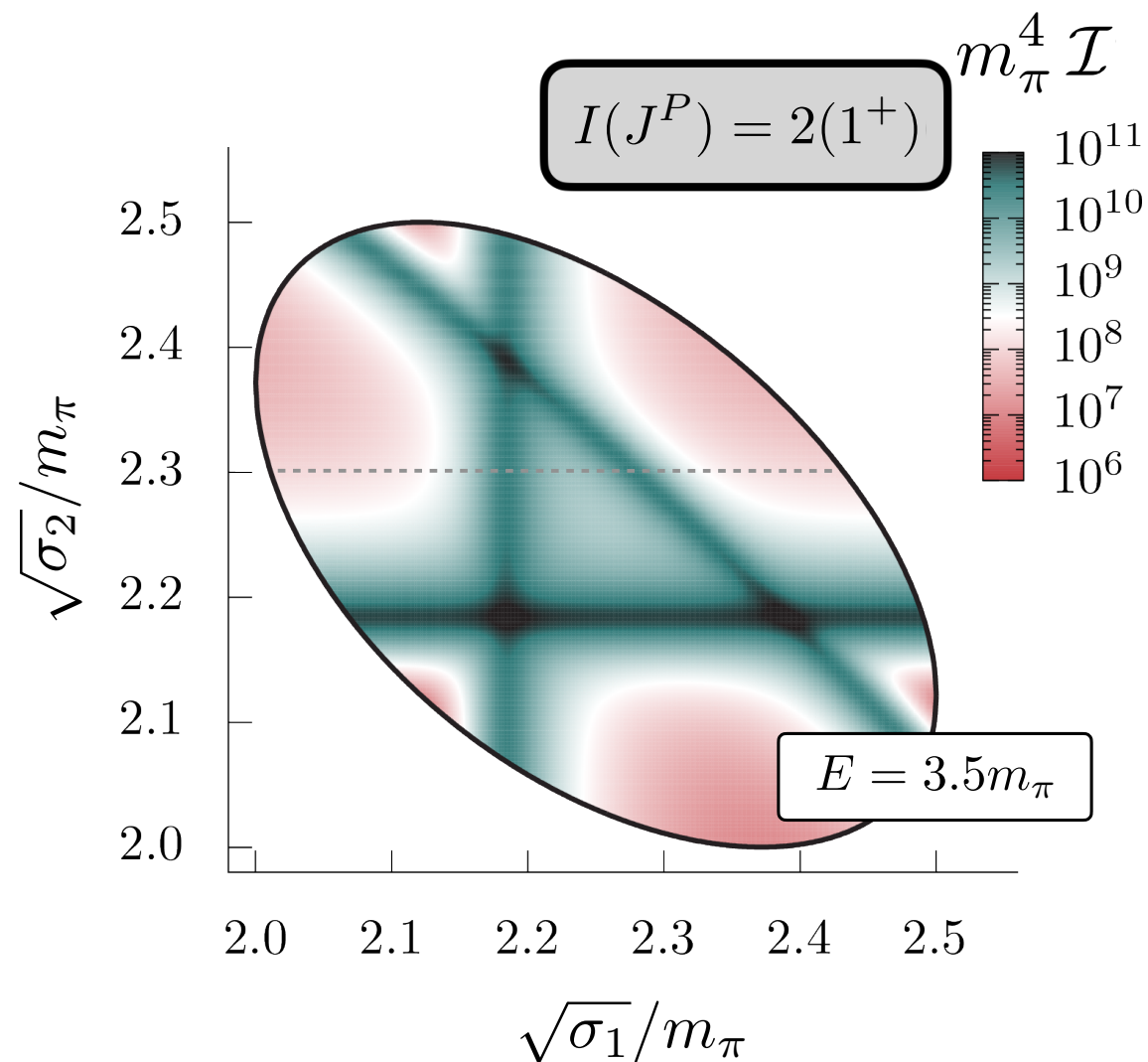


Dalitz Distributions

Dalitz distributions exhibit rich **symmetry features** and two-body subchannel physics



Final Thoughts



1. 3B integral equations produce **asymmetric** amplitudes
2. Symmetrized amplitudes lead naturally to **Dalitz distributions**
3. These symmetries are only visible through **post-analysis** of 3B amplitudes
4. Results already applied, see [2510.24894]

Next Step: Particles with **intrinsic spin**

Relevant References

3B Finite-volume formalism

- [M. Hansen, S. Sharpe, Phys. Rev. D 90, 116003 (2014)]
- [Mai, Döring, Eur. Phys. J. A 53, 240 (2017)]

3B Integral Equations

- [Hansen, Sharpe, Phys. Rev. D 92, 114509 (2015)]
- [A. Jackura, Phys. Rev. D 108, 034505 (2023)]

3B Partial Wave Projection

- [A. Jackura, R. Briceño, Phys. Rev. D. 109, 096030 (2024)]
- [Briceño, Costa, Jackura. Phys. Rev. D 111, 036029 (2025)]

3B Lattice Calculations

- [M. Hansen et al. [HadSpec] Phys. Rev. Lett. 126, (2021) 012001]
- [R. Briceño, M. Hansen, A. Jackura, R. Edwards, C. Thomas [Hadspec], arXiv:2510.24894v1]
- [S. Dawid, Z. Draper, et. al. Phys.Rev.D 112 (2025) 1, 014505]
- [Yan, Mai, et. al. Phys.Rev.Lett. 136 (2026) 14, 141901]

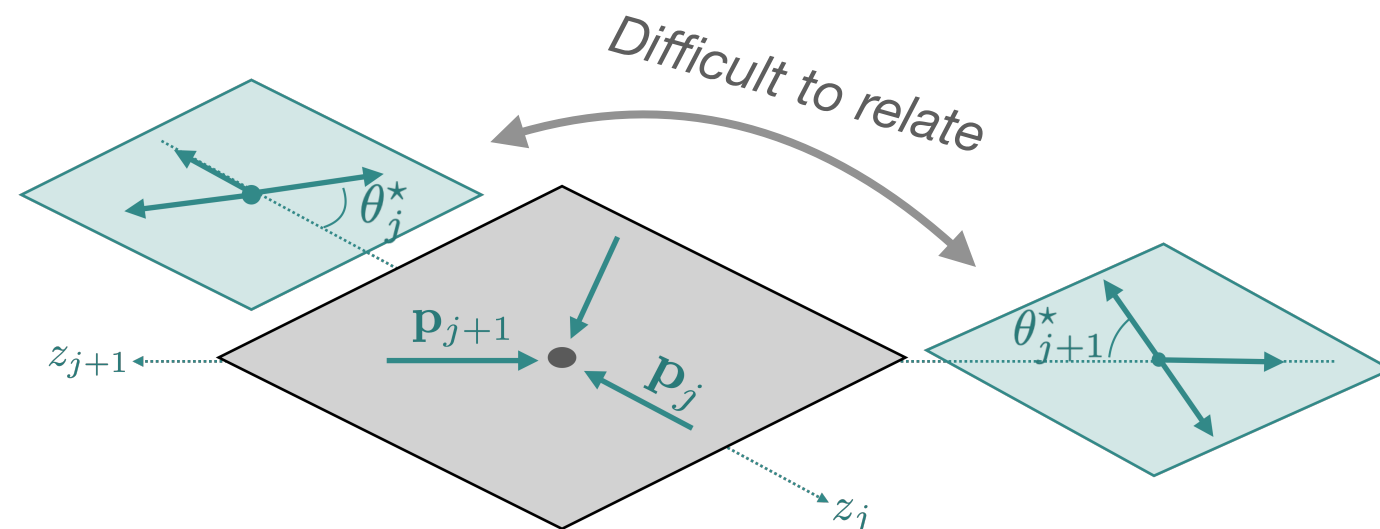
Backup Slides

We need to symmetrize...

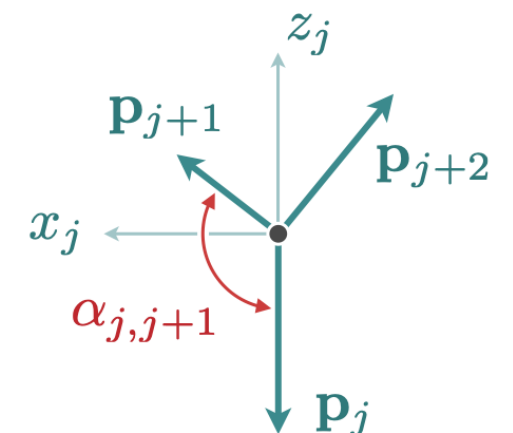
Key Concept

Difficult to symmetrize in pair-spectator basis

- Requires relating angular kinematics in different Lorentz frames



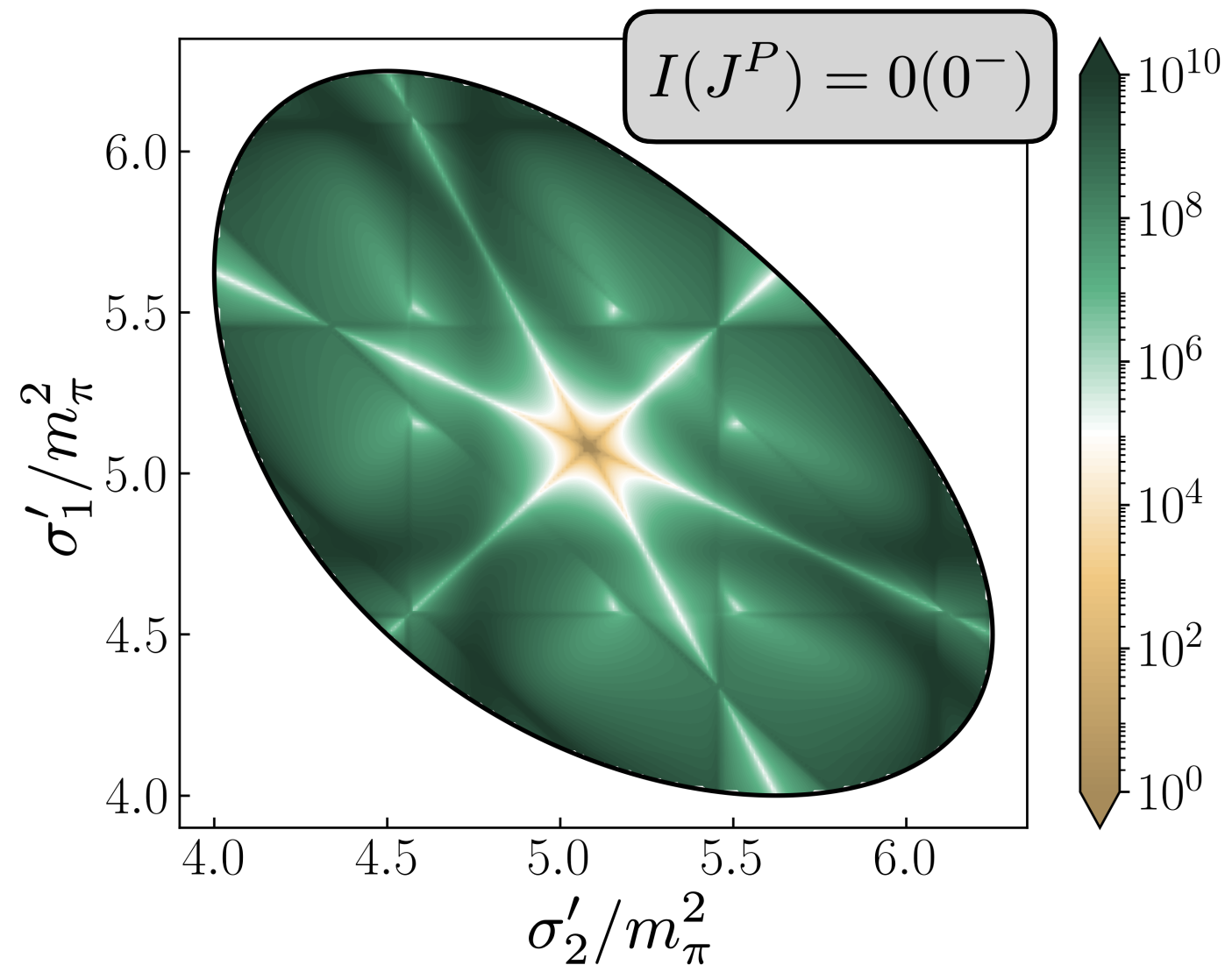
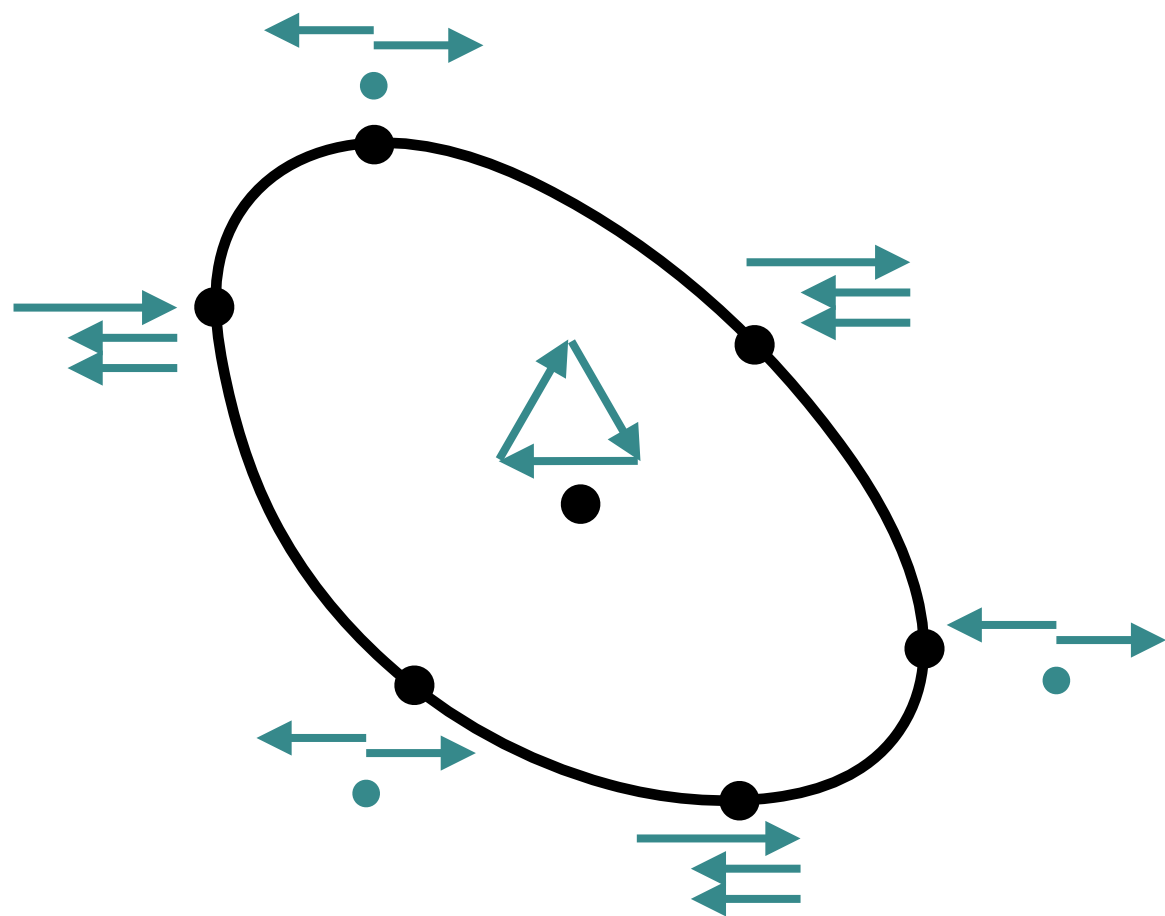
Rotating between spectators



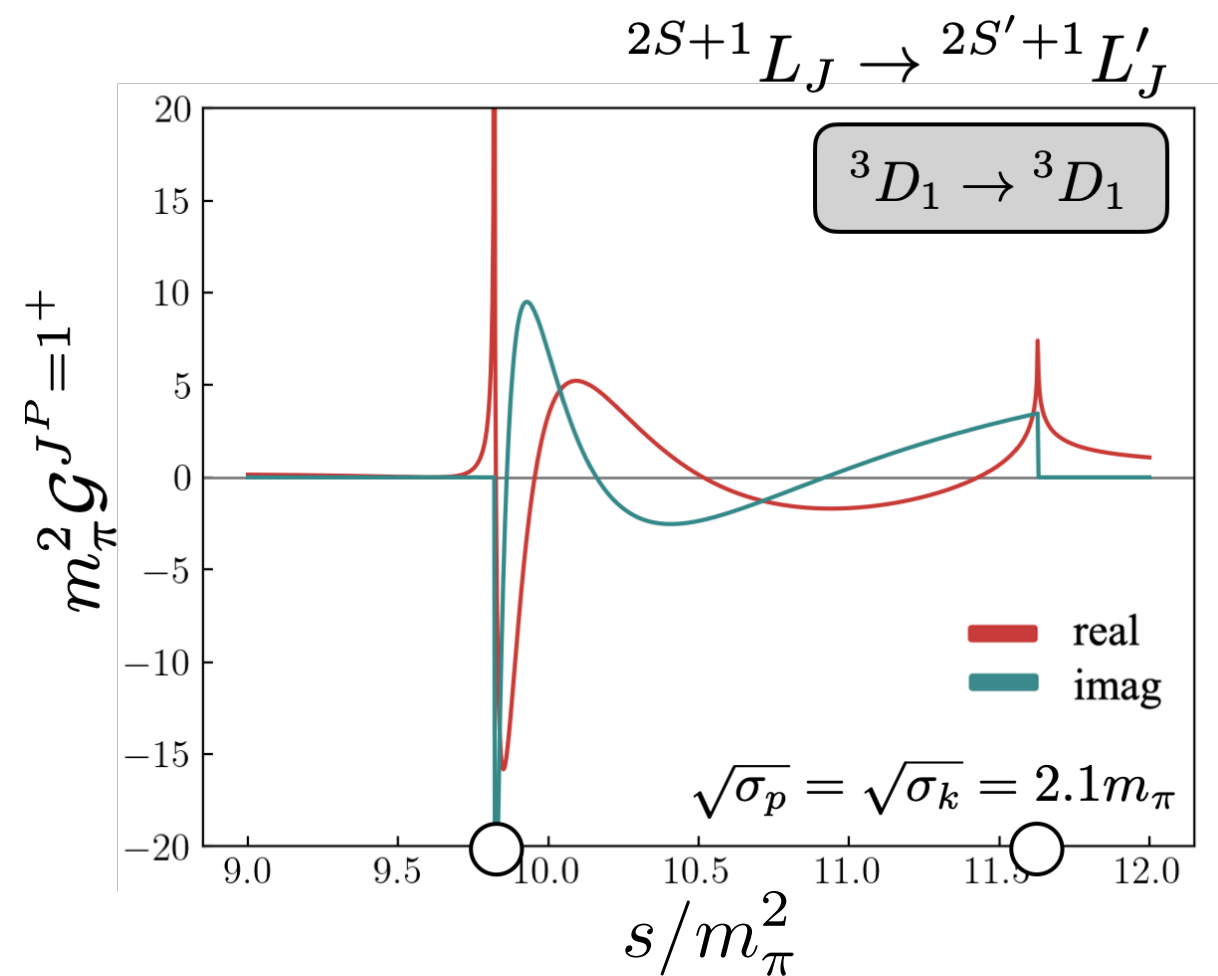
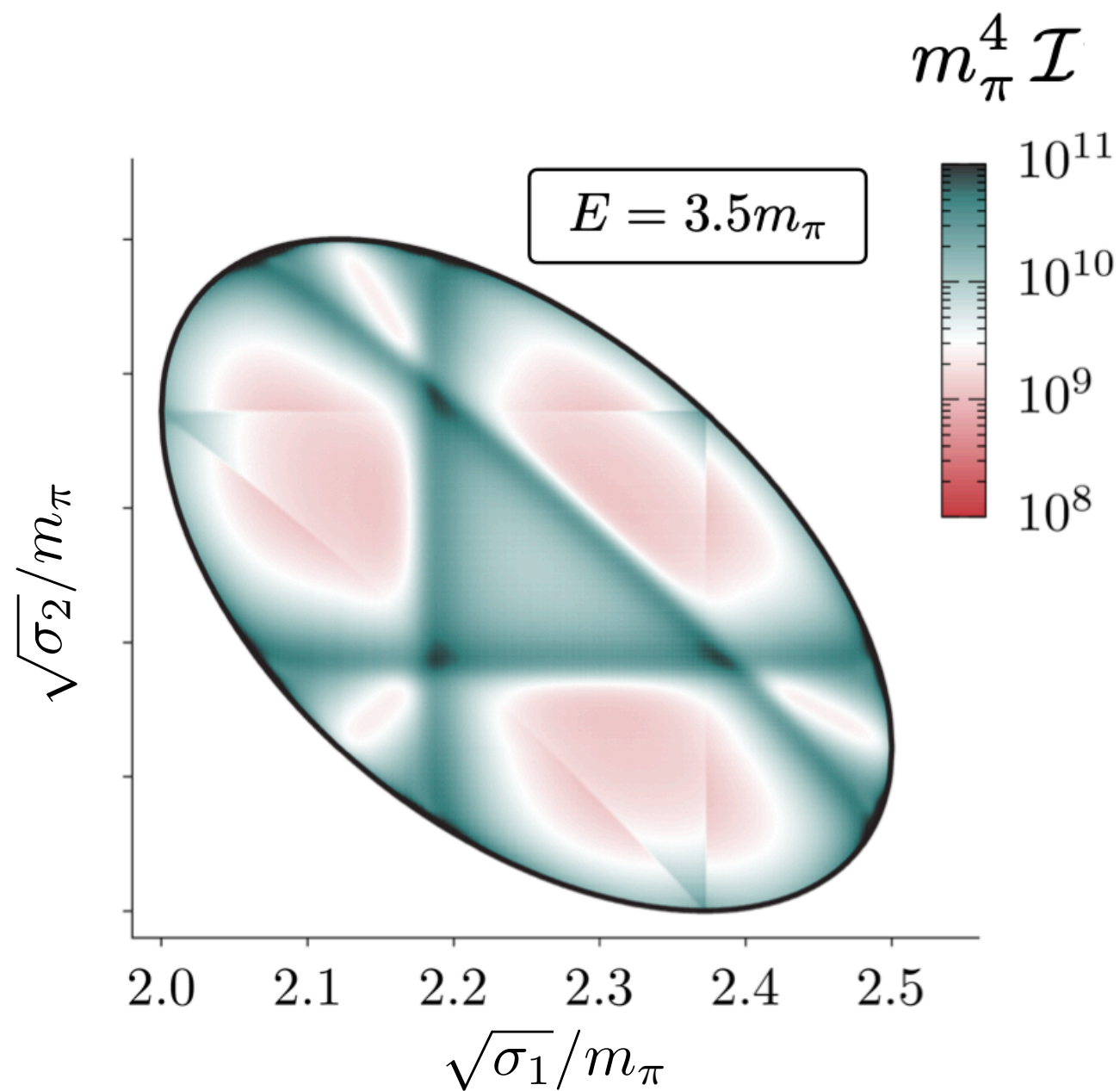
$$|p_j, Jm_J, L_j S_j\rangle \propto \int d\cos\theta_j^* \sum_{\lambda\mu} Y_{S_j\lambda}(\theta_j^*) d_{\lambda\mu}^J(\alpha_{j,j+1}) \\ \times Y_{S_{j+1}\mu}^*(\theta_{j+1}^*) |p_{j+1}, Jm_J, L_{j+1} S_{j+1}\rangle$$

Dalitz Distributions

Nodes of vanishing intensity correspond to **symmetric** momenta cfigs.



Log singularities in Dalitz Plots



Subchannel Contribution

$I_{3\pi}^G$	J^{PC}	$([\pi\pi]_{\ell}^I \pi)_L$
3^-	0^{-+}	$([\pi\pi]_S^2 \pi)_S$
	1^{-+}	none
	1^{++}	$([\pi\pi]_S^2 \pi)_P$
2^-	0^{--}	$([\pi\pi]_S^2 \pi)_S, ([\pi\pi]_P^1 \pi)_P$
	1^{--}	$([\pi\pi]_P^1 \pi)_P$
	1^{+-}	$([\pi\pi]_S^2 \pi)_P, ([\pi\pi]_P^1 \pi)_S, ([\pi\pi]_P^1 \pi)_D$
1^-	0^{-+}	$([\pi\pi]_S^{0,2} \pi)_S, ([\pi\pi]_P^1 \pi)_P$
	1^{-+}	$([\pi\pi]_P^1 \pi)_P$
	1^{++}	$([\pi\pi]_S^{0,2} \pi)_P, ([\pi\pi]_P^1 \pi)_S, ([\pi\pi]_P^1 \pi)_D$
0^-	0^{--}	$([\pi\pi]_P^1 \pi)_P$
	1^{--}	$([\pi\pi]_P^1 \pi)_P$
	1^{+-}	$([\pi\pi]_P^1 \pi)_S, ([\pi\pi]_P^1 \pi)_D$

Initial kinematics

