One-loop perturbation theory estimates for exponential clover-improved Wilson fermions

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July 31, 2026

Motivation

Large scale simulations necessary to study key physical phenomena

Wilson fermions:

- No doublers but explicit breaking of chiral symmetry and $O(a)$ cutoff effects.
- Symanzik improvement program to remove them systematically (clover term).

$$D = D_W + ac_{\mathrm{sw}} \tfrac{i}{4} \sigma_{\mu\nu} \hat{F}_{\mu\nu}$$

Numerical stability:

Diagonal part of the $O(a)$-improved Wilson–Dirac operator

$$\mathrm{diag}(D) = M_0 + ac_{\mathrm{sw}} \tfrac{i}{4} \sigma_{\mu\nu} \hat{F}_{\mu\nu}, \quad M_0 = m_0 + \frac{4}{a},$$

becomes problematic for coarse lattices and small π masses in large scale simulations.

Motivation

Exponential clover-improved Wilson fermions: [Francis et al. 2020]
Replace the diagonal part by

$$\mathrm{diag}(D) = M_0 + ac_{\mathrm{sw}}\tfrac{i}{4}\sigma_{\mu\nu}\hat{F}_{\mu\nu} \quad \longrightarrow \quad \mathrm{diag}(D) = M_0 \exp\left\{\frac{ac_{\mathrm{sw}}}{M_0}\frac{i}{4}\sigma_{\mu\nu}\hat{F}_{\mu\nu}\right\}$$

- Guarantees invertibility (even-odd preconditioning)
- Compatible with Symanzik improvement program
- Wider parameter range $(g_0^2, m_{\mathrm{q},f})$

Motivation

Changing the action changes $O(a)$-improvement

$$(A^{ij}_\mu)_R = Z_\mathrm{A}\left(1 + ab_\mathrm{A} m_{\mathrm{q},ij} + a\bar{b}_\mathrm{A}\mathrm{Tr}(M_\mathrm{q})\right)\left(A^{ij}_\mu + ac_\mathrm{A}\partial_\mu P^{ij}\right)$$

$$m_\mathrm{q} = m_0 - m_\mathrm{crit}$$

Goal: One loop estimate of these quantities with the new action

$$\begin{aligned} c_\mathrm{A} &= c_\mathrm{A}^{(1)} g_0^2 + c_\mathrm{A}^{(2)} g_0^4 + \dots \\ m_\mathrm{crit} &= m_\mathrm{crit}^{(1)} g_0^2 + m_\mathrm{crit}^{(2)} g_0^4 + \dots \end{aligned}$$

Introduction

Action

Gauge action: Plaquette action or "Lüscher–Weisz" action [Lüscher, Weisz, 85]

Fermion action: Exponential clover Wilson action [Francis et al. 2020]

$$S_F[U, \bar{\psi}, \psi] = a^4 \sum_x \bar{\psi}_f(x)(D_W + \delta D_{\rm eSW} + m_0)\psi_f(x),$$

$$D_W + \delta D_{\rm eSW} = \sum_{\mu=0}^{3} \frac{1}{2} \left\{ \gamma_\mu \left(\nabla_\mu^* + \nabla_\mu \right) - a \nabla_\mu^* \nabla_\mu \right\} + M_0 \exp \left\{ \sum_{\mu,\nu} \frac{a c_{\rm sw}}{M_0} \frac{i}{4} \sigma_{\mu\nu} \hat{F}_{\mu\nu} \right\} - M_0.$$

At first order in the expansion the two actions are equivalent

$$\delta D_{\rm eSW} = \delta D_{\rm SW} + \frac{1}{2! M_0} \left[\sum_{\mu,\nu} a c_{\rm sw} \frac{i}{4} \sigma_{\mu\nu} \hat{F}_{\mu\nu} \right]^2 + \ldots$$

Schrödinger Functional

Boundary conditions: [Lüscher et al. 92; Sint et al. 95]

- Periodic (up to a phase θ) in space

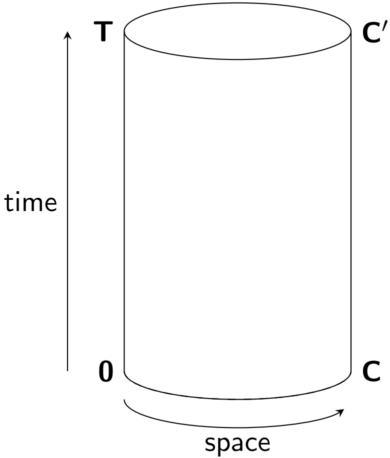
$$\psi(x+L\hat{k}) = \mathrm{e}^{i\theta}\psi(x), \quad \bar{\psi}(x+L\hat{k}) = \bar{\psi}(x)\mathrm{e}^{-i\theta}$$

- Dirichlet in time

$$P_+\psi(x)|_{x_0=0} = \rho, \quad P_-\psi(x)|_{x_0=T} = \rho'$$
$$\bar{\psi}(x)P_-|_{x_0=0} = \bar{\rho}, \quad \bar{\psi}(x)P_+|_{x_0=T} = \bar{\rho}'$$

- Quark boundary fields: Boundary values as external sources!

$$\zeta(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}(\mathbf{x})}, \quad \bar{\zeta}(\mathbf{x}) = \frac{\delta}{\delta\rho(\mathbf{x})}$$
$$\zeta'(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}'(\mathbf{x})}, \quad \bar{\zeta}'(\mathbf{x}) = \frac{\delta}{\delta\rho'(\mathbf{x})}$$



$$P_\pm = \tfrac{1}{2}(1 \pm \gamma_0)$$

PCAC relation in the Schrödinger Functional

- PCAC Ward Identity:

$$\langle \partial_0 A_0^a(x) \mathcal{O} \rangle = 2m \langle P^a(x) \mathcal{O} \rangle$$

- Bilinear boundary quark sources:

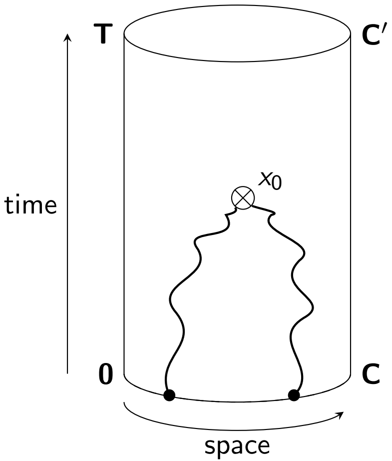
$$\mathcal{O}^a = a^6 \sum_{\mathbf{u},\mathbf{v}} \bar{\zeta}(\mathbf{u}) T^a \gamma_5 \zeta(\mathbf{v})$$

- Correlation functions: [Lüscher et al. 97]

$$f_{\rm A}(x_0) = -\tfrac{1}{3} \sum_a \langle A_0^a(x) \mathcal{O}^a \rangle$$

$$f_{\rm P}(x_0) = -\tfrac{1}{3} \sum_a \langle P^a(x) \mathcal{O}^a \rangle$$

$$\Rightarrow \partial_0 f_{\rm A}(x_0) = 2m f_{\rm P}(x_0)$$



Determination of c_{A} and m_{crit}

Using the $O(a)$-improved PCAC relation ($m_{\mathrm{q}} = 0$), we define the current quark mass as

$$m = \frac{1}{2}\frac{\partial_0 f_{\mathrm{A}}(x_0) + a c_{\mathrm{A}} \partial_0^2 f_{\mathrm{P}}(x_0)}{f_{\mathrm{P}}(x_0)} \quad \text{and} \quad \Delta m = m|_{\theta} - m|_{\theta=0}$$

Non-perturbative determination of c_{A} and m_{crit}: [Lüscher et al. 97]

- Compute c_{A} by imposing invariance of the current quark mass on kinematical variables

$$\Delta m = \Delta m^{(\mathrm{tree})} \Rightarrow c_{\mathrm{A}}$$

- Determine m_{crit} by fixing the current quark mass to zero

$$m|_{m_0 = m_{\mathrm{crit}}} = 0 \Rightarrow m_{\mathrm{crit}}$$

One-loop strategy

One-loop strategy

Perturbative expansion of the axial current correlation function

$$f_{\mathrm{A}}(x_0) = \sum_{k=0}^{\infty} g_{\mathrm{R}}^{2k} f_{\mathrm{A}}^{(k)}(x_0),$$

by setting $m_0^{(0)} = 0$, at one-loop we have [Lüscher, Weisz 96]

$$\partial_0 f_{\mathrm{A}}^{(1)}(x_0) = \sum_n \partial_0 f_{\mathrm{A},n}^{(1)}(x_0) + m_{\mathrm{crit}}^{(1)} \partial_{m_0} \partial_0 f_{\mathrm{A}}^{(0)} + a c_{\mathrm{A}}^{(1)} \partial_0^2 f_{\mathrm{P}}^{(0)}(x_0) + O(a^2)$$

Perturbative expansion of the axial current correlation function

$$f_{\mathrm{A}}(x_0) = \sum_{k=0}^{\infty} g_{\mathrm{R}}^{2k} f_{\mathrm{A}}^{(k)}(x_0),$$

Boundary improvement $\tilde{c}_t^{(1)}$,
renormalization constants $Z_A^{(1)}, Z_\zeta^{(1)}$
...

by setting $m_0^{(0)} = 0$, at one-loop we have [Lüscher, Weisz 96]

$$\partial_0 f_{\mathrm{A}}^{(1)}(x_0) = \underbrace{\sum_n \partial_0 f_{\mathrm{A},n}^{(1)}(x_0)}_{w_1} + m_{\mathrm{crit}}^{(1)} \underbrace{\partial_{m_0} \partial_0 f_{\mathrm{A}}^{(0)}}_{w_3} + a c_{\mathrm{A}}^{(1)} \underbrace{\partial_0^2 f_{\mathrm{P}}^{(0)}(x_0)}_{w_4} + O(a^2)$$

One-loop strategy

Perturbative expansion of the axial current correlation function

$$f_{\mathrm{A}}(x_0) = \sum_{k=0}^{\infty} g_{\mathrm{R}}^{2k} f_{\mathrm{A}}^{(k)}(x_0),$$

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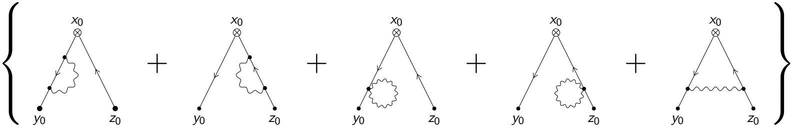
w_1 is computed numerically (Double precision `Fortran 90 code` [Vilaseca, 2014])

$w_1 = \partial_0$



One-loop calculation of $m_{\mathrm{crit}}^{(1)}$ and $c_{\mathrm{A}}^{(1)}$

The PCAC relation implies

$$\partial_0 f_{\mathrm{A}}^{(1)}(x_0) = \underbrace{\sum_n \partial_0 f_{\mathrm{A},n}^{(1)}(x_0)}_{w_1} + m_{\mathrm{crit}}^{(1)} \underbrace{\partial_0 \partial_{m_0} f_{\mathrm{A}}^{(0)}}_{w_3} + a c_{\mathrm{A}}^{(1)} \underbrace{\partial_0^2 f_{\mathrm{P}}^{(0)}(x_0)}_{w_4} = O(a^2)$$

$$\boxed{m_0^{(0)} = 0 \Rightarrow \partial_0 f_{\mathrm{A}}^{(1)} = O(a^2)}$$

With $\theta = 0$ we can compute $m_{\mathrm{crit}}^{(1)}$ * [Lüscher, Weisz 96]

$$a m_{\mathrm{crit}}^{(1)} = -\left(\frac{w_1}{w_3}\right)_{\theta=0} + O(a^3)$$

While for $c_{\mathrm{A}}^{(1)}$ we need $\theta \neq 0$ [Lüscher, Weisz 96]

$$c_{\mathrm{A}}^{(1)} = \frac{-w_1 + w_3 \left(\frac{w_1}{w_3}\right)_{\theta=0}}{w_4} + O(a)$$

*$\theta = 0$ is a good choice, as it removes order $O(a^2)$ effects from $m_{\mathrm{crit}}^{(1)}$ ($w_4|_{\theta=0} = O(a^2)$)

The case of exponential clover-improved Wilson fermions

Exponential clover-improved Wilson fermions:

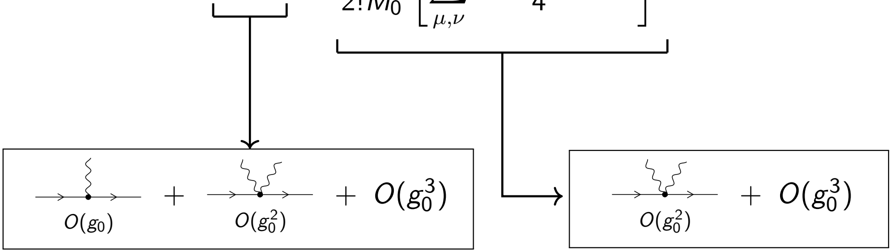
The expansion of the exponential clover term yields

$$\delta D_{\mathrm{eSW}} = \delta D_{\mathrm{SW}} \; + \; \frac{1}{2!\, M_0} \left[\sum_{\mu,\nu} a c_{\mathrm{sw}} \frac{i}{4} \sigma_{\mu\nu} \hat{F}_{\mu\nu} \right]^2 \; + \; \ldots$$

Exponential clover-improved Wilson fermions:

The expansion of the exponential clover term yields at one loop

$$\delta D_{\mathrm{eSW}} = \delta D_{\mathrm{SW}} + \frac{1}{2!\,M_0}\left[\sum_{\mu,\nu} a c_{\mathrm{sw}} \frac{i}{4}\sigma_{\mu\nu}\hat{F}_{\mu\nu}\right]^2 + \ldots$$



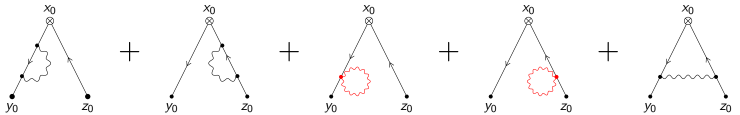
The procedure for the computation of $c_{\mathrm{A}}^{(1)}$ and $m_{\mathrm{crit}}^{(1)}$ is identical

$$am_{\mathrm{crit}}^{(1)}\Big|_{\mathrm{eSW}} = -\left(\frac{w_1^{\mathrm{eSW}}}{w_3}\right)_{\theta=0} + O(a^3)$$

$$c_{\mathrm{A}}^{(1)}\Big|_{\mathrm{eSW}} = \frac{-w_1^{\mathrm{eSW}} + w_3\left(\frac{w_1^{\mathrm{eSW}}}{w_3}\right)_{\theta=0}}{w_4} + O(a)$$

we just need to compute one new contribution (numerically)

$$w_1^{\mathrm{eSW}} = \partial_0 \Big\{ \quad \Big\}$$



$c_{\mathrm{A}}^{(1)}$ and $m_{\mathrm{crit}}^{(1)}$ computation

Numerical analysis

Setup: $m_0^{(0)} = 0$, $c_{\rm sw}^{(0)} = 1$, $L/a \in \{4, 6, \ldots, 48\}$, $\theta = 0$ and $\theta' \neq 0^*$

We compute numerically $w_1^{\rm eSW}$, w_3 and w_4

For a general 1-loop estimate, the asymptotic expansion on a/L follows

$$Q^{(1)}(a/L) = Q^{(1)} + \sum_{k=1}^{\infty} \left[a_k^{(Q)} + b_k^{(Q)} \log(a/L)\right] \left(\frac{a}{L}\right)^k \quad \text{with} \quad Q = c_{\rm A}, m_{\rm crit}$$

We extrapolate to $a/L \to 0$ using the method described in [Bode et al. 2000]

* Cross-checked the invariance of $c_{\rm A}$ for different values of θ' after taking the $a/L \to 0$ extrapolation

Numerical analysis

Extrapolation $\rightarrow$ Fit data to the asymptotic expansion to some order $n \rightarrow \{Q^{(1)}, a_k, b_k\}$ for range of L/a

Error sources

- Roundoff error (negligible).
- Systematic error in $Q^{(1)}$:
  1. Perform two independent fits by extending the expansion to
  $$F_1(a/L) = (a/L)^{n+1} \quad \text{or} \quad F_2(a/L) = \log(a/L)(a/L)^{n+1}$$
  obtaining A_1 or A_2.
  2. Construct the residue estimator
  $$\mathcal{R}_i(a/L) = A_i F_i(a/L)$$
  3. Fit each $\mathcal{R}_i$ with the original ansatz (order n) to obtain $\{Q_i^{(1)}\}$
  4. Estimate the uncertainty as
  $$\max_i \left| Q_i^{(1)} \right| .$$

$c_{\mathrm{A}}^{(1)}$ and $m_{\mathrm{crit}}^{(1)}$ numerical results

Results: [PRELIMINARY] $\quad$ † Determination from this work $\quad$ * Determination from [Taniguchi, Ukawa 98]

- The extra contribution to 1-loop order of the new exponential clover term changes the value of $m_{\mathrm{crit}}^{(1)}$ (expected)

$m_{\mathrm{crit}}^{(1)}/C_{\mathrm{F}}$	Plq	LW
SW*	$-0.2025565(1)$	$-0.15092010(3)$
SW†	$-0.2025565(5)$	$-0.1509201(5)$
eSW†	$-0.2186026(9)$	$-0.1637884(9)$

- The value for $c_A^{(1)}$ using the clover term and the "new" exponential clover is the same in the $a/L \to 0$ limit (as stated in [Francis et al. 2020])

$c_{\mathrm{A}}^{(1)}/C_{\mathrm{F}}$	Plq	LW
SW*	$-5.680(2) \times 10^{-3}$	$-4.51(1) \times 10^{-3}$
SW†	$-5.676(13) \times 10^{-3}$	$-4.51(1) \times 10^{-3}$
eSW†	$-5.676(12) \times 10^{-3}$	$-4.51(1) \times 10^{-3}$

Outlook

Outlook

- Determination of $c_{\mathrm{A}}^{(1)}$ for exponential clover-improved Wilson fermions agrees with the standard clover
- First estimate of $m_{\mathrm{crit}}^{(1)}$ for exponential clover-improved Wilson fermions
- Follow similar techniques to compute $c_V^{(1)}$ and $\tilde{c}_t^{(1)}$ [Sint, Weisz, 97]
- Non-perturbative estimates of improvement coefficients and renormalization constants (talk by J. Kuhlmann on Monday $\rightarrow Z_{\mathrm{A}}$, Z_{V} & c_{A})
- Employ new determination techniques (see [Fritzsch et al. 2026]) with exponential clover fermions setup for heavy quark physics applications

Thanks for your attention!

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(arXiv:2606.17236).

Extra slides

Perturbation theory in the SF

Consider $\psi_{\rm cl}$ the classical solution of the Dirac equation, and define the matrix [Lüscher et al. 96]

$$H(x) = a^3 \sum_{\mathbf{y}} \frac{\delta\psi_{\rm cl}(x)}{\delta\rho(\mathbf{y})}.$$

Then, the 2-point functions are given by

$$f_{\rm A}(x_0) = -\frac{1}{2}\left\langle H(x)^\dagger \gamma_0 H(x)\right\rangle, \quad f_{\rm P}(x_0) = -\tfrac{1}{2}\left\langle H(x)^\dagger H(x)\right\rangle,$$

with the expansion

$$H(x) = H^{(0)}(x) + g_0 H^{(1)}(x) + g_0^2 H^{(2)}(x) + O(g_0^3).$$

Perturbation theory in the SF

Using the expansion of $H(x)$ in $f_{\rm A}$, we obtain to one-loop

$$
\begin{aligned}
f_{\rm A}(x_0) &= f_{\rm A}^{(0)}(x_0) + g_0^2 f_{\rm A}^{(1)}(x_0) \\
&= -\frac{1}{2}\mathrm{Tr}\left\{H^{(0)}(x)^\dagger\gamma_0 H^{(0)}(x)\right\} - \frac{g_0^2}{2}\Bigg[\left\langle \mathrm{Tr}\left\{H^{(0)}(x)^\dagger\gamma_0 H^{(2)}(x)\right\}\right\rangle_{\rm G} \\
&\qquad + \left\langle \mathrm{Tr}\left\{H^{(2)}(x)^\dagger\gamma_0 H^{(0)}(x)\right\}\right\rangle_{\rm G} + \left\langle \mathrm{Tr}\left\{H^{(1)}(x)^\dagger\gamma_0 H^{(1)}(x)\right\}\right\rangle_{\rm G}\Bigg],
\end{aligned}
$$

where we have $H^{(1)}(x) \sim A_\mu(x)$ and $H^{(2)}(x) \sim A_\mu(x)A_\nu(x) + \text{boundary}$.