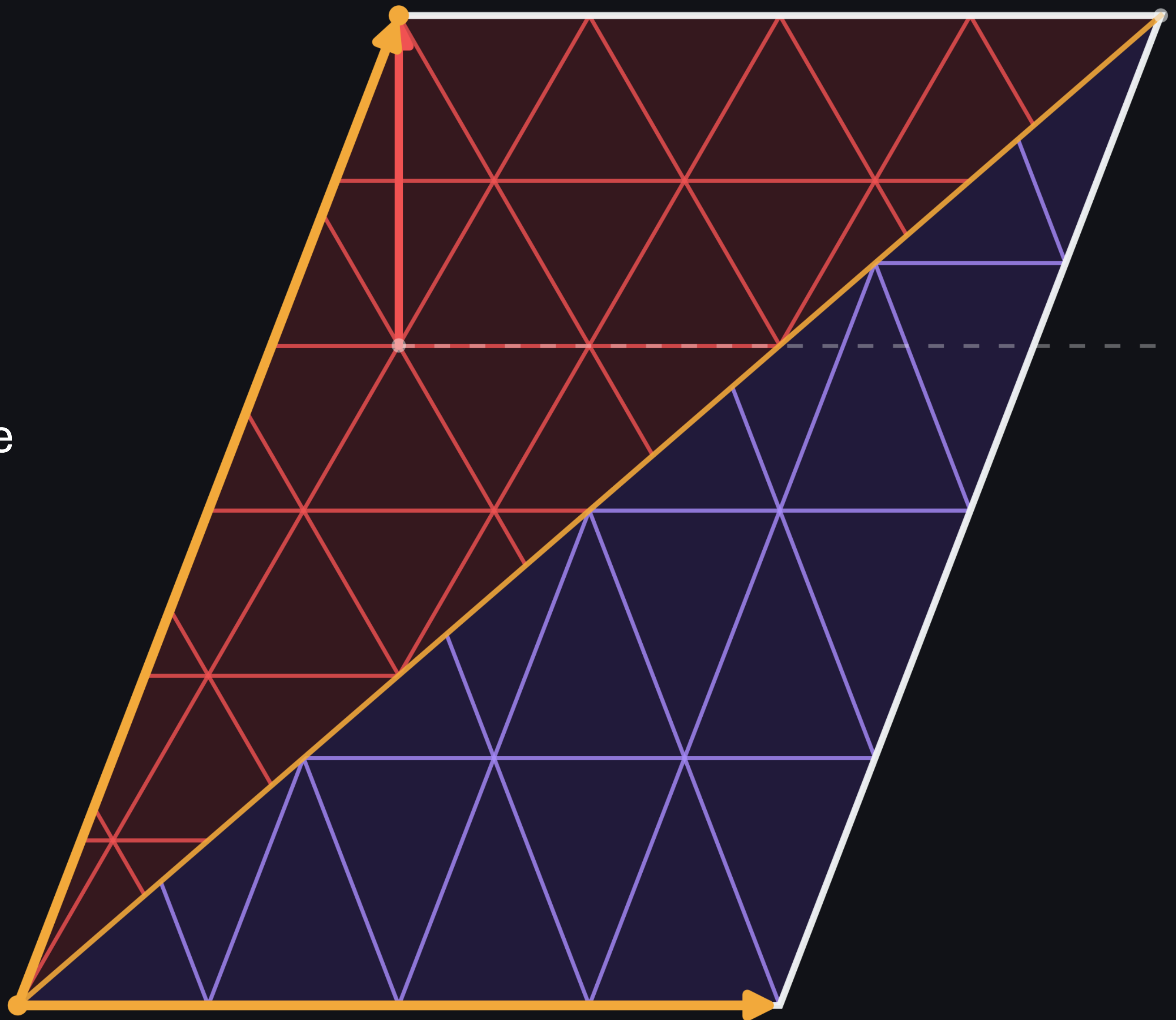
Tuning Couplings to Geometries for Lattice Field Theories

A step toward curved space QFT via the Affine Conjecture

LATTICE 2026



Rohan Misra | Boston University

Tuning Couplings to Geometries for Lattice Field Theories

A step toward curved space QFT via the Affine Conjecture

LATTICE 2026

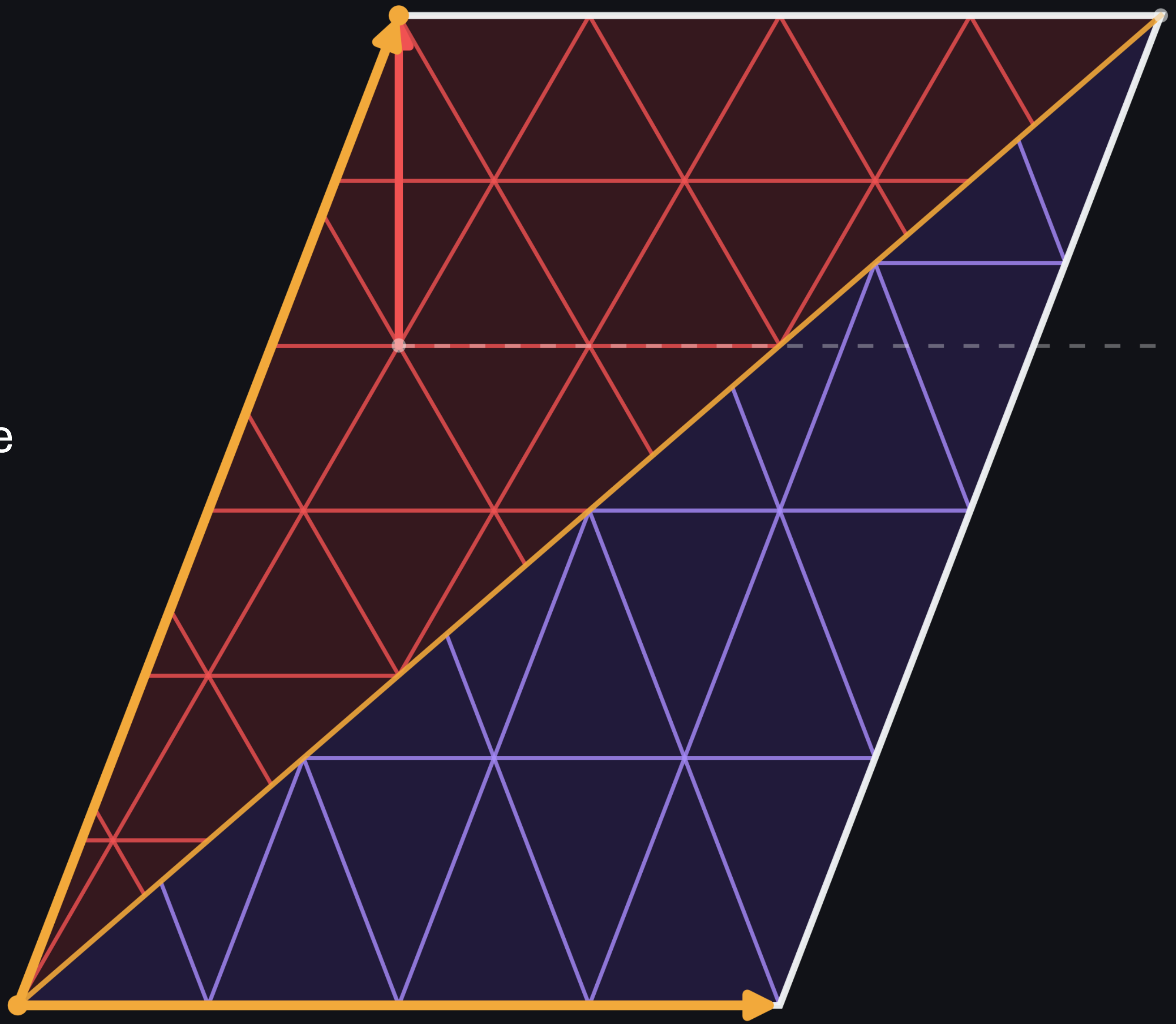
George Fleming

Kostas Orginos

Nobuyuki Matsumoto

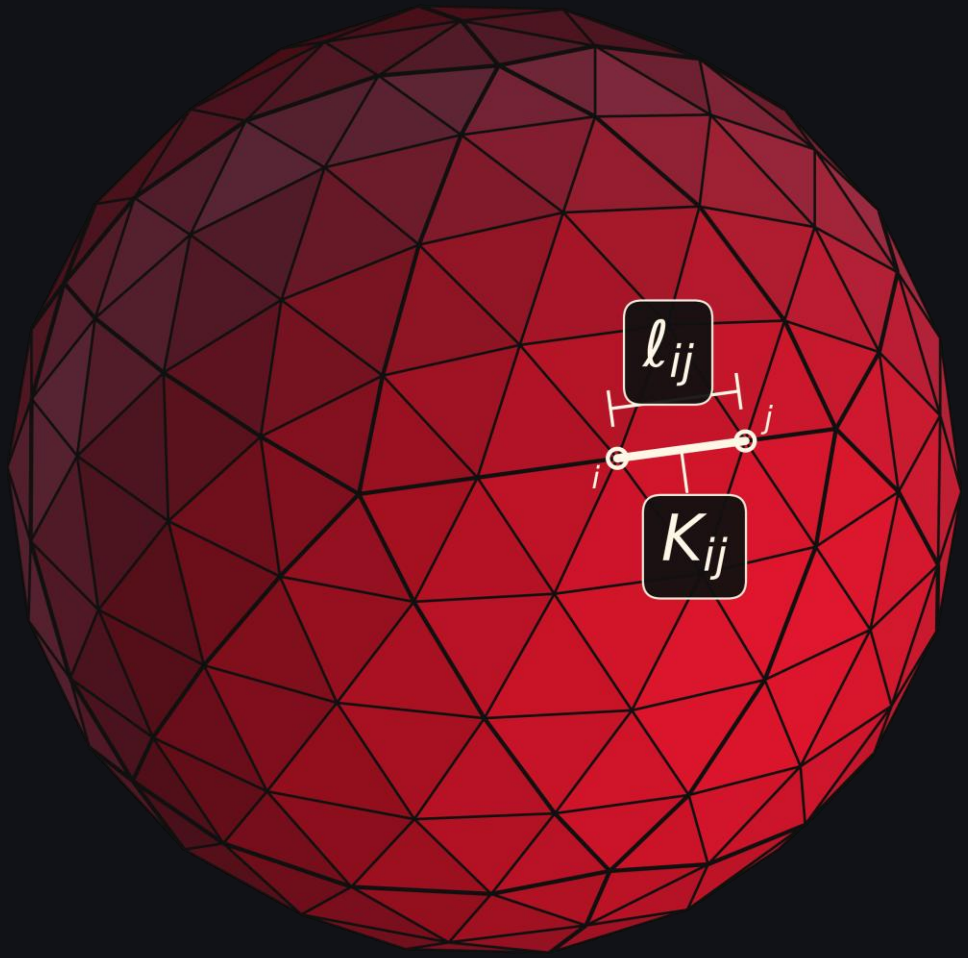
Richard Brower

Rohan Misra | Boston University



How to put the Ising CFT on a Turtle

$$S = -\sum_{<i,j>} k_{ij} s_i s_j$$

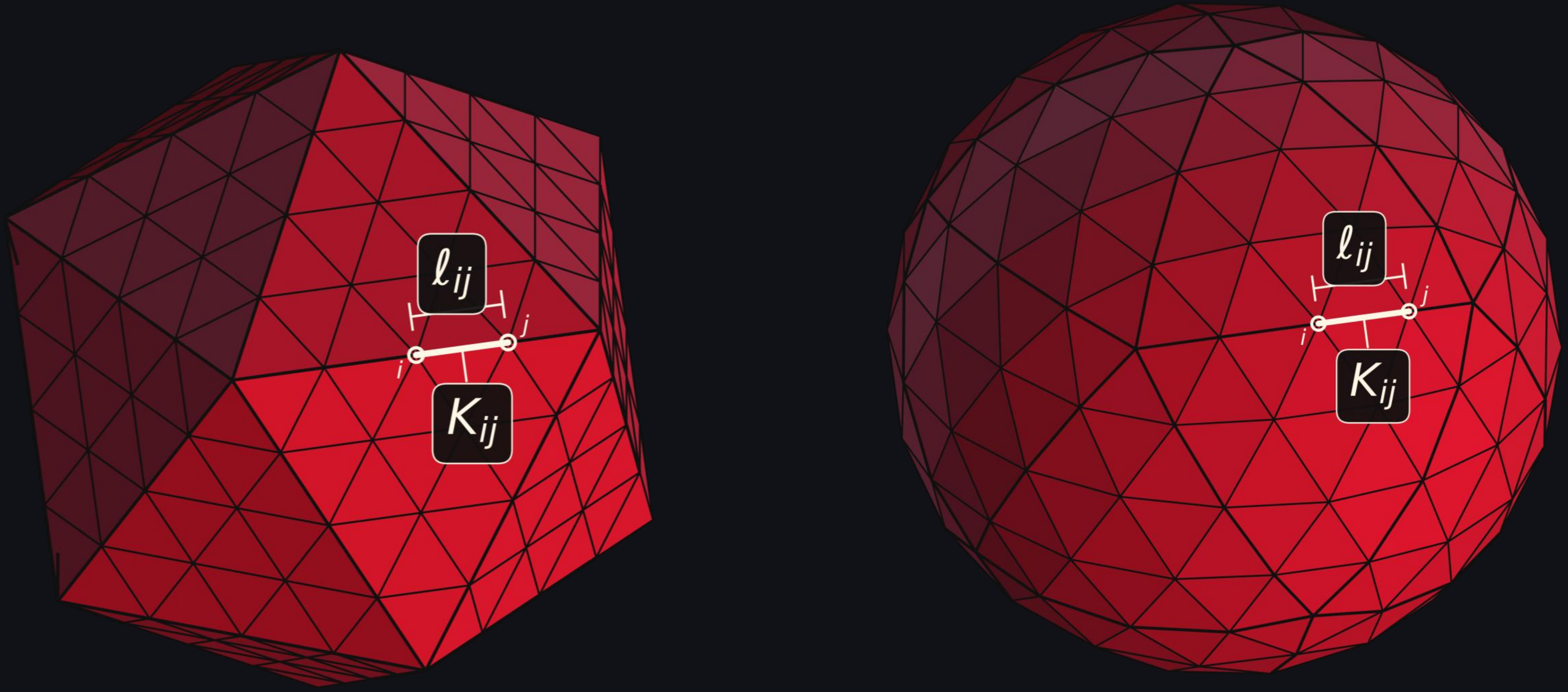


Outline

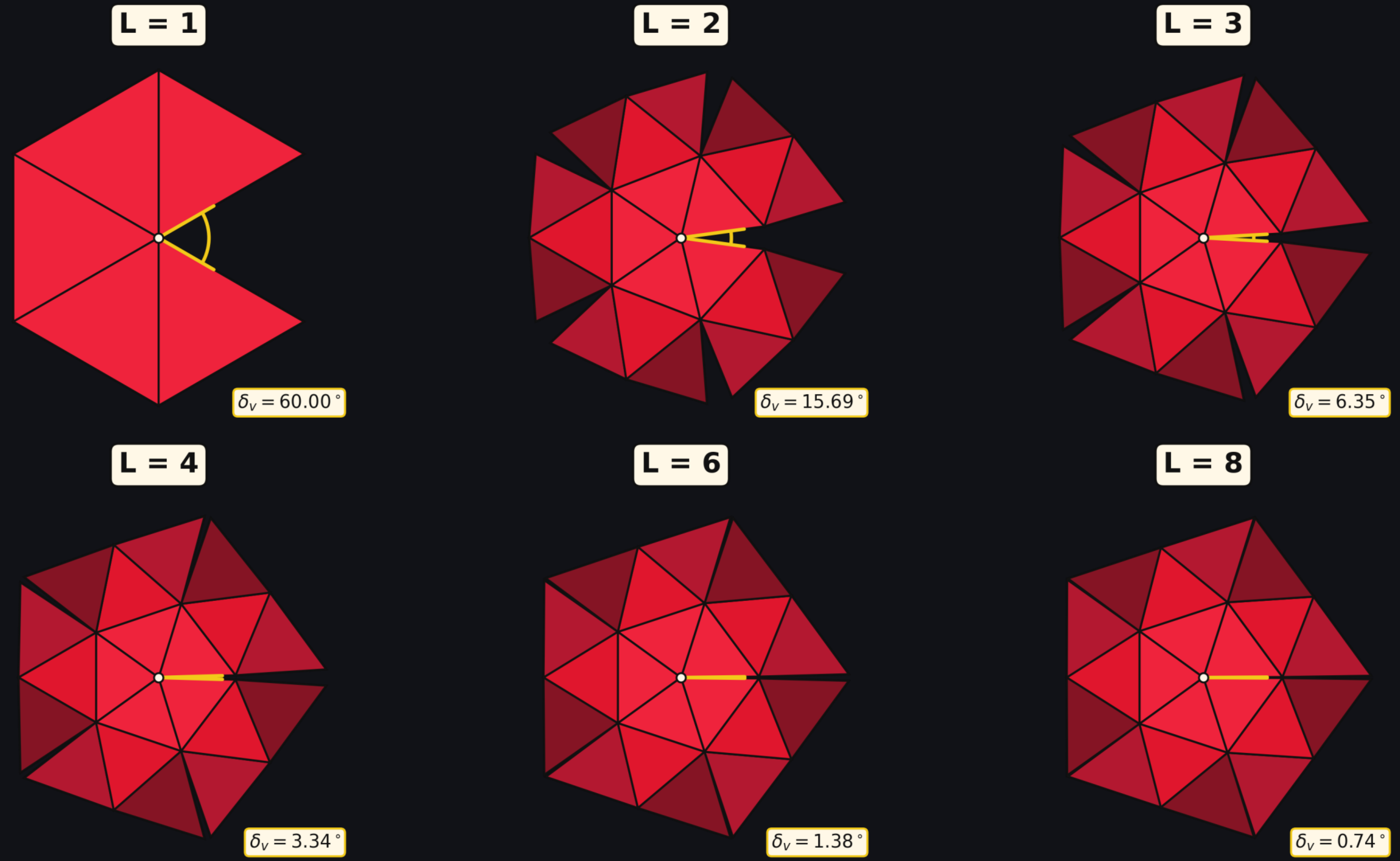
- How to put spin models on a sphere… And why it is difficult.
- The tangent plane and affine transformation of simplices
- Quantum couplings and geometry as stresses and strains
- An analytical coupling-geometry map for critical 2d Ising, and results on S2
- A plan to approximate the map numerically
- Does it work? (maybe!)
- Future things

Discretizing S2

Chapter 1



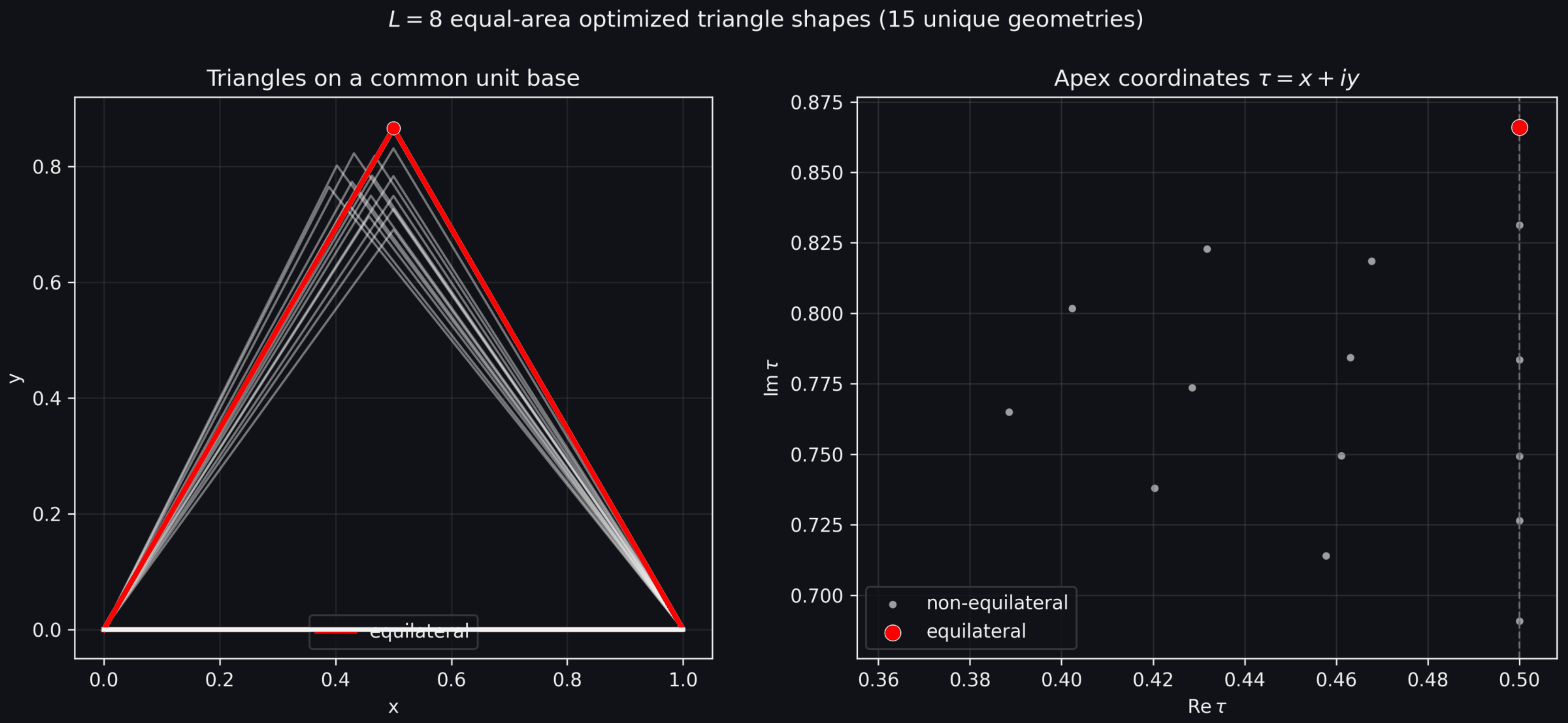
ℓ_{ij}
i
j
K_{ij}
ℓ_{ij}
i
j
K_{ij}


L = 1
$\delta_v = 60.00^\circ$
L = 2
$\delta_v = 15.69^\circ$
L = 3
$\delta_v = 6.35^\circ$
L = 4
$\delta_v = 3.34^\circ$
L = 6
$\delta_v = 1.38^\circ$
L = 8
$\delta_v = 0.74^\circ$

Ch.1 Summary

- Refining the discretization allows an approach to the continuum manifold but…
- It prescribes a non-uniform simplicial lattice with finely tuned lengths

Ch.1 Summary



$L = 8$ equal-area optimized triangle shapes (15 unique geometries)
Triangles on a common unit base
equilateral
y
x
0.0
0.2
0.4
0.6
0.8
1.0
Apex coordinates $\tau = x + iy$
Im τ
Re τ
0.875
0.850
0.825
0.800
0.775
0.750
0.725
0.700
0.36
0.38
0.40
0.42
0.44
0.46
0.48
0.50
non-equilateral
equilateral

Ch.1 Summary

- Refining the discretization allows an approach to the continuum manifold but…
- It prescribes a non-uniform simplicial lattice with finely tuned lengths
- But… Lengths do not enter the action. Only couplings do.

$$S = -\sum_{<i,j>} k_{ij} s_i s_j$$

- How do we choose the couplings to stress the lattice into the correct shape?

Analytical Solutions for 2d Ising

Chapter 2

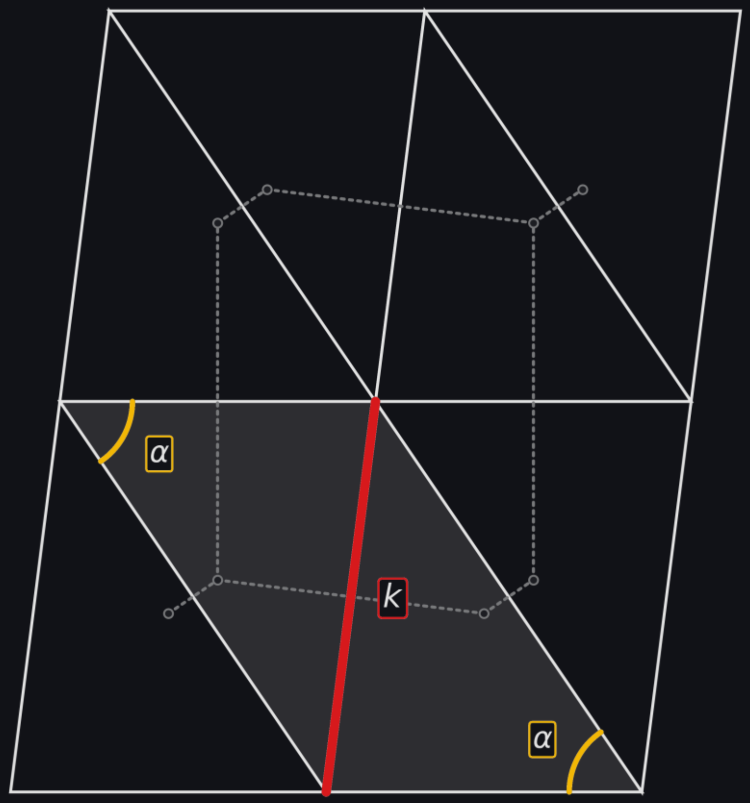
Based on Brower, Owen: 2209.15546, 2407.00459

The Coupling-Geometry map for 2d critical Ising

- For a lattice of identical scalene triangles (generated by an affine transformation):

The Coupling-Geometry map for 2d critical Ising

- For a lattice of identical scalene triangles (generated by an affine



$$\sinh(2k) = \cot(\alpha)$$

The Coupling-Geometry map for 2d critical Ising

- For a lattice of identical scalene triangles (generated by an affine

$$\sinh(2k_1) = \cot(\alpha_1)$$

$$\sinh(2k_2) = \cot(\alpha_2)$$

$$\sinh(2k_3) = \cot(\alpha_3)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi$$

Comparison to the Modular Ising CFT

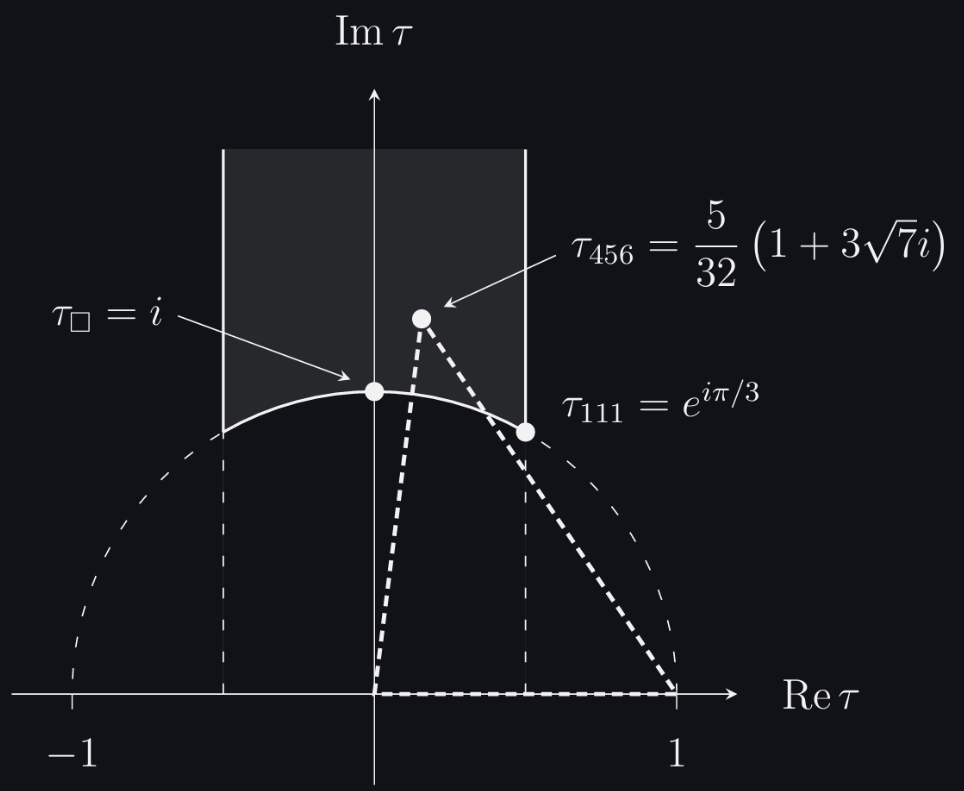


Figure 9: The fundamental domain of the modular parameter τ.

$$\langle\sigma(0)\sigma(z)\rangle = \left|\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z|\tau)}\right|^{1/4} \frac{\sum_{\nu=1}^{4}|\vartheta_\nu(z/2|\tau)|}{\sum_{\nu=2}^{4}|\vartheta_\nu(0|\tau)|}$$

Comparison to the Modular Ising CFT

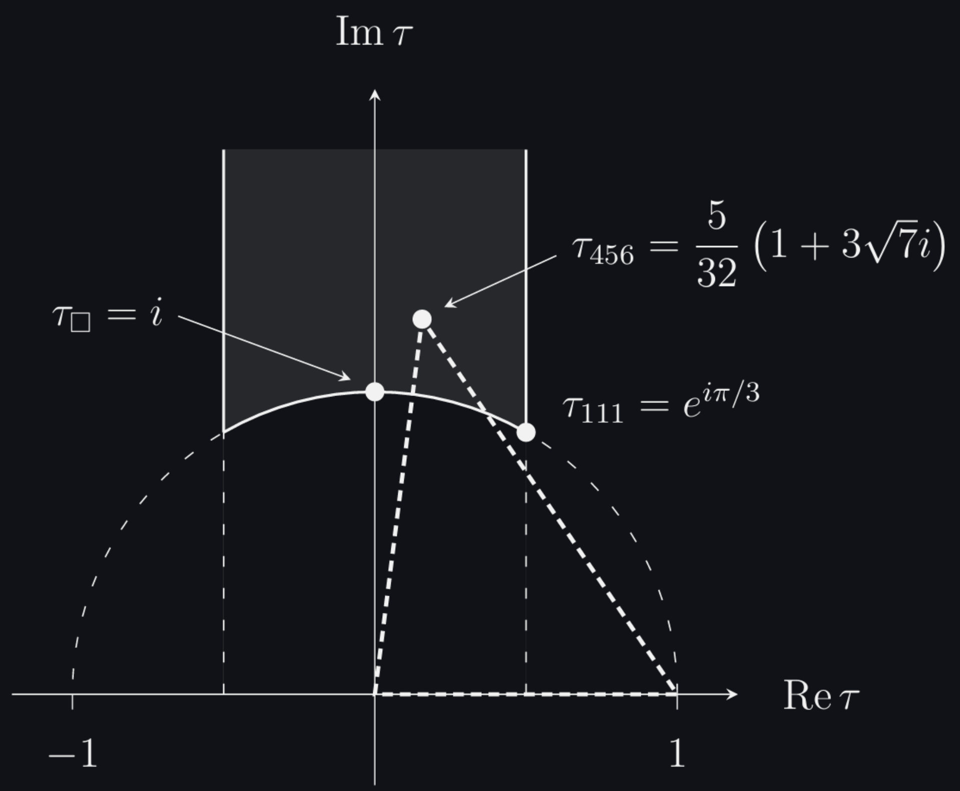
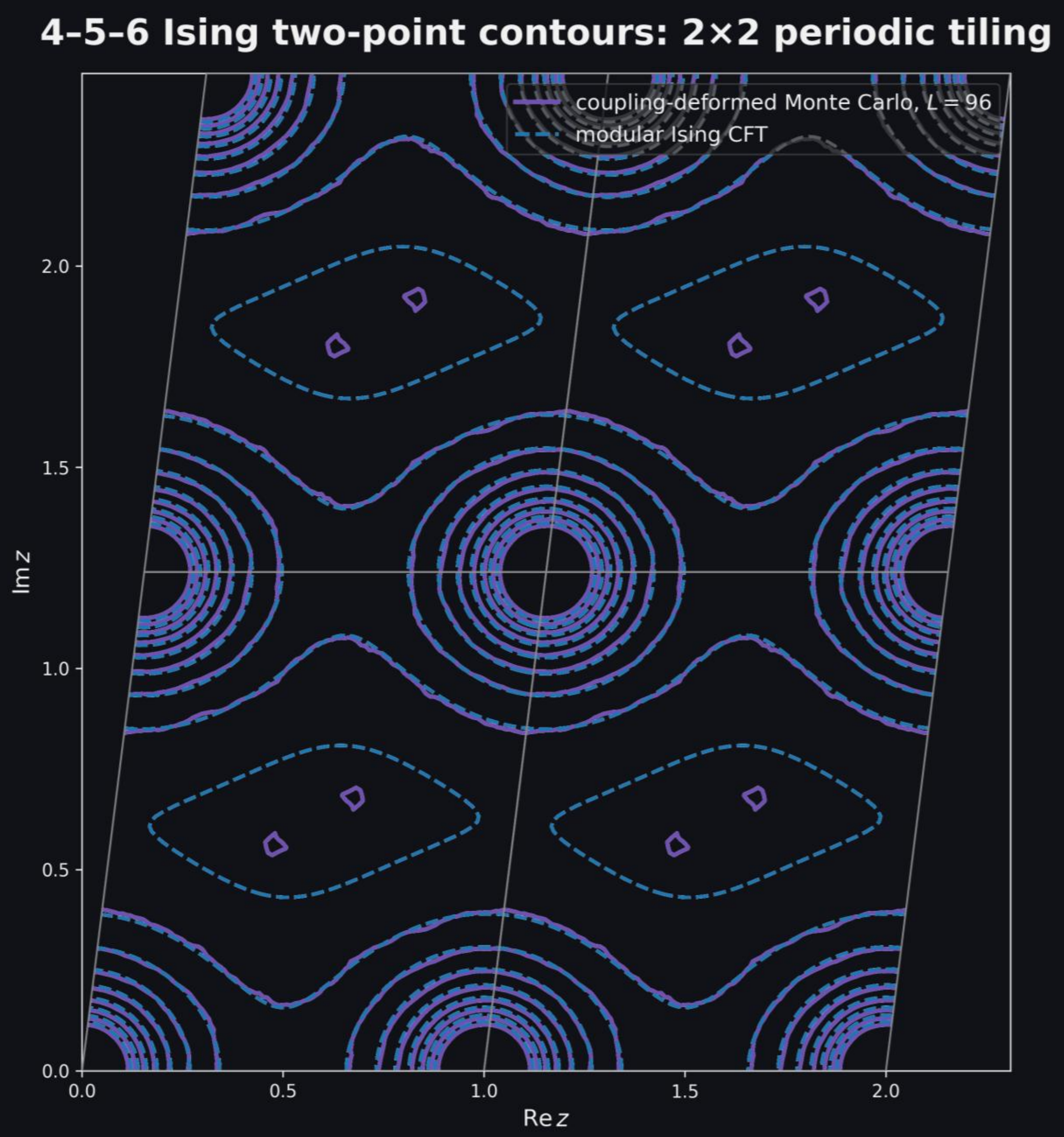
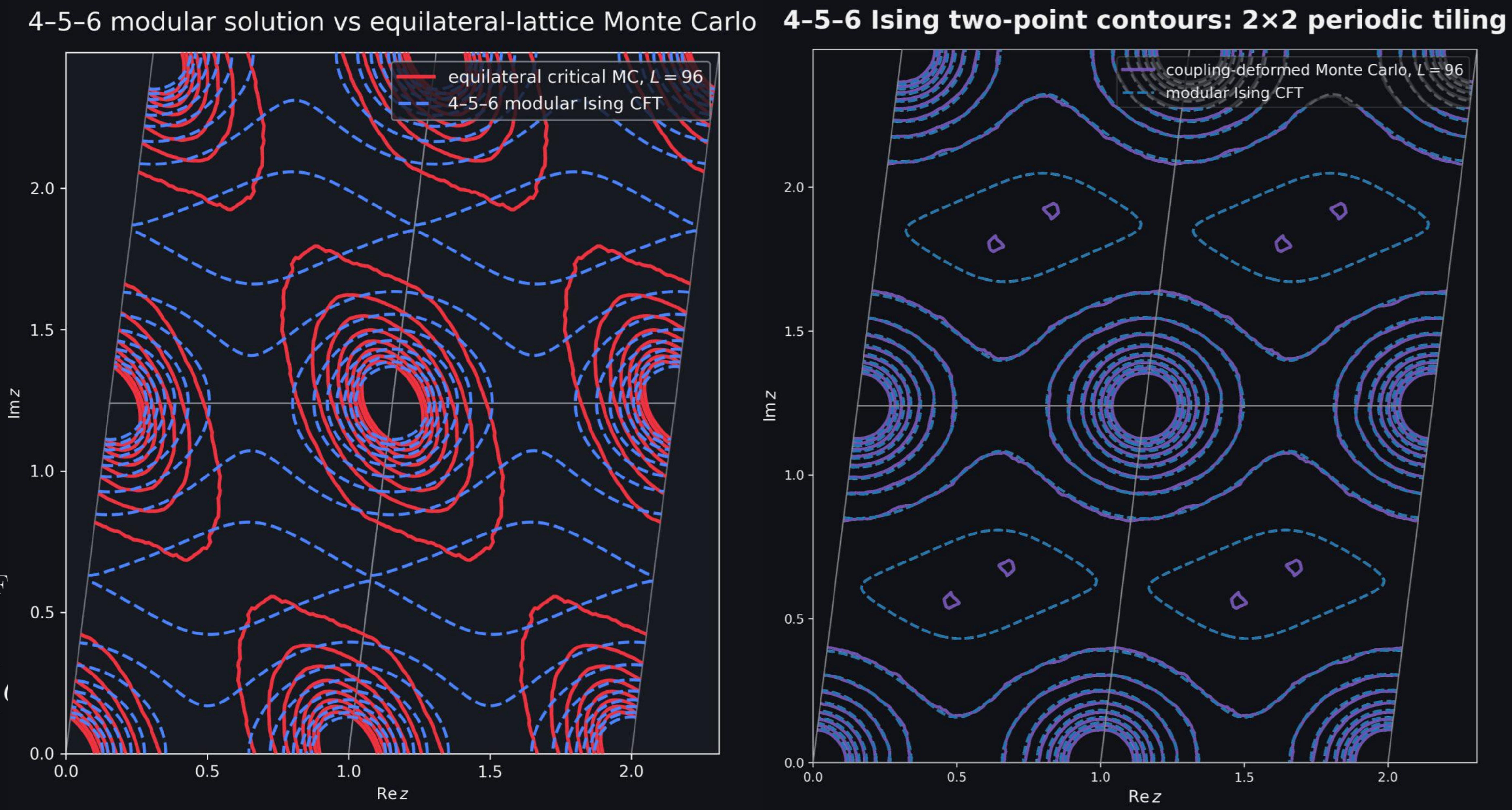


Figure 9: The fundamental domain of the modular parameter τ

$$\langle \sigma(0)\sigma(z)\rangle = \left|\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z|\tau)}\right|^{1/4} \frac{\sum_{\nu=1}^{4} |\vartheta_\nu(z/2|\tau)|}{\sum_{\nu=2}^{4} |\vartheta_\nu(0|\tau)|}$$

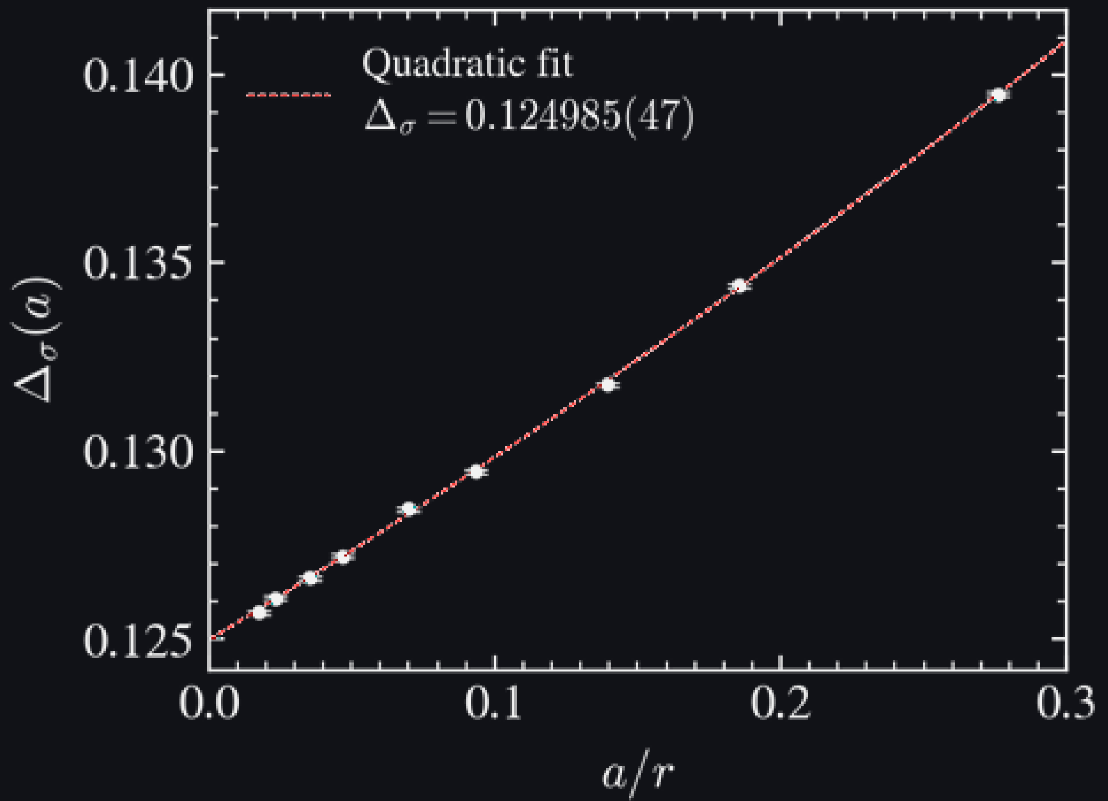


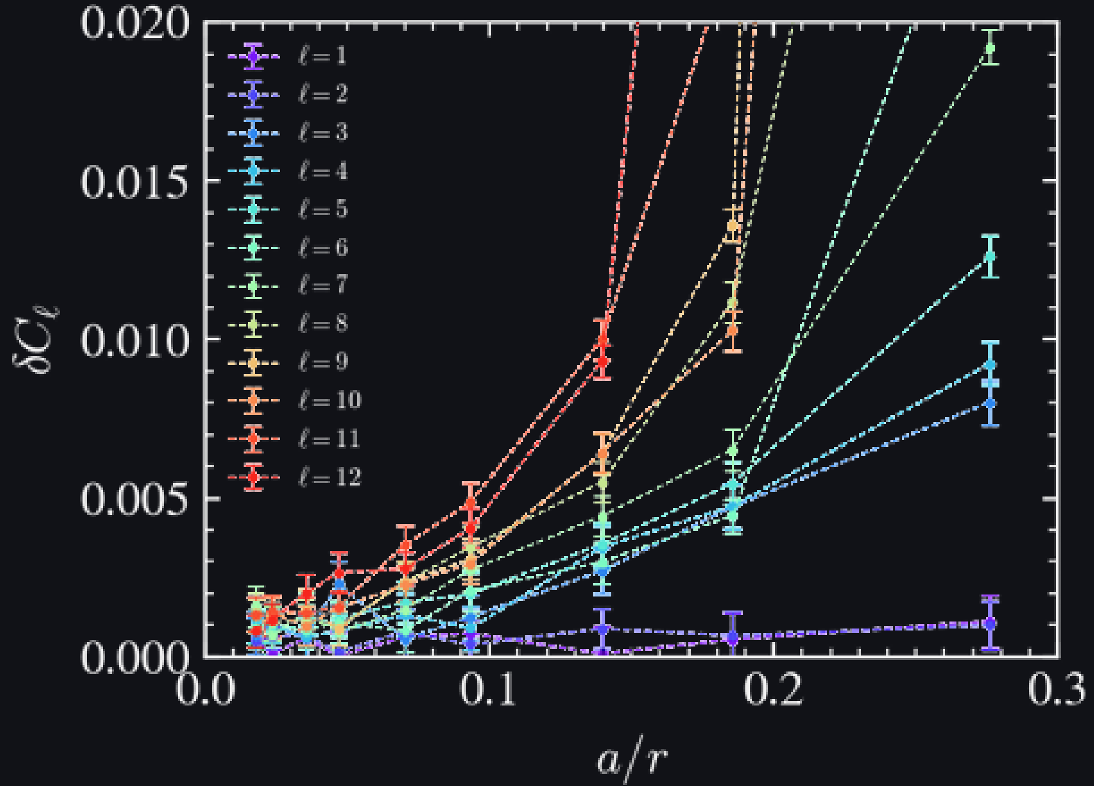
Comparison to the Modular Ising CFT



F

Tests on the Sphere





Ch.2 Summary

- The analytical map for the 2d, flat, affine-transformed, critical Ising model seems to faithfully reproduce a critical Ising model on the 2-sphere.*

$$\sinh(2k) = \cot(\alpha)$$

- However, the method to get this map is NOT general.
- Can we make a model-blind numerical method to estimate this map?

Moving Beyond Analytics

Chapter 3

Comparison to the Modular Ising CFT - Reprise

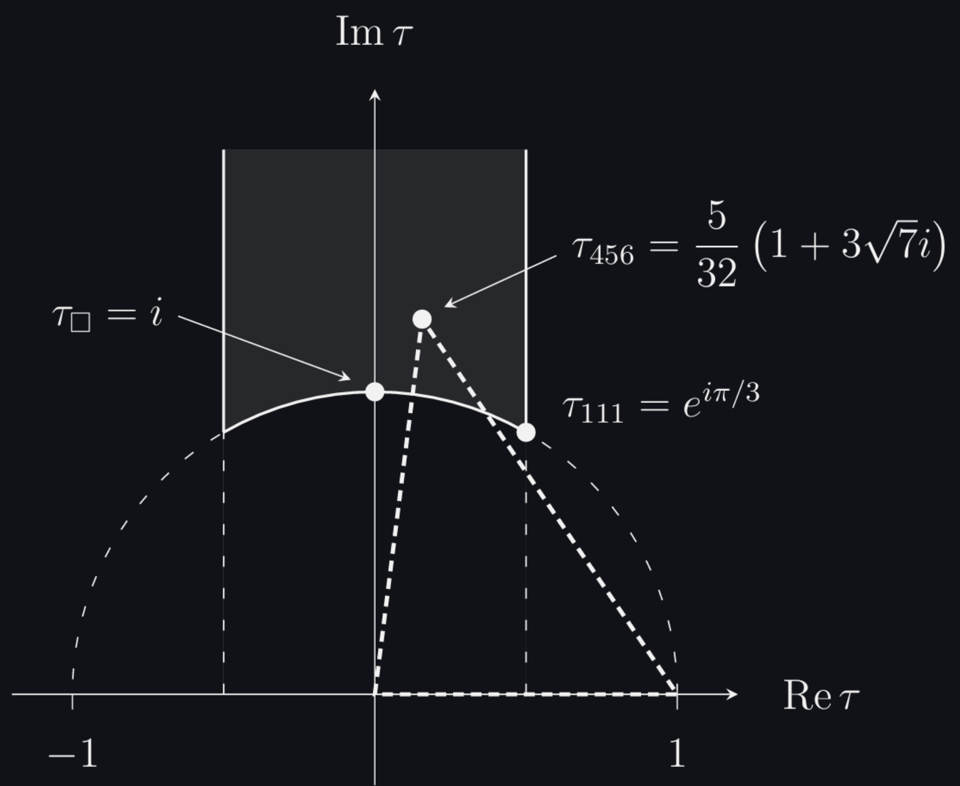
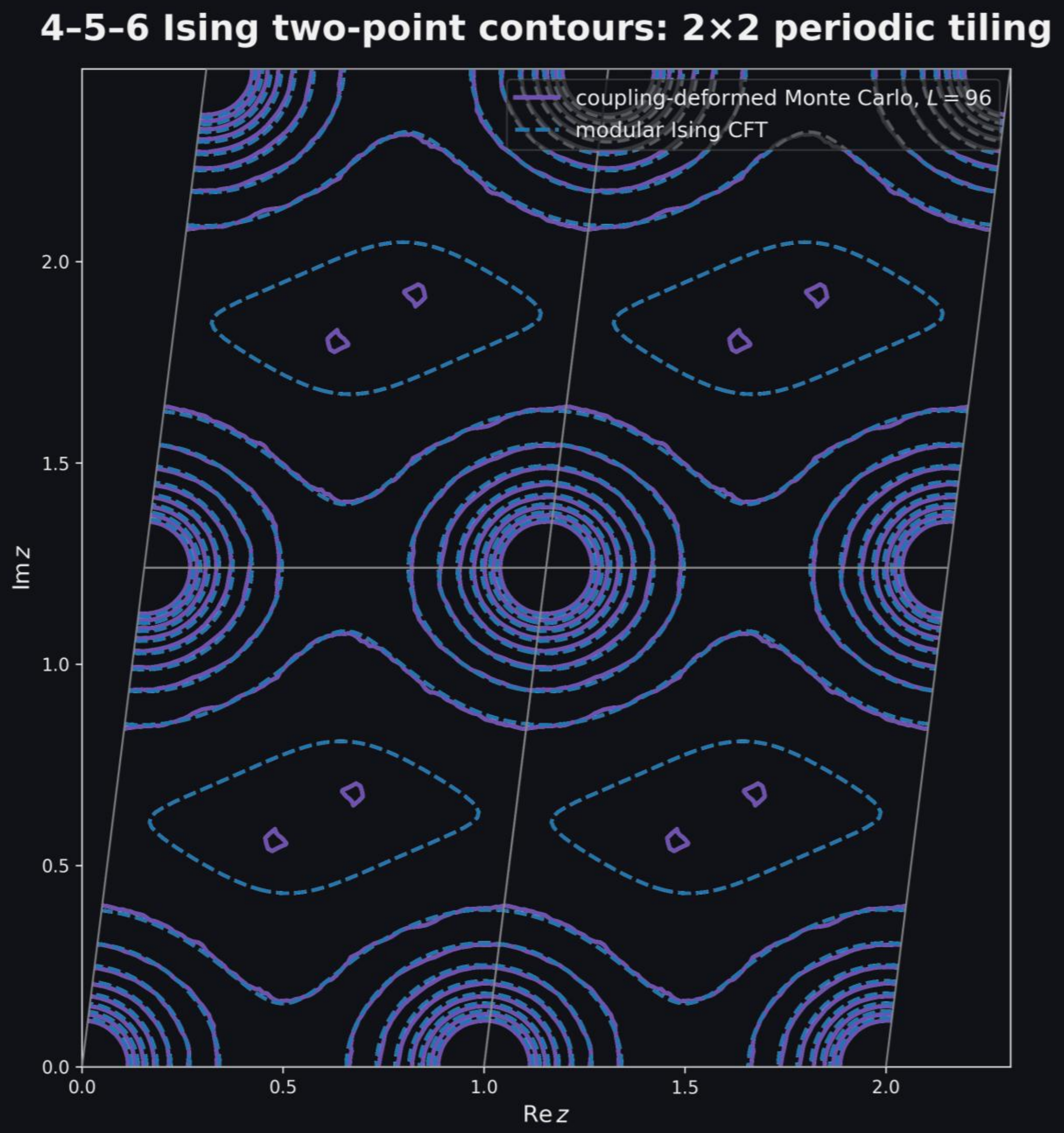


Figure 9: The fundamental domain of the modular parameter τ

$$\langle \sigma(0)\sigma(z) \rangle = \left| \frac{\vartheta_1'(0|\tau)}{\vartheta_1(z|\tau)} \right|^{1/4} \frac{\sum_{\nu=1}^{4} |\vartheta_\nu(z/2|\tau)|}{\sum_{\nu=2}^{4} |\vartheta_\nu(0|\tau)|}$$



Can we replace the modular solution?

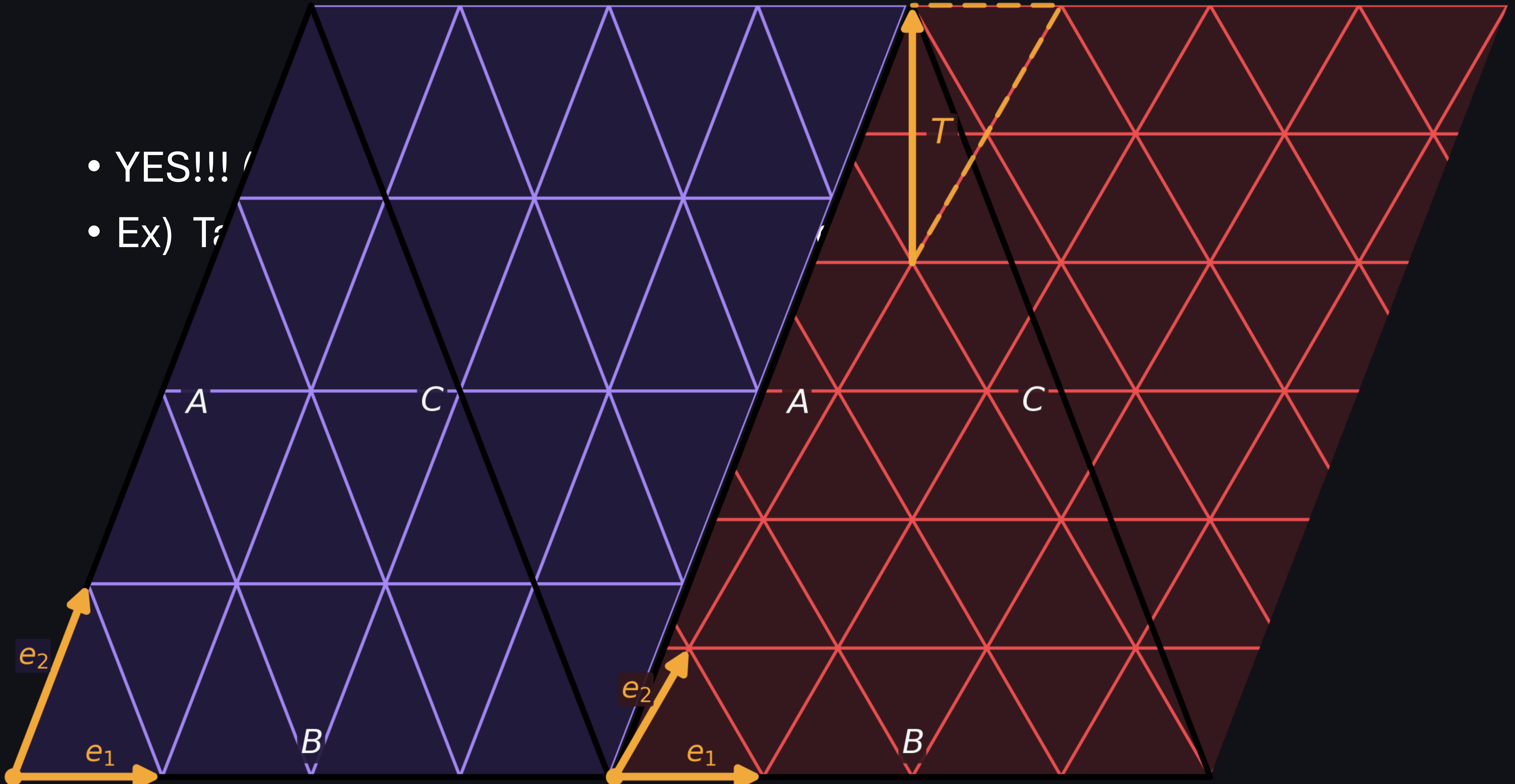
- YES!!! (maybe.)

Can we replace the modular solution?

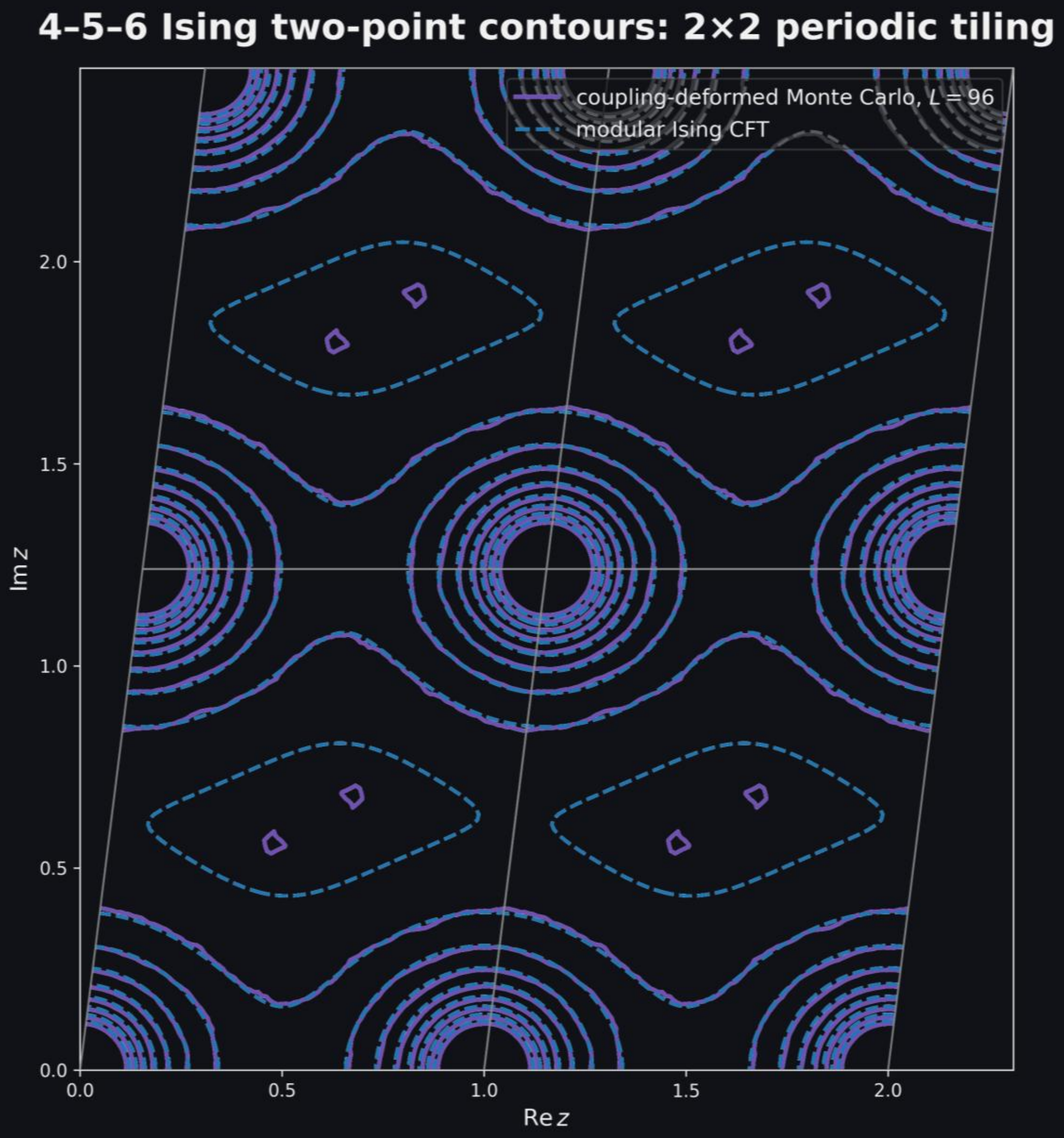
- YES!!! (maybe.)
- Ex) Target: Isosceles triangle with height 30% larger than the base

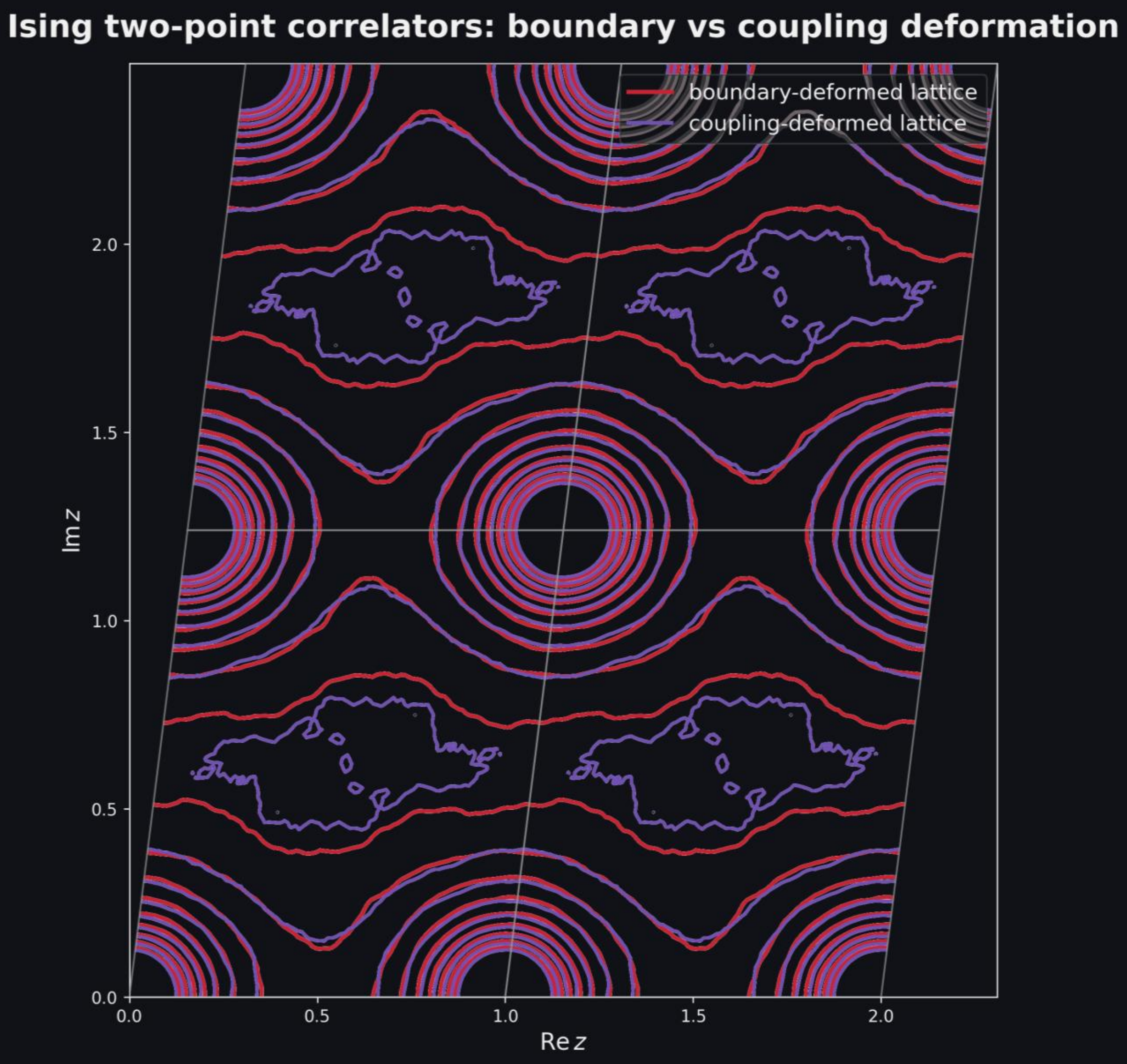
Can we replace the modular solution?

- YES!!! (
- Ex) Ta



Boundary deformed MC – Modular solution





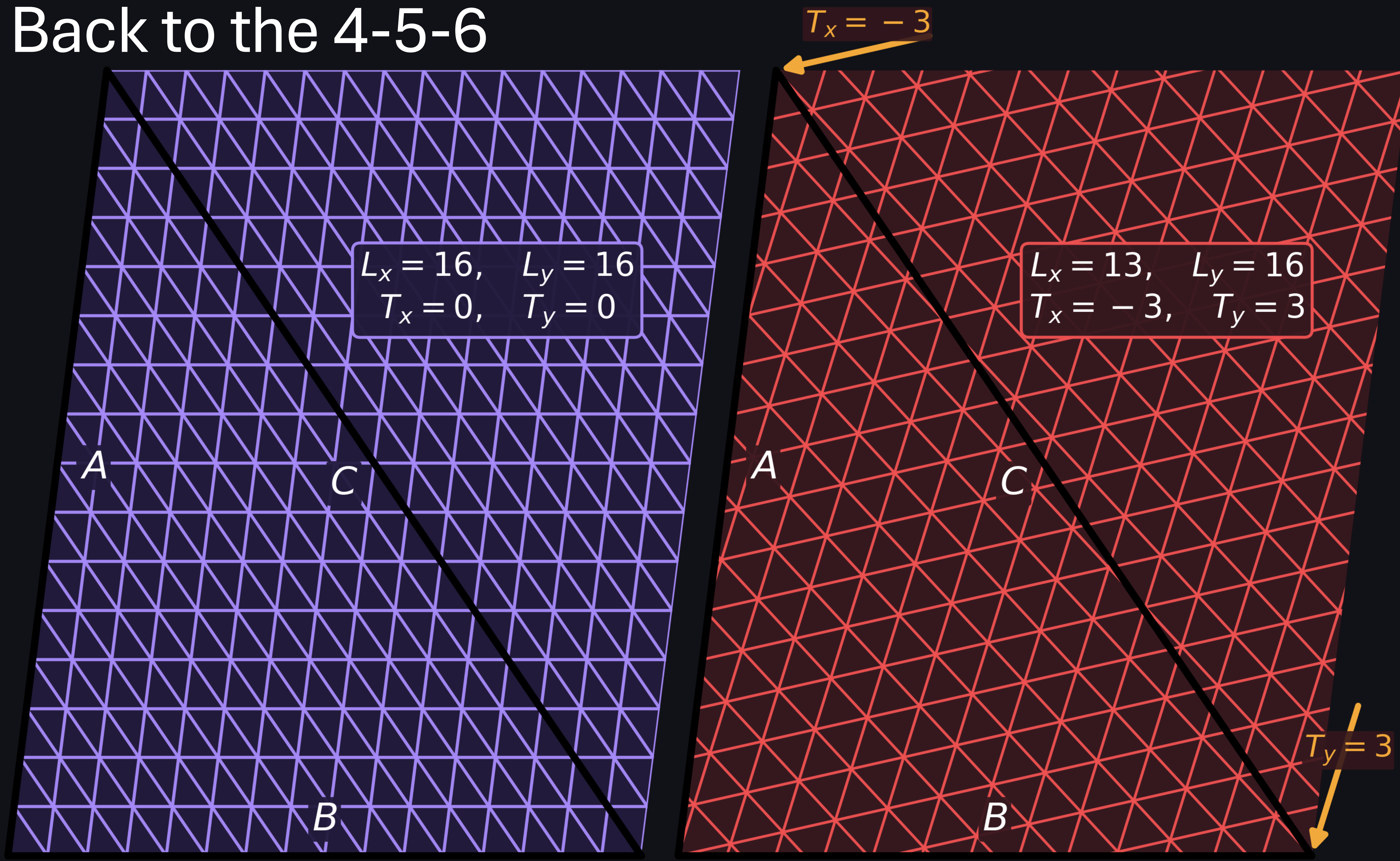
Ch.3 Summary

- If we run MC on boundary deformed lattices we can guarantee that its behavior matches some geometric target.
- If we then also run coupling deformed systems, we can compare the two.
- Say we run many tests and match some number of targets, a model fit will give us an approximation to the coupling-geometry map.

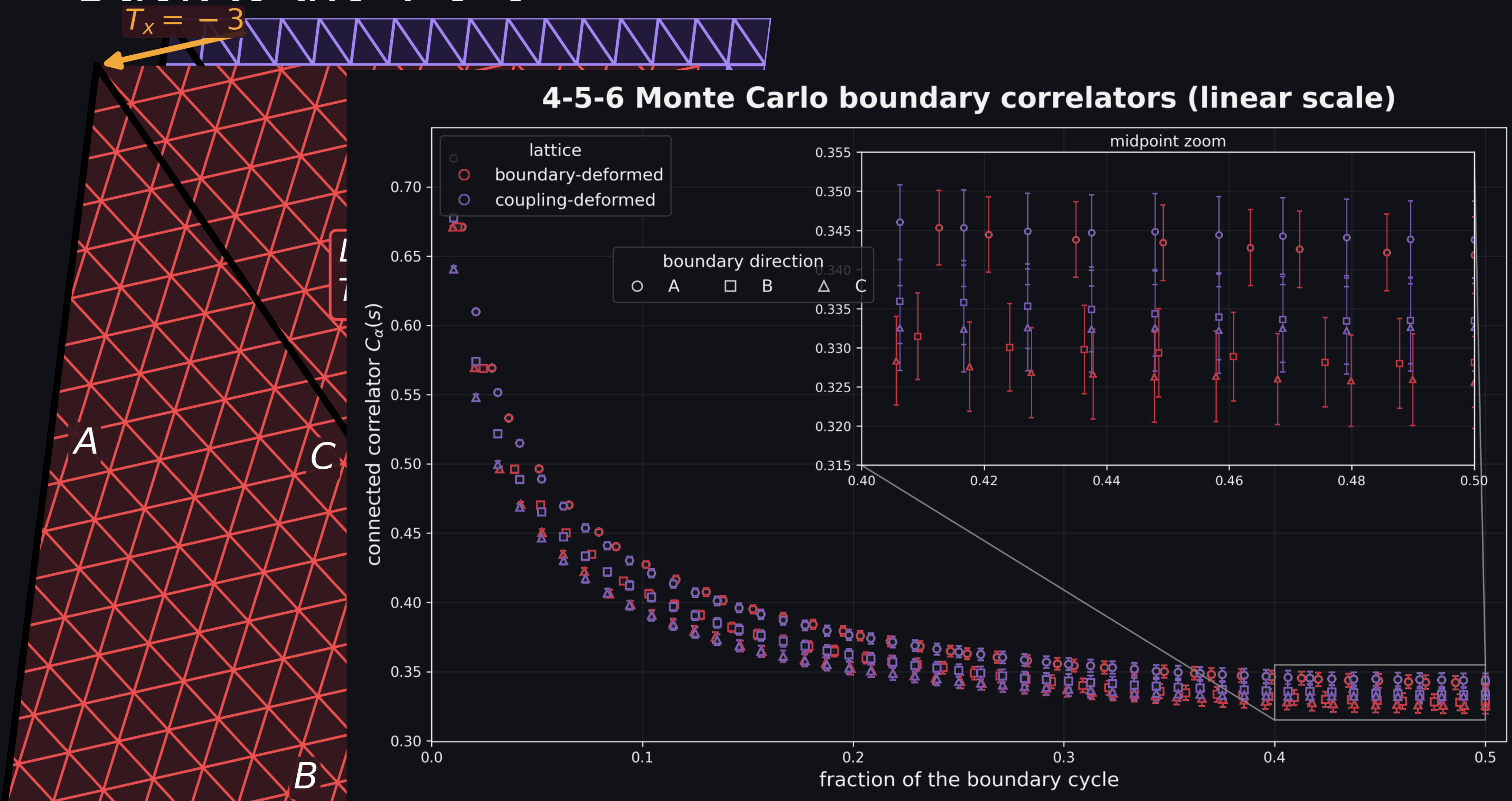
Scoring

Chapter 4

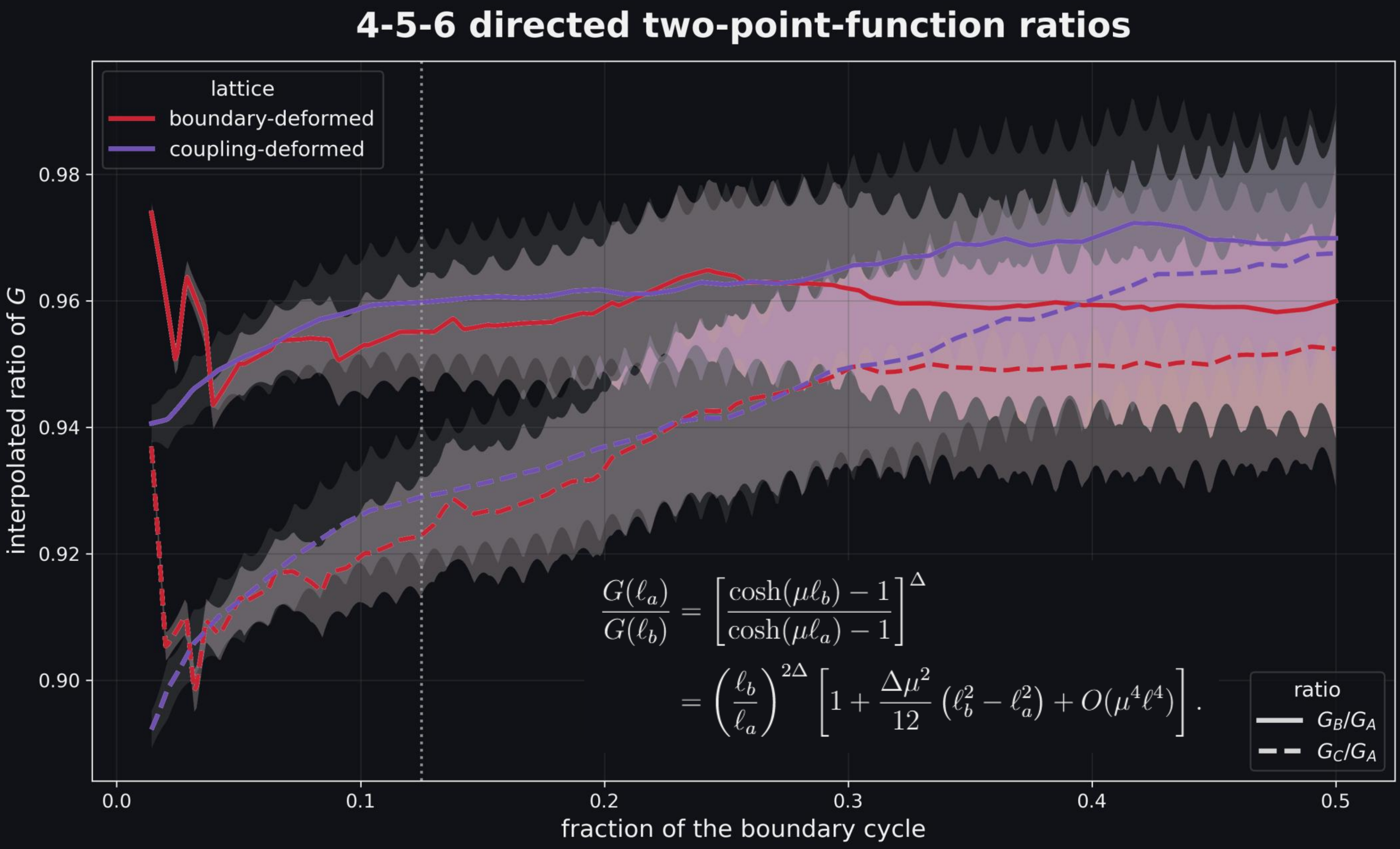
Back to the 4-5-6



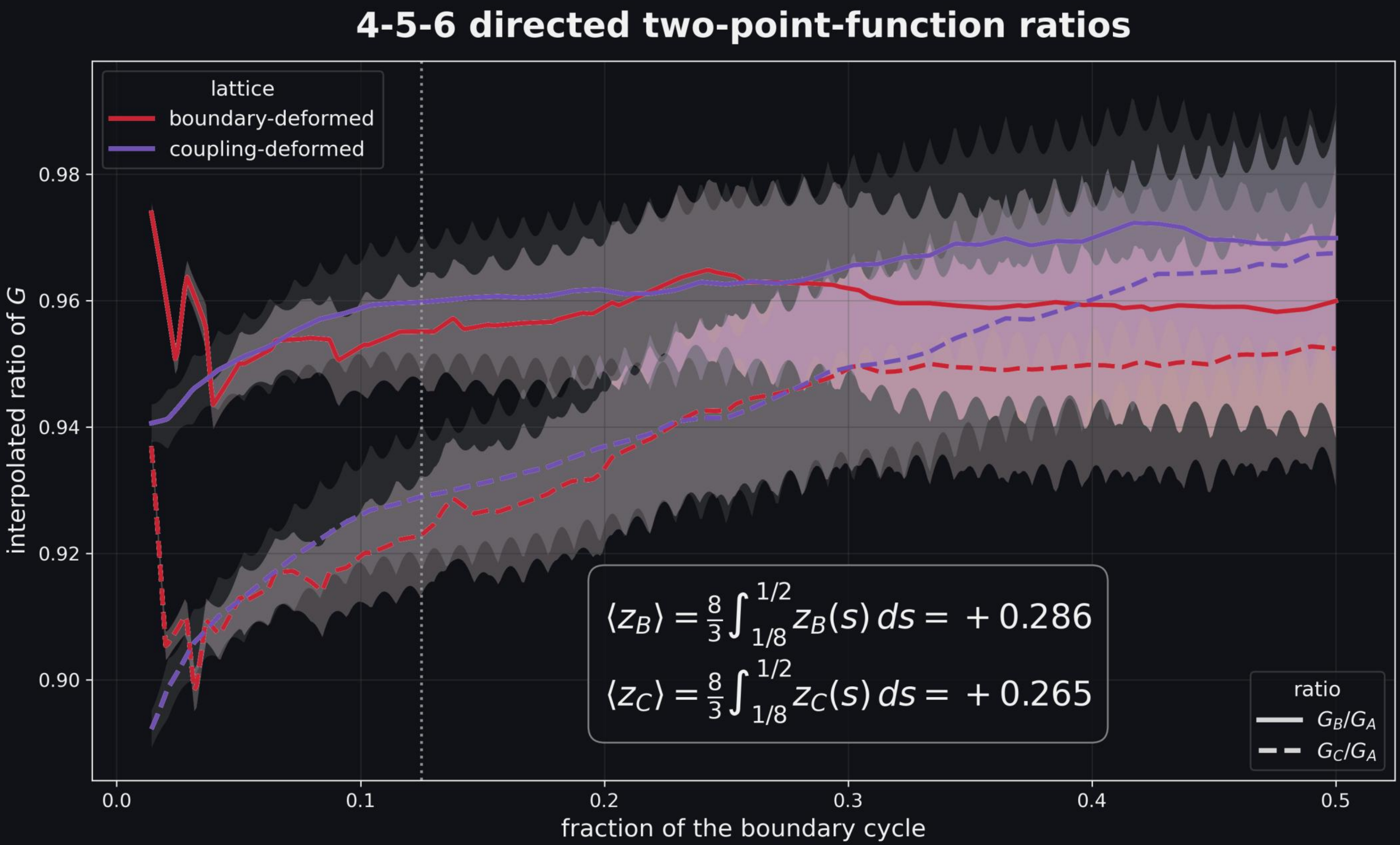
Back to the 4-5-6



Scoring


4-5-6 directed two-point-function ratios
lattice
boundary-deformed
coupling-deformed
ratio
G_B/G_A
G_C/G_A
interpolated ratio of G
fraction of the boundary cycle
0.98
0.96
0.94
0.92
0.90
0.0
0.1
0.2
0.3
0.4
0.5
$$\frac{G(\ell_a)}{G(\ell_b)} = \left[\frac{\cosh(\mu\ell_b)-1}{\cosh(\mu\ell_a)-1}\right]^{\Delta} = \left(\frac{\ell_b}{\ell_a}\right)^{2\Delta}\left[1+\frac{\Delta\mu^2}{12}\left(\ell_b^2-\ell_a^2\right)+O(\mu^4\ell^4)\right].$$

Scoring

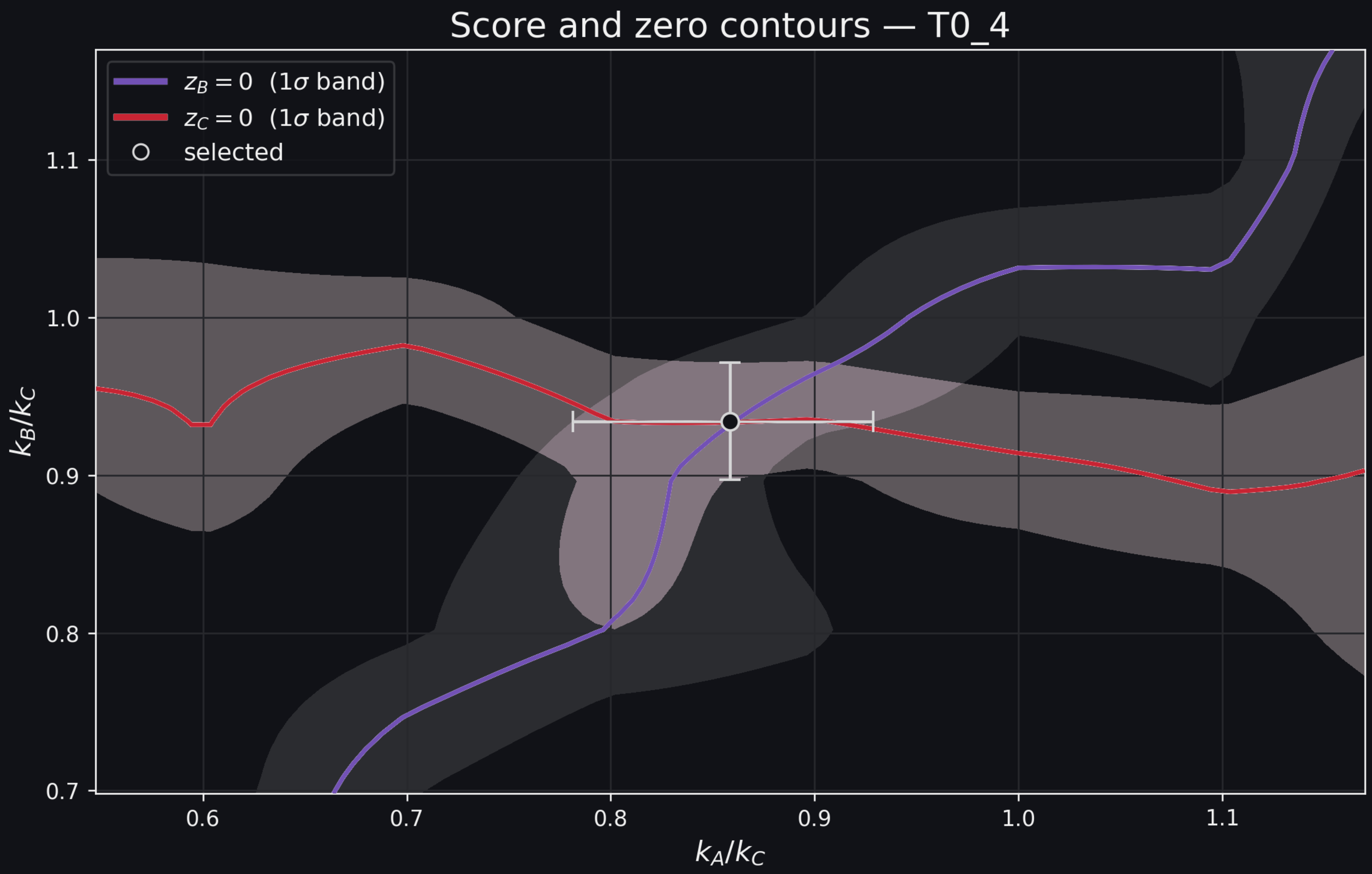

4-5-6 directed two-point-function ratios
lattice
boundary-deformed
coupling-deformed
0.98
0.96
0.94
0.92
0.90
interpolated ratio of G
0.0
0.1
0.2
0.3
0.4
0.5
fraction of the boundary cycle
$\langle z_B \rangle = \frac{8}{3}\int_{1/8}^{1/2} z_B(s)\,ds = +0.286$
$\langle z_C \rangle = \frac{8}{3}\int_{1/8}^{1/2} z_C(s)\,ds = +0.265$
ratio
G_B/G_A
G_C/G_A

A Method?

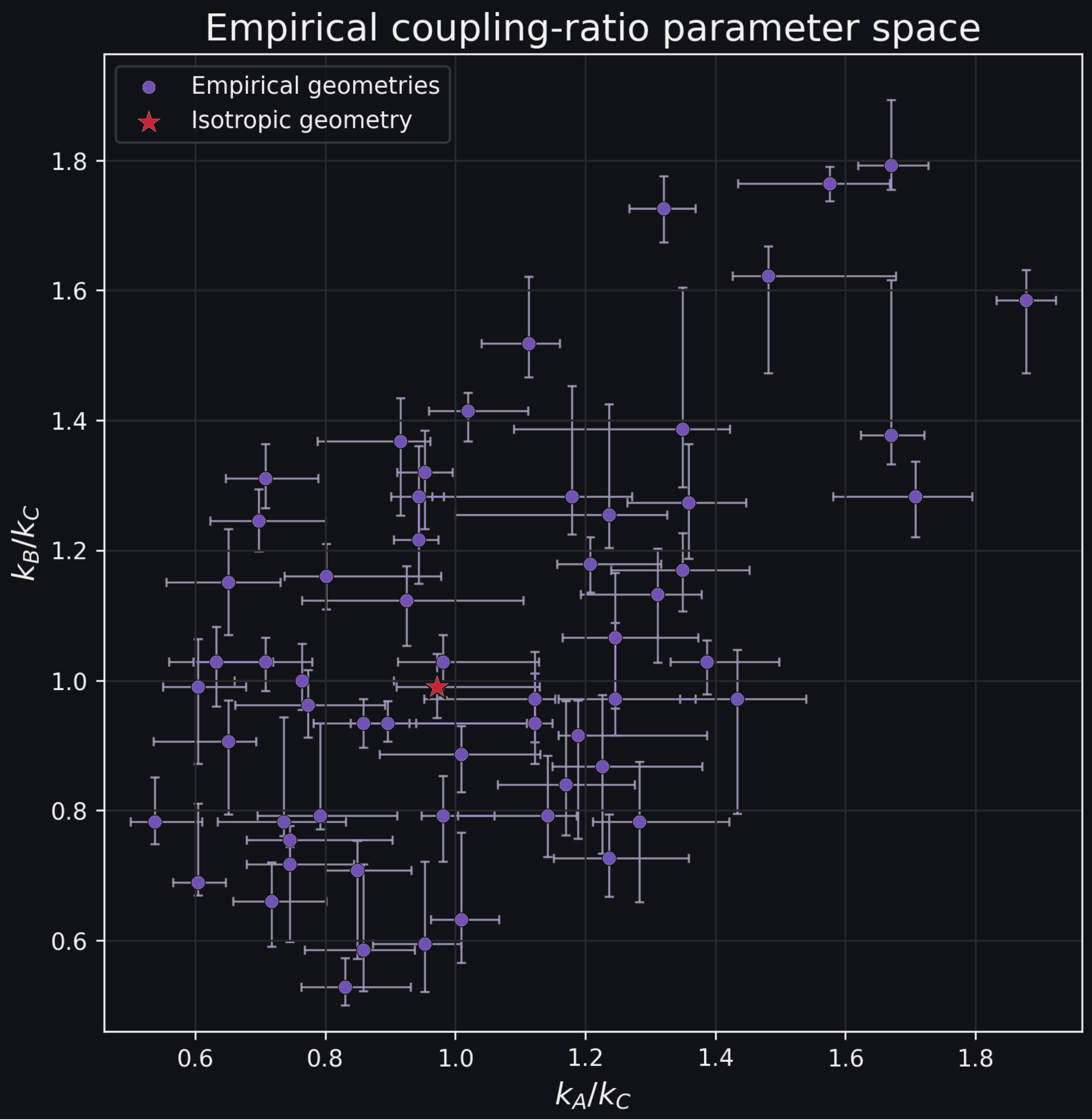
- Choose a set of target geometries (boundary deformations) and run MC
- Run MC over a large sweep of coupling parameter space
  - Could tune to the critical point first
- Generate scores to compare the sweep of coupling deformed lattices to the target geometries
- Collect a list of coupling – geometry pairs
- Perform a fit

Chapter 5
Results

$$S = -k_C \sum_i \left[\frac{k_A}{k_C} s_i s_{i+A} + \frac{k_B}{k_C} s_i s_{i+B} + s_i s_{i+C} \right]$$



$\tau = 0.527 + 0.812i, \qquad k_A/k_C = 0.858^{+0.070}_{-0.077}, \qquad k_B/k_C = 0.934^{+0.037}_{-0.037}$



Bars show the local 1σ overlap intervals.

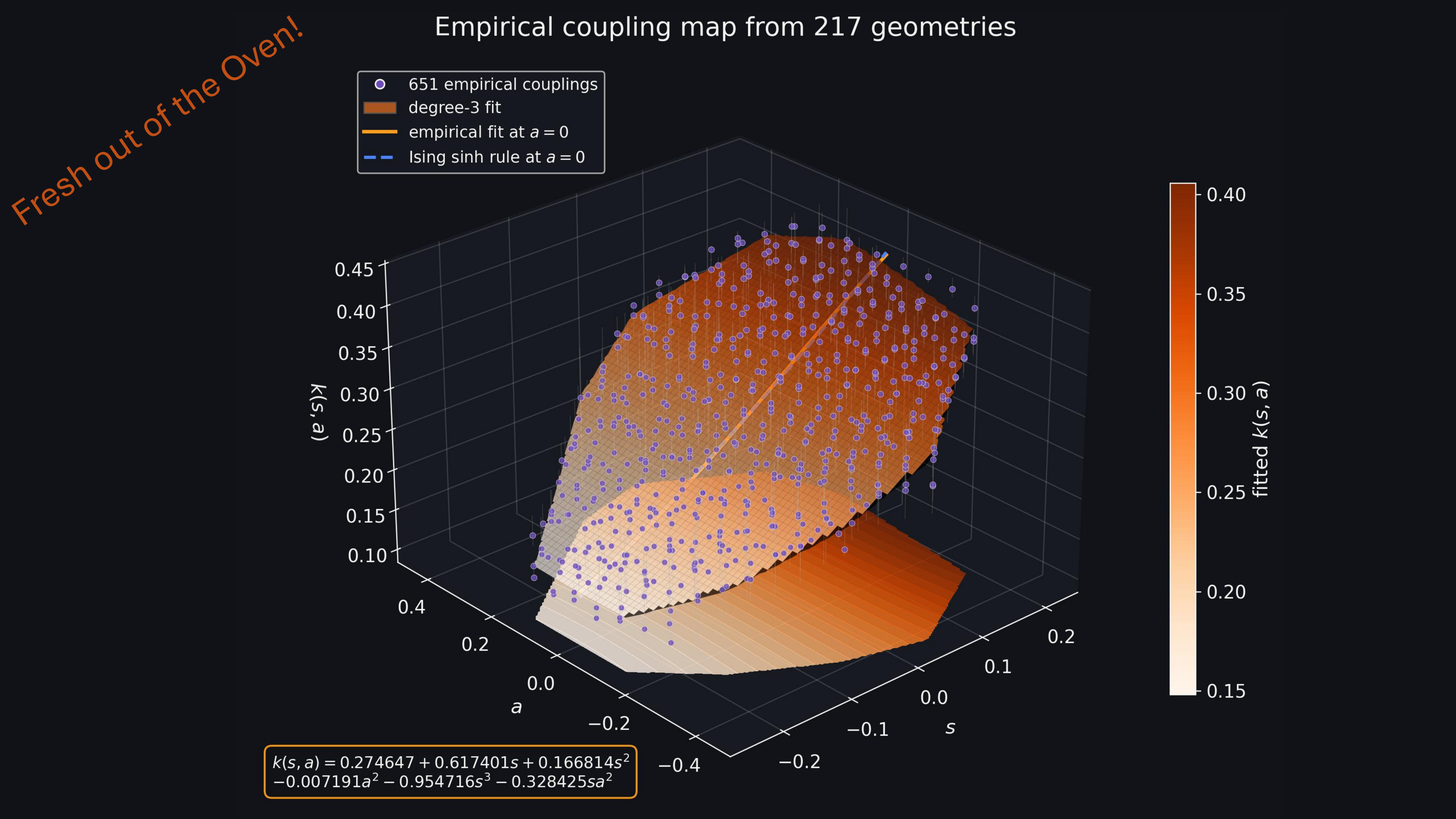
Fit Model

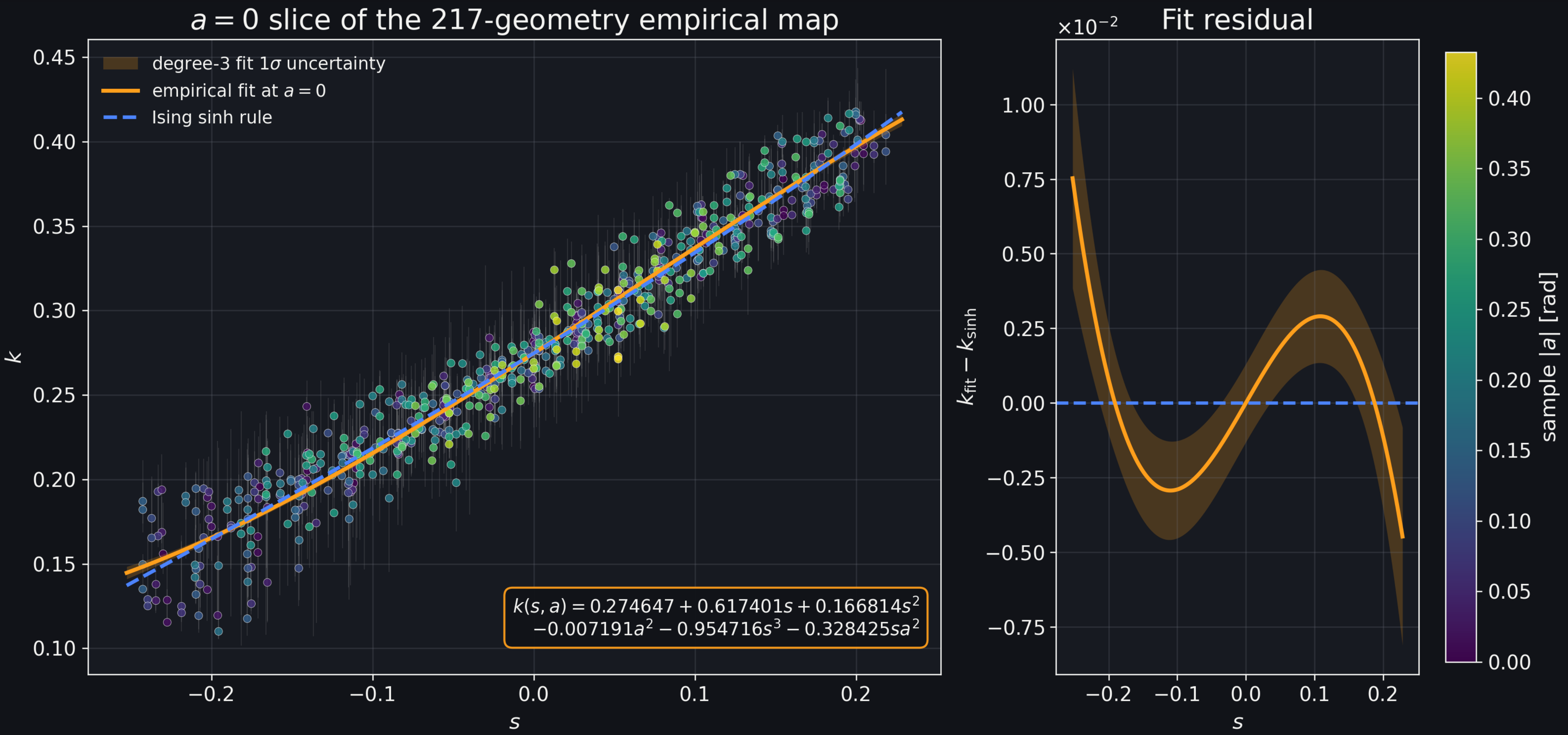
$$\delta_i \equiv \alpha_i - \frac{\pi}{3}, \qquad \delta_i + \delta_j + \delta_k = 0$$

$$s_{ij} \equiv \frac{1}{2}(\delta_i + \delta_j), \qquad a_{ij} \equiv \frac{1}{2}\delta_i - \delta_j,$$

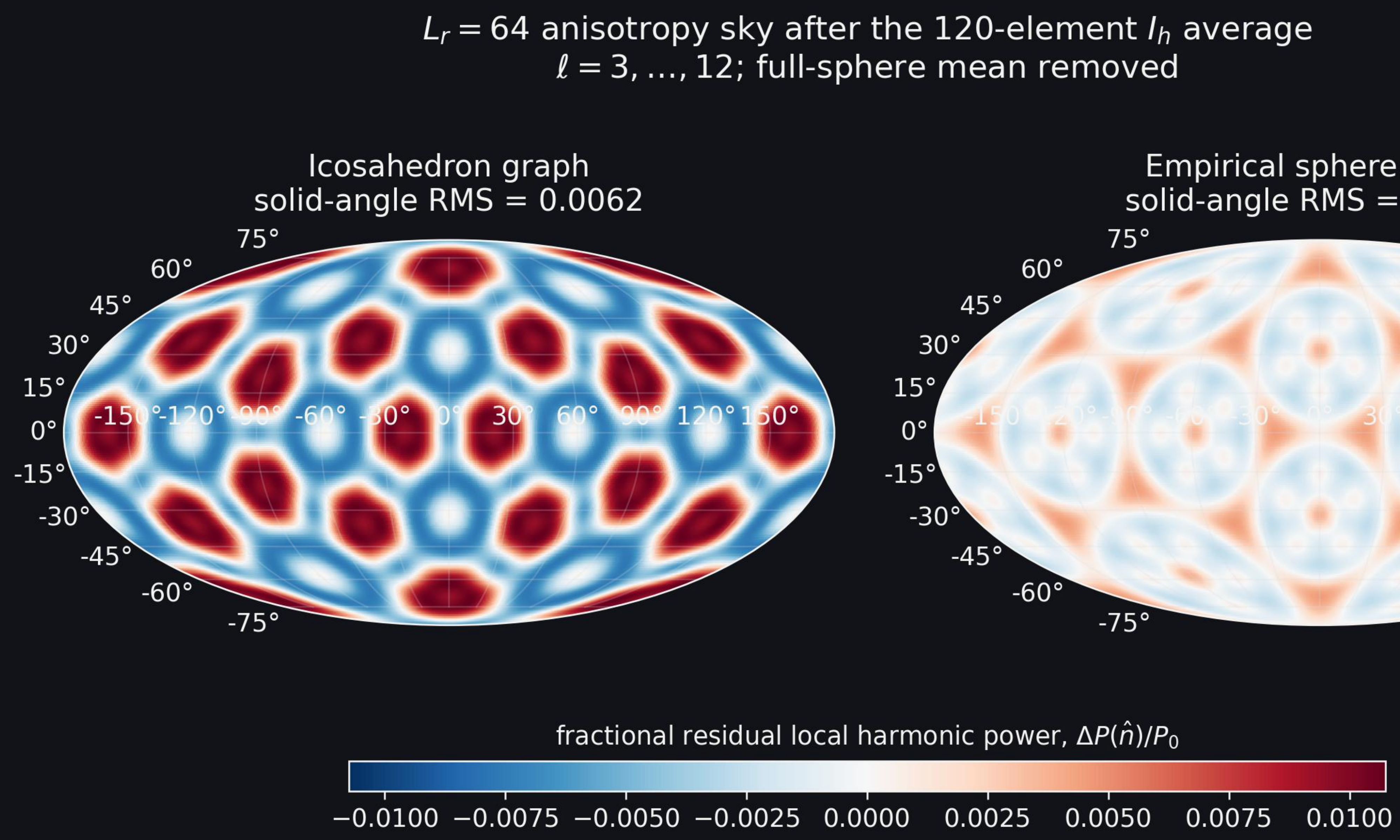
$$i \leftrightarrow j: \qquad k_{ij} \mapsto k_{ji}, \qquad s_{ij} \mapsto s_{ij}, \qquad a_{ij} \mapsto -a_{ij},$$

$$k_{ij} = \sum_{m,n \geq 0} c_{mn}\, s_{ij}^m a_{ij}^n, \qquad c_{mn} = 0 \quad \text{for odd } n$$


Fresh out of the Oven!
Empirical coupling map from 217 geometries
651 empirical couplings
degree-3 fit
empirical fit at $a = 0$
Ising sinh rule at $a = 0$
$k(s, a)$
a
s
fitted $k(s, a)$
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.4
0.2
0.0
−0.2
−0.4
0.2
0.1
0.0
−0.1
−0.2
$k(s, a) = 0.274647 + 0.617401s + 0.166814s^2$
$-0.007191a^2 - 0.954716s^3 - 0.328425sa^2$



$a = 0$ slice of the 217-geometry empirical map
degree-3 fit 1σ uncertainty
empirical fit at $a = 0$
Ising sinh rule
$k(s,a) = 0.274647 + 0.617401s + 0.166814s^2$
$-0.007191a^2 - 0.954716s^3 - 0.328425sa^2$
k
s
Fit residual
$\times 10^{-2}$
$k_{\mathrm{fit}} - k_{\mathrm{sinh}}$
sample $|a|$ [rad]



$L_r = 64$ anisotropy sky after the 120-element I_h average
$\ell = 3, \ldots, 12$; full-sphere mean removed
Icosahedron graph
solid-angle RMS = 0.0062
Empirical sphere graph
solid-angle RMS = 0.0017
75°
60°
45°
30°
15°
0°
-15°
-30°
-45°
-60°
-75°
-150° -120° -90° -60° -30° 0° 30° 60° 90° 120° 150°
fractional residual local harmonic power, $\Delta P(\hat{n})/P_0$
−0.0100 −0.0075 −0.0050 −0.0025 0.0000 0.0025 0.0050 0.0075 0.0100

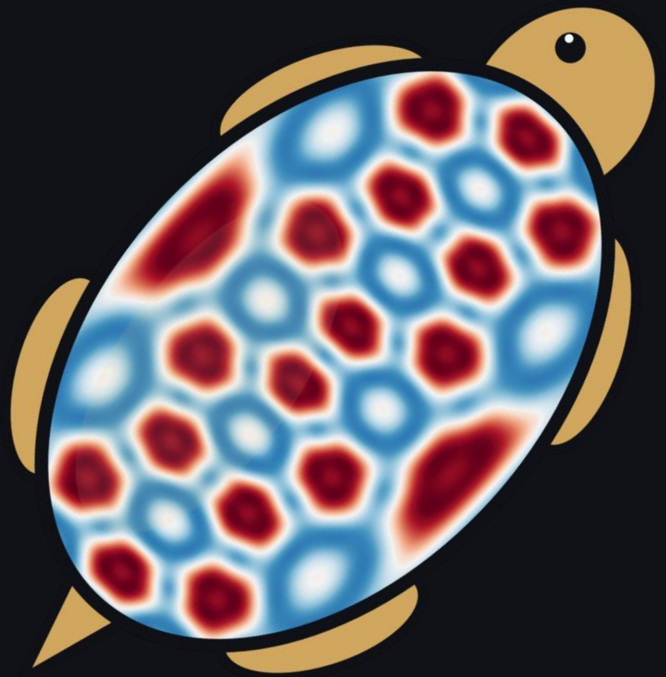
Future Applications

(after it works!)

Future applications

- Ising CFT on RxS2 for radial quantization
  - Compare conformal data with the CFT bootstrap and Fuzzy Sphere
- Phi-4 Theory
- O(N) model on RxS2
- Gauge Theories (Sounds hard!)
  - Do see Nobuyuki's talk on QED3! (Friday, Theory, 14:20-14:40)
- Extend method to 3d affine transformations to unlock S3

Thank You!



FEAR THE
TURTLE