Analyzing the entanglement property of Clifford-circuits augmented MPS

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LATTICE 2026 @University of Maryland

Tensor network & Quantum circuit

- Simulation in the Hamiltonian formalism is a promising approach
  👍 free from the sign problem　　👍 analyze wave functions directly

- Quantum computer:
  large-scale calculation still looks a long way off

- Tensor network (MPS):
  well-established tool for low-entanglement states

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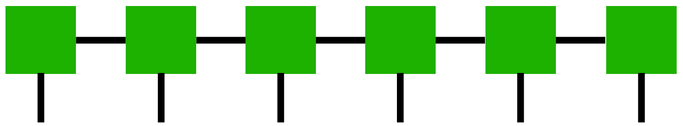
This talk:
Enlarge the applicability of tensor networks
by combining them with a “quantum circuit”
→ CAMPS



Clifford circuit

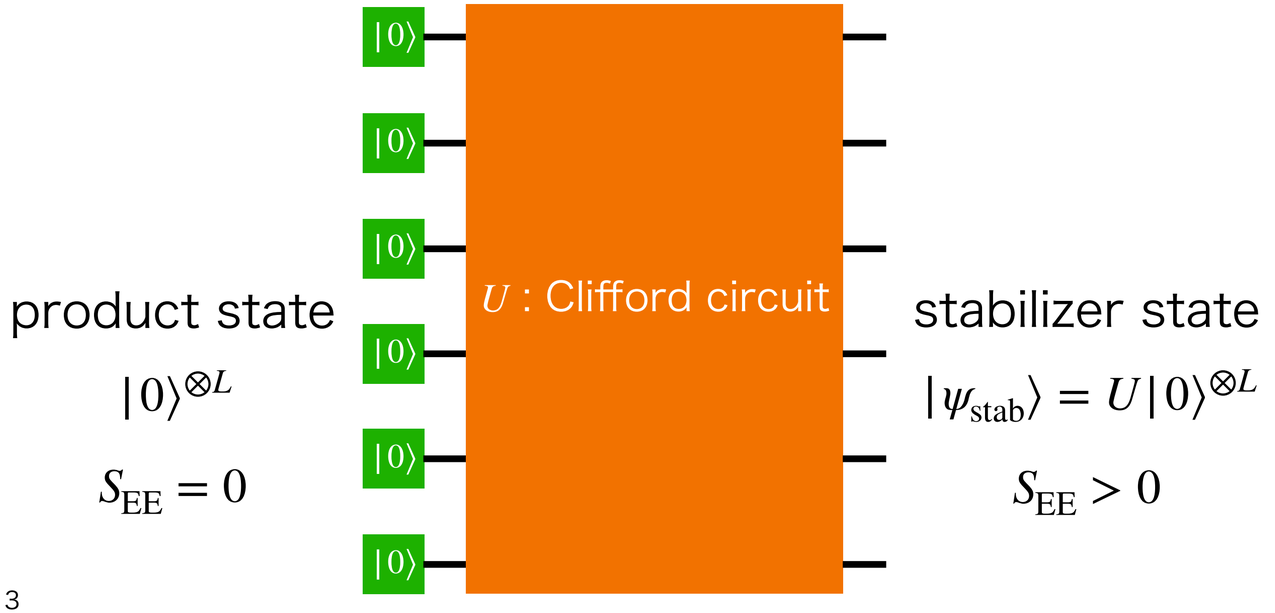
💡 Clifford circuit is a special class of quantum circuit
that can be simulated classically in polynomial time [Gottesman (1998)]
despite generating large entanglement

Clifford gate: $\left\{ \mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \mathbf{CNOT}_{1,2} = |0\rangle\langle 0|_1 \otimes I_2 + |1\rangle\langle 1|_1 \otimes X_2 \right\}$

$\{\text{Clifford}\} \cup \{\mathbf{T}\}$ = universal gate set

non-stabilizerness (quantum magic)

- amount of non-Clifford gates ($\mathbf{T}$)
- deviation from stabilizer state
- “hardness” of classical simulation



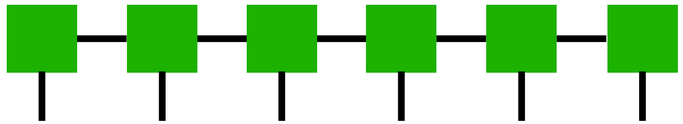
CAMPS

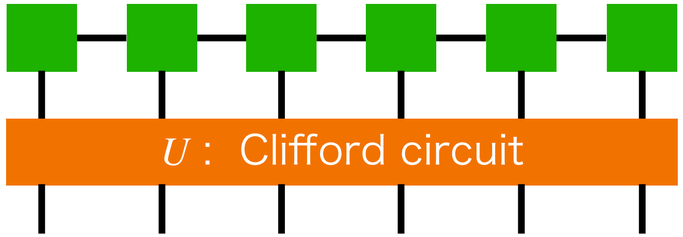
Clifford-circuits Augmented Matrix Product State

[Huang, Qian & Qin (2024)]

- Apply an optimal Clifford circuit U
  that reduces the EE of MPS $|\psi\rangle$ as much as possible

$$S_{\mathrm{EE}}\left(U|\psi\rangle\right) \leq S_{\mathrm{EE}}\left(|\psi\rangle\right)$$

$|\psi\rangle =$ 

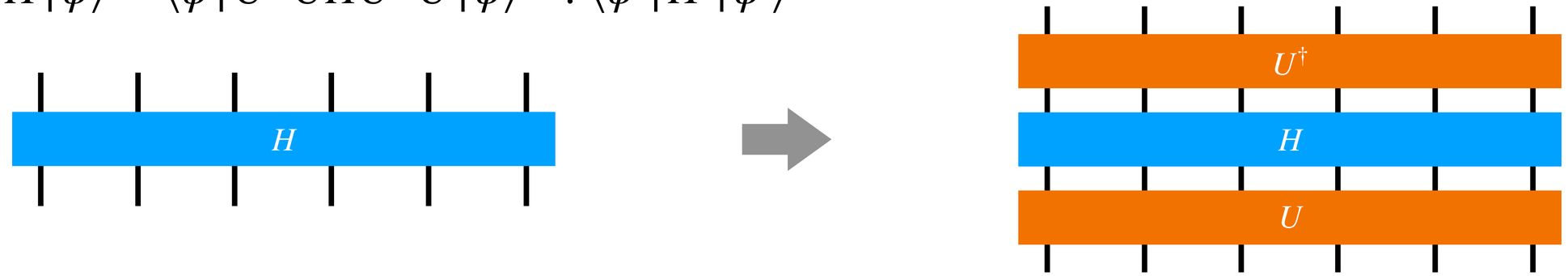
$|\psi'\rangle = U|\psi\rangle =$ 

- Clifford circuit generates the EE as a (dis)entangler
- Tensor network (MPS) generates non-stabilizerness
- Remaining EE is called non-stabilizerness EE: $\mathrm{NsEE} = \min_{U\in\{\mathrm{Clifford}\}} \sum_{\mathrm{cuts}} S_{\mathrm{EE}}\left(U|\psi\rangle\right)$

Transformation of Hamiltonian

- Hamiltonian H is also transformed by the Clifford circuit U

$$\langle\psi|H|\psi\rangle = \langle\psi|U^\dagger UHU^\dagger U|\psi\rangle =: \langle\psi'|H'|\psi'\rangle$$



💡 The Clifford group is a normalizer of the Pauli group

—> transforms Pauli strings to Pauli strings

—> The Hamiltonian, written as Pauli strings, is kept as Pauli strings

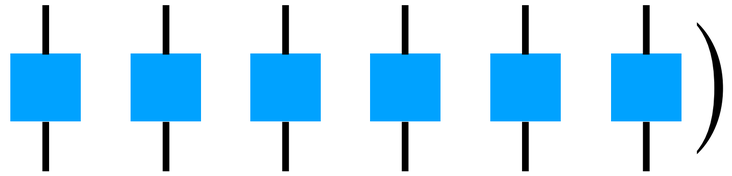
- Keep track of the transformation $H \to UHU^\dagger$ by the tableau algorithm

[Aaronson & Gottesman (2004)]

DMRG with CAMPS

- Pauli string Hamiltonian of the spin-1/2 chain (qubits) of length L

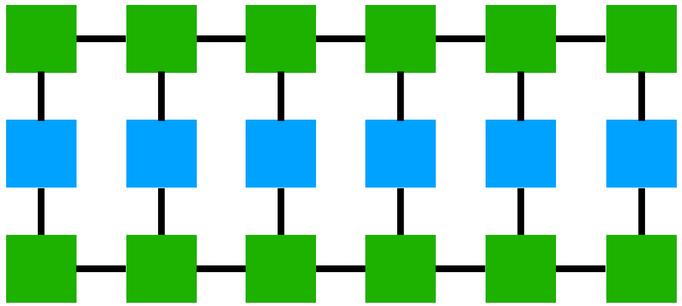
$$H = \sum_{\alpha=1}^{N_{PS}} c_\alpha P_\alpha = \sum_{\alpha=1}^{N_{PS}} c_\alpha$$



$$= p_{\alpha,n} \in \{I, X, Y, Z\}$$

- Energy

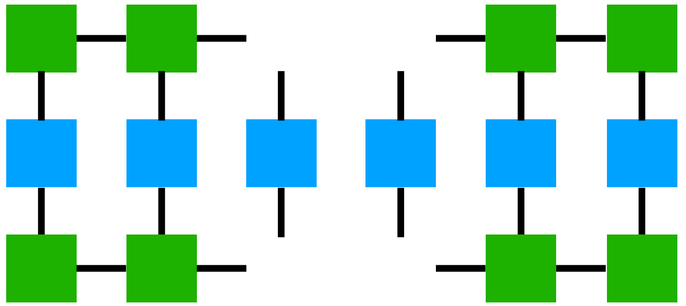
$$E = \langle\psi|H|\psi\rangle = \sum_{\alpha=1}^{N_{PS}} c_\alpha$$



(1) 2-site optimization:

find the ground state $\phi =$

of $\tilde{H} = \sum_{\alpha=1}^{N_{PS}} c_\alpha$

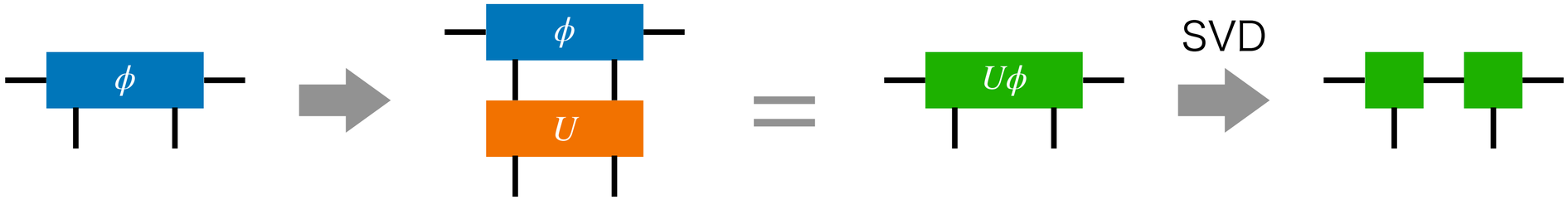


DMRG with CAMPS

(2) apply 2-site Clifford circuits U on ϕ and then perform SVD

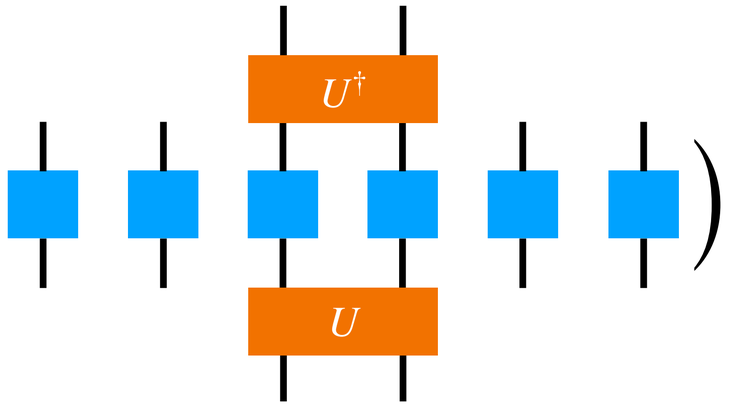
—> select one of the 20 circuits that gives the smallest EE

[Huang, Qian & Qin (2024)]
[E. Fux et al. (2025)]



(3) transform the Hamiltonian by U with the tableau algorithm

$$H \to \sum_{\alpha=1}^{N_{PS}} c_\alpha U P_\alpha U^\dagger = \sum_{\alpha=1}^{N_{PS}} c_\alpha$$



Numerical cost

MPS: $O(LD^3)$

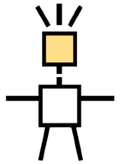
CAMPS: $O(N_{PS}LD^3)$

$D \sim \text{const}, L^{\alpha c}, \cdots$

$N_{PS} \sim L^{d_{\text{space}}}$

CAMPS-DMRG with ITensor library

- We developed a library of CAMPS-DMRG based on ITensor (Julia version)
- The ease of use of ITensor is retained!



[Fishman et al. (2022)]

✅ User-friendly input of the Hamiltonian

✅ Obtain the ground state as a CAMPS by DMRG

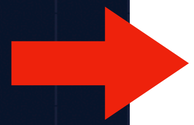
✅ Compute expectation values of any Pauli string operator

…to be published soon

Use CAMPS by a simple rewrite

DMRG with MPS & MPO

```
using ITensors, ITensorMPS

### generate spin-1/2 sites of size N
sites = siteinds( "S=1/2", N; conserve_qns=false )

### define the Hamiltonian
os = OpSum( )
for j = 1 : N-1
    os += -1.0, "Z", j, "Z", j+1
    os += -g, "X", j
end
os += -g, "X", N

### convert the Hamiltonian to MPO
H = MPO( os, sites )

### set the DMRG parameters
nsweeps = 20
maxdim = [ 4, 4, 8, 8, 16, 16, 32, 32 ]
cutoff = [ 1E-12 ]

### random initial state
psi0 = random_mps( sites; linkdims=maxdim[1] )

E, psi = dmrg( H, psi0; nsweeps, maxdim, cutoff )
```



DMRG with CAMPS

```
using ITensors, ITensorMPS, CAMPS_ITensor

### generate spin-1/2 sites of size N
sites = siteinds( "S=1/2", N; conserve_qns=false )

### define the Hamiltonian
os = OpSum( )
for j = 1 : N-1
    os += -1.0, "Z", j, "Z", j+1
    os += -g, "X", j
end
os += -g, "X", N

### convert the Hamiltonian to tableau form
H = PSH( os, sites )

### set the DMRG parameters
nsweeps = 20
maxdim = [ 4, 4, 8, 8, 16, 16, 32, 32 ]
cutoff = [ 1E-12 ]

### random initial state
psi0 = random_mps( sites; linkdims=maxdim[1] )

E, Cpsi, Hout = dmrg_camps( H, psi0; nsweeps, maxdim, cutoff )
```

1d CFT with open boundary

- EE increases with the system size: $S_{\mathrm{EE}} \sim \frac{c}{6} \log L + b$

  —> bond dimension: $D \sim L^{\frac{c}{6}}$

[Laflorencie et al (2006)]
[Cardy & Tonni (2016)]

In the following results…

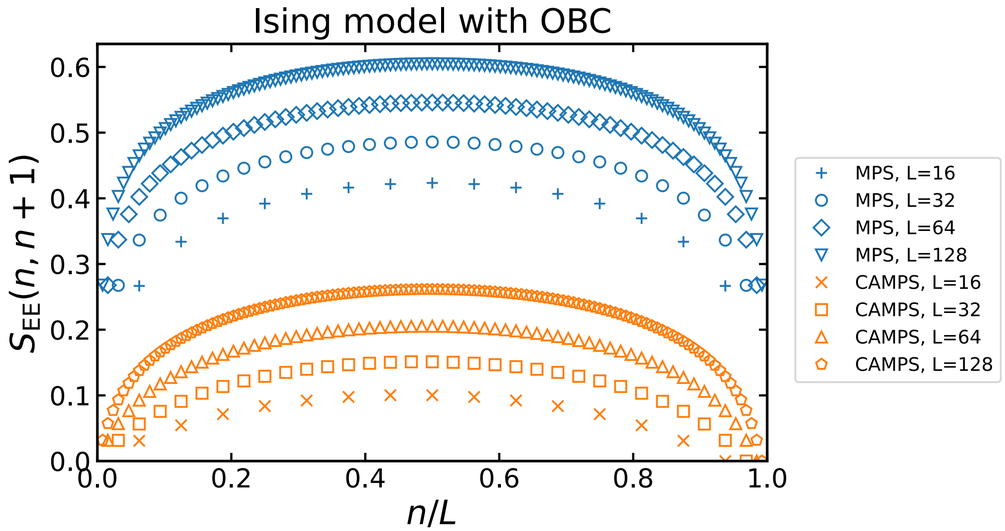
- SVD truncation cutoff is $\varepsilon = 10^{-12}$
- Sweep is continued until $\delta E/E < 10^{-10}$
- The same CPU (Intel Core Ultra 9 285K) is used

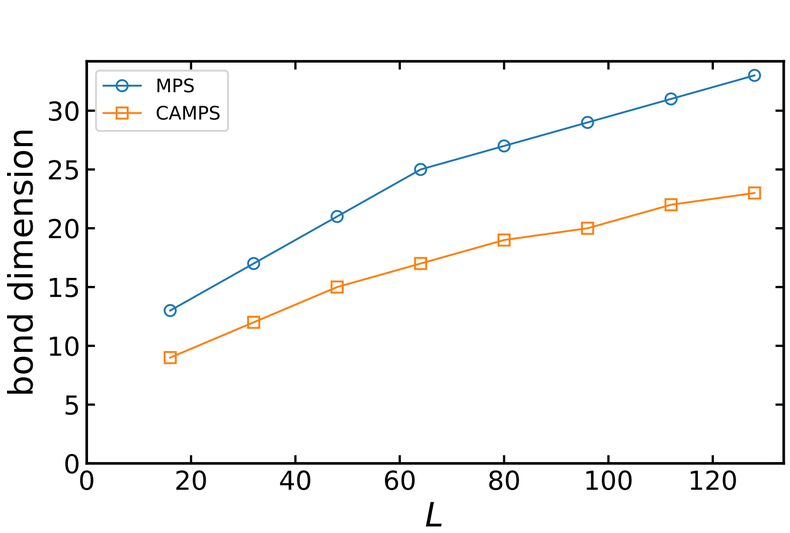
Critical Ising model (OBC)

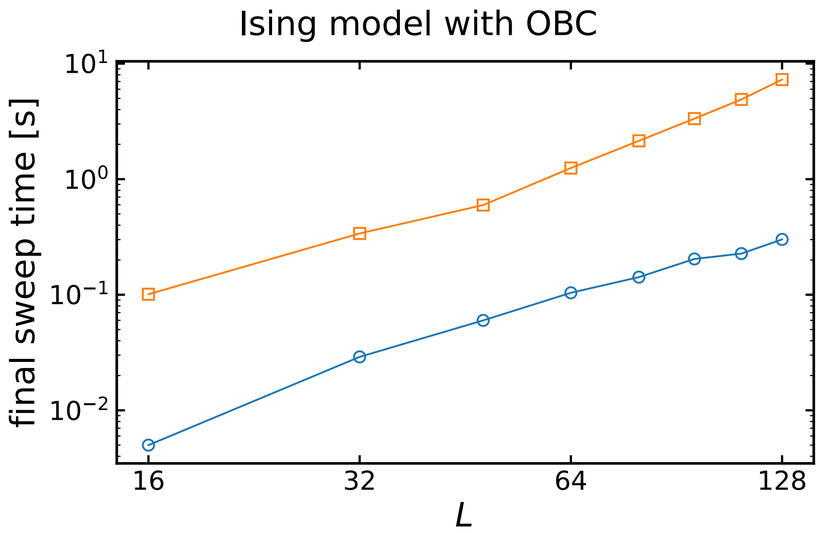
EE at a cut between sites n and $n+1$

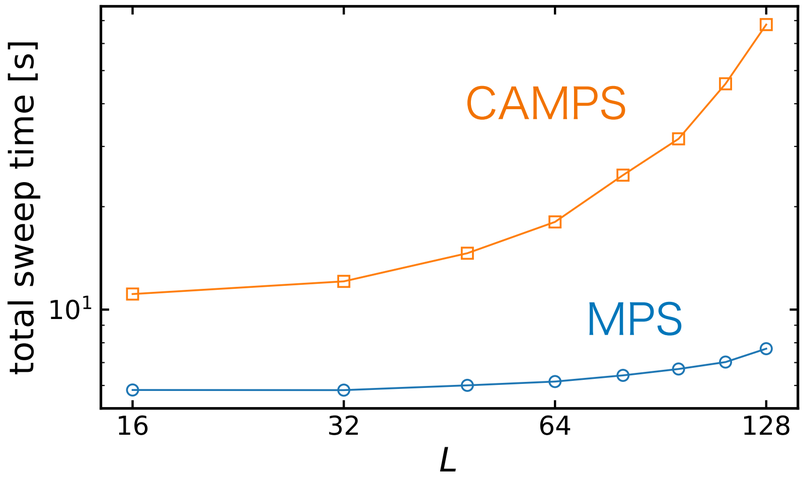
$$H = -\sum_{n=1}^{L-1} Z_n Z_{n+1} - g \sum_{n=1}^{L} X_n$$

- Critical point $g = 1$ ⋯ Ising CFT with $c = 1/2$
- The boundary entropy is eliminated









CAMPS has smaller bond dimensions, but the total cost is high.

Critical Ising model (OBC)

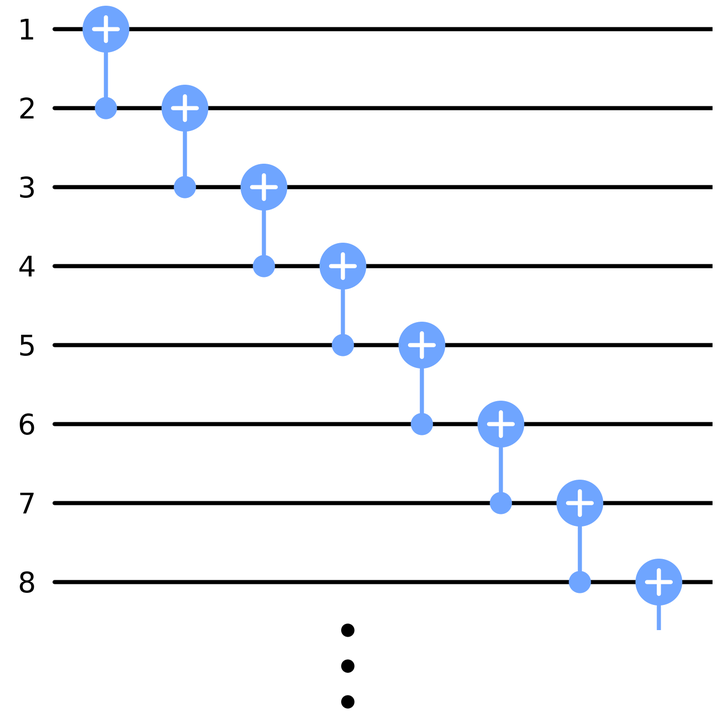
- The resulting Clifford circuit U transforms H as

$$Z_n Z_{n+1} \to Z_n, \quad X_n \to \begin{cases} X_1 & (n=1) \\ X_{n-1}X_n & (n>1) \end{cases},$$

$$H = -\sum_{n=1}^{L-1} Z_n Z_{n+1} - \sum_{n=1}^{L} X_n$$

$$\longrightarrow \quad H' = -\sum_{n=1}^{L-1} Z_n - \left(X_1 + \sum_{n=2}^{L} X_{n-1}X_n \right)$$

$$U = \mathrm{CNOT}_{L,L-1} \cdots \mathrm{CNOT}_{3,2}\,\mathrm{CNOT}_{2,1}$$

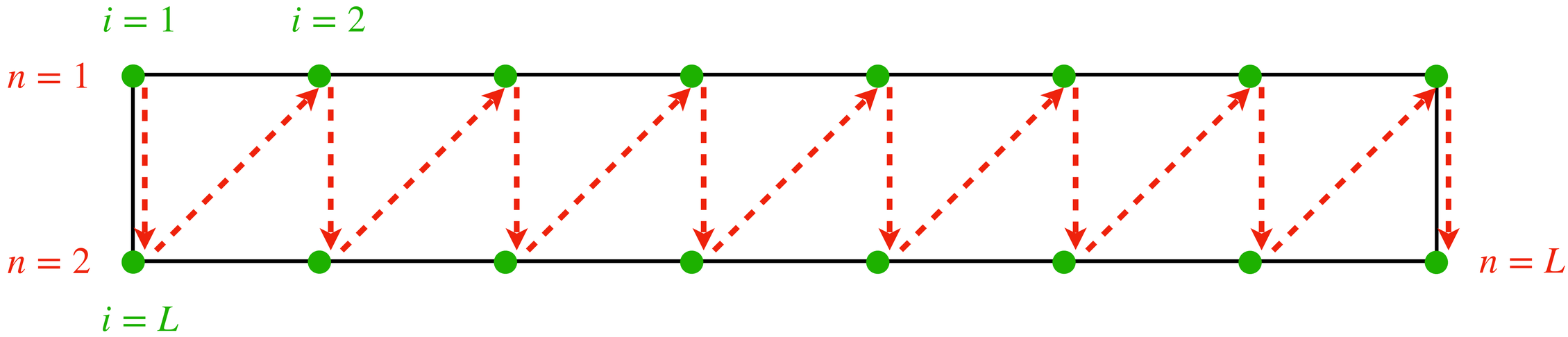


💡 unitary analog of Kramers-Wannier duality

Nontrivial duality transformation is automatically obtained!

1d CFT with periodic boundary

- Assign the periodic lattice sites i to the MPS indices n in zigzag order



- Local interactions with PBC become “nonlocal” on the MPS indices
  —> bond dimension increases rapidly

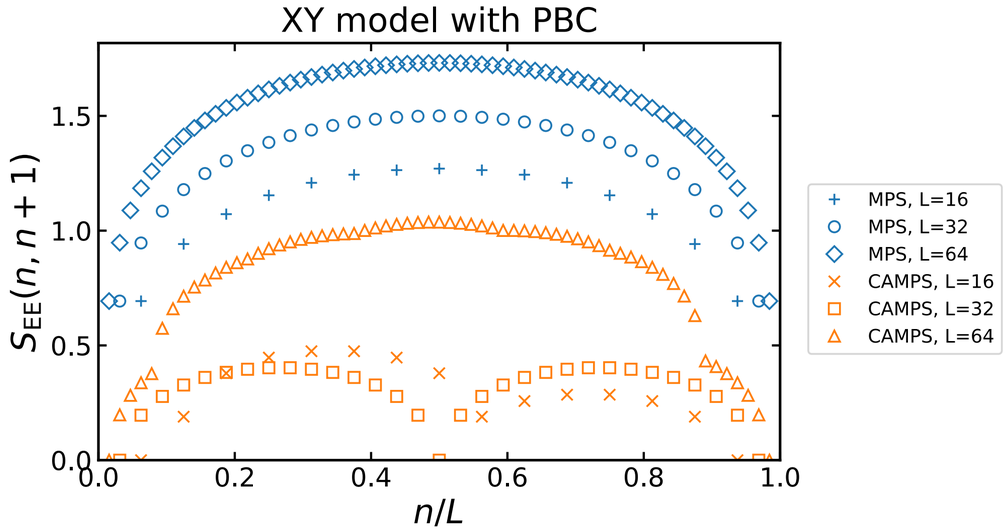
- Does the Clifford circuit reorganize the interactions to be “local” on the MPS?

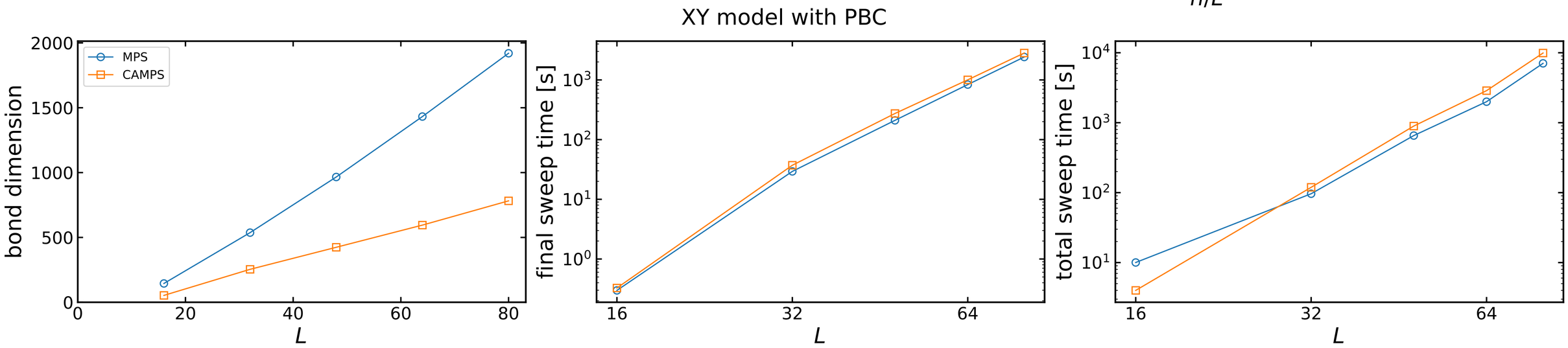
XY model with PBC

$$H = \sum_{n=1}^{L} \left(X_n X_{n+1} + Y_n Y_{n+1} \right) \qquad n = L+1 \sim 1$$

- compact boson CFT with $c = 1$

  <—> two copies of Ising CFT with $c = 1/2$



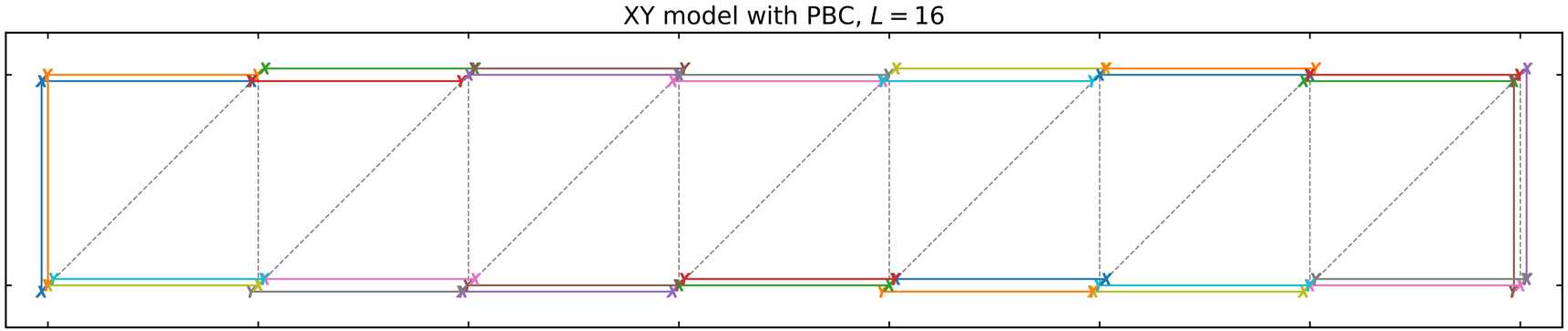


The total costs of MPS and CAMPS are comparable!

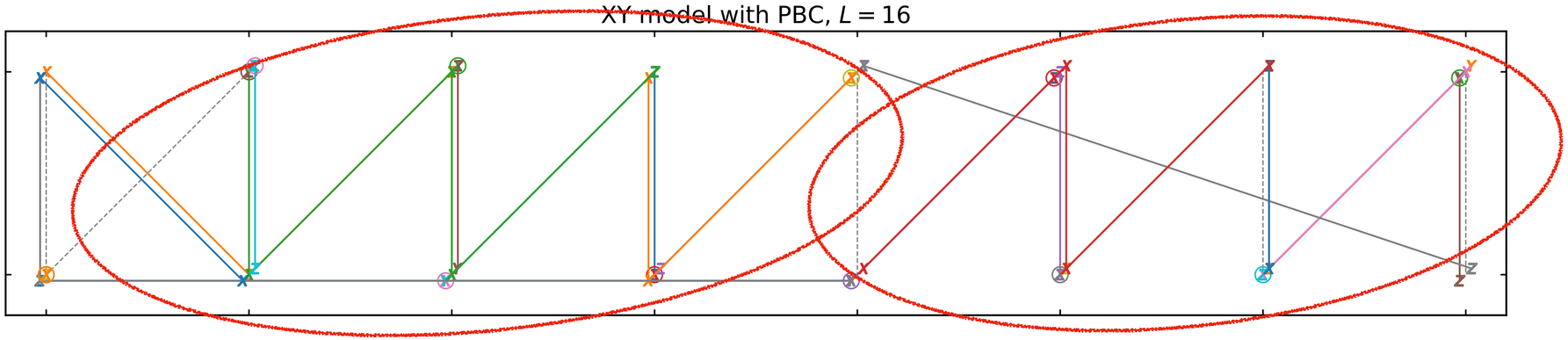
XY model with PBC

- Separate into two (or more) blocks of Ising-like interactions
- The two blocks are connected by a few long-range interactions

original interactions with PBC



transformed interactions



Gauge theory example

- 1-flavor Schwinger model in spin representation

$$H = \frac{J}{2}\sum_{n=1}^{L-2}\sum_{k=n+1}^{L-1}(L-k)\,Z_nZ_k + \frac{J}{2}\sum_{n=1}^{L-1}\left[\left(\frac{1}{2}+\frac{\theta}{\pi}\right)(L-n) + \frac{(-1)^L-(-1)^n}{4}\right]Z_n$$
$$+\frac{w}{2}\sum_{n=1}^{L-1}\left(X_nX_{n+1}+Y_nY_{n+1}\right) + \frac{m_{\mathrm{lat}}}{2}\sum_{n=1}^{L}(-1)^{n+1}Z_n$$

$$J = \frac{g^2a}{2}$$
$$w = \frac{1}{2a}$$
$$m_{\mathrm{lat}} = m - \frac{g^2a}{8}$$

- critical end point: $\theta = \pi$, $m \approx 0.33356$ in continuum limit $a \to 0$

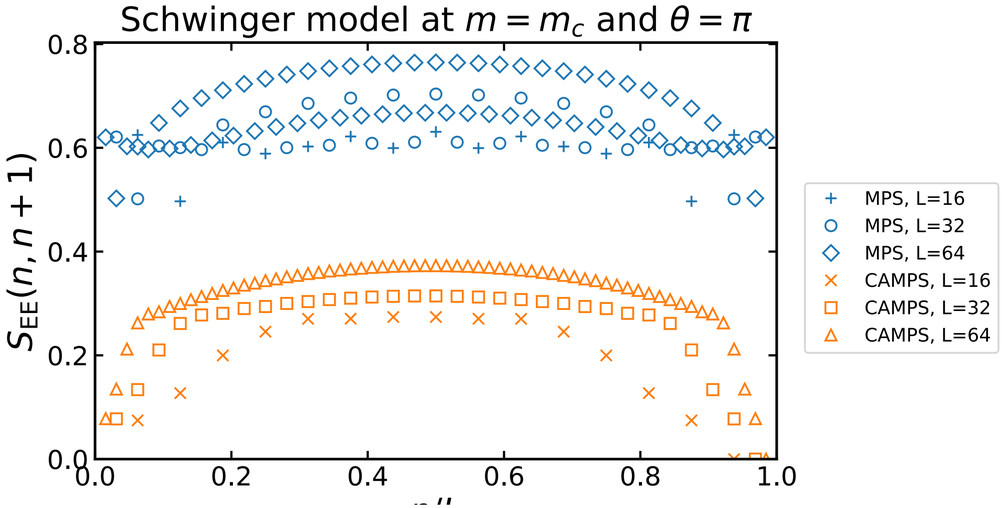
  —> Ising CFT with $c = 1/2$

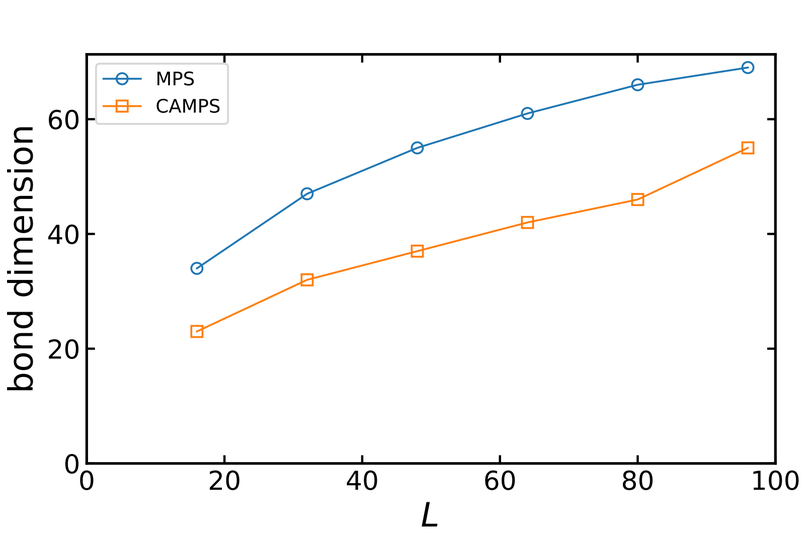
[Fujii et al. (2025)] [Cruz et al. (2025)]

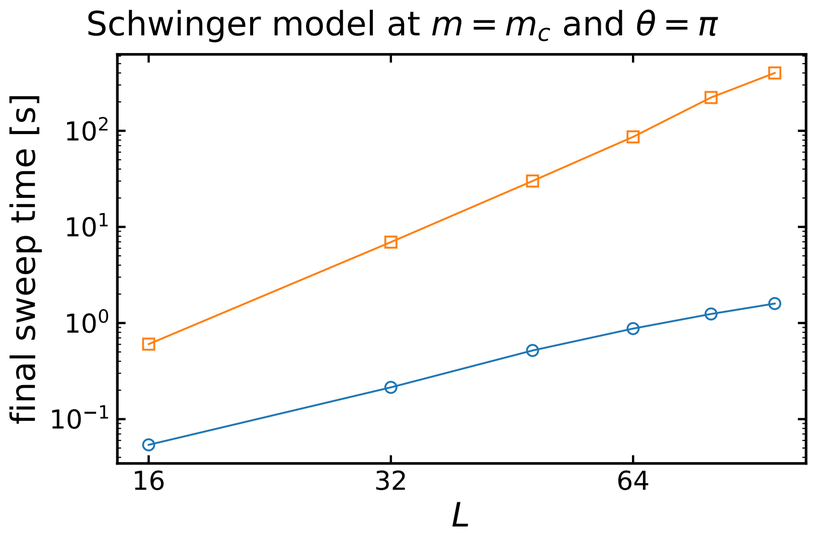
Schwinger model at the critical point

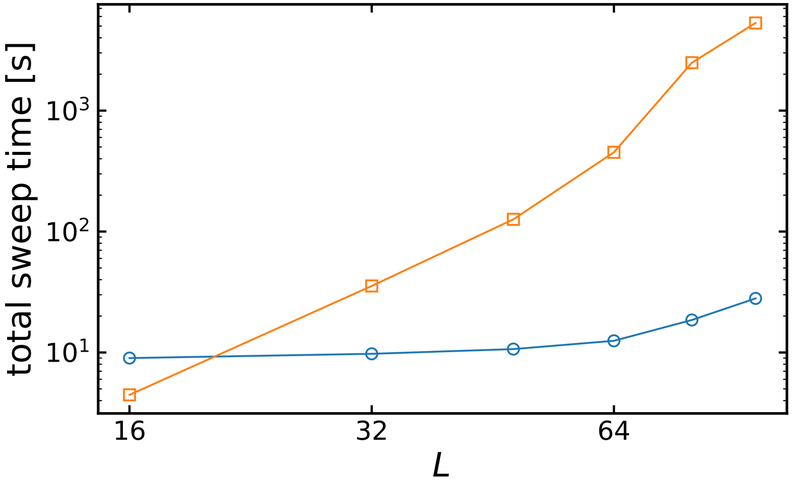
- $ga = 0.2,\ \ \theta = \pi,\ \ m = 0.33356$ (critical point)
- The sweep time grows as $O(L^3D^3)$

  since the Hamiltonian has $N_{PS} \sim O(L^2)$ terms









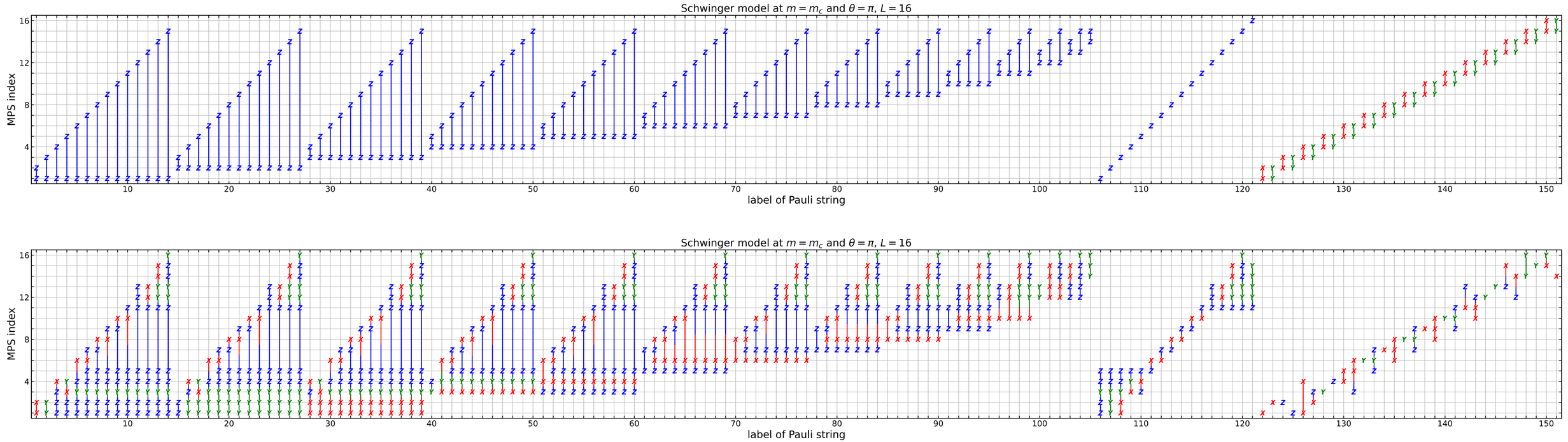
Integrating out the gauge field is not good for CAMPS.

Schwinger model at the critical point

- The kinetic and mass terms of the fermion are affected by the Clifford circuit
  —> larger contribution to entanglement than the gauge part

gauge part: $O(L^2)$ terms

fermion part: $O(L)$ terms



Summary and Prospect

- We developed a CAMPS-DMRG library based on ITensor.

- CAMPS reveals the non-stabilizerness (quantum magic) of the state and duality transformation of the model.

- Clifford circuits reduce the entanglement,
  while operating many Pauli strings requires additional numerical cost.

- Integrating out the gauge field yields many Pauli strings.
  —> We should use other formulations.

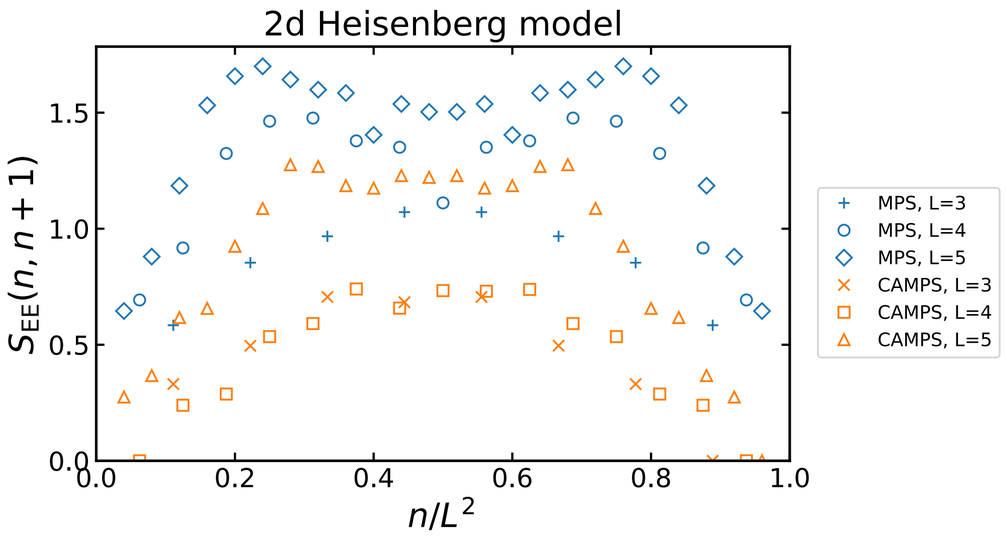
Summary and Prospect

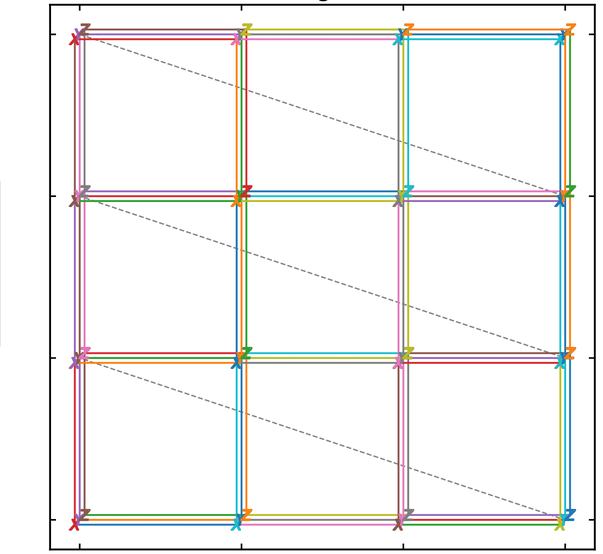
- We find that CAMPS is effective with the periodic boundary condition.
  —> There are a lot of potential applications beyond 1d open chains.

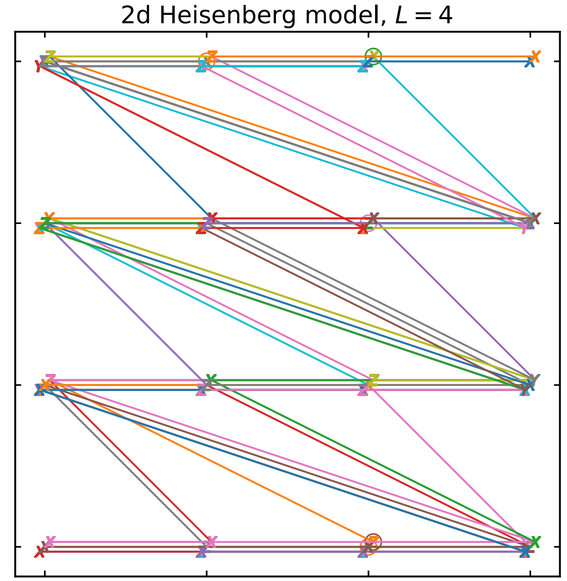
- CAMPS has better parallelizability than MPS,
  especially for operations on the many Pauli strings.
  —> The factor N_{PS} in the cost $O(N_{PS}LD^3)$ will be milder

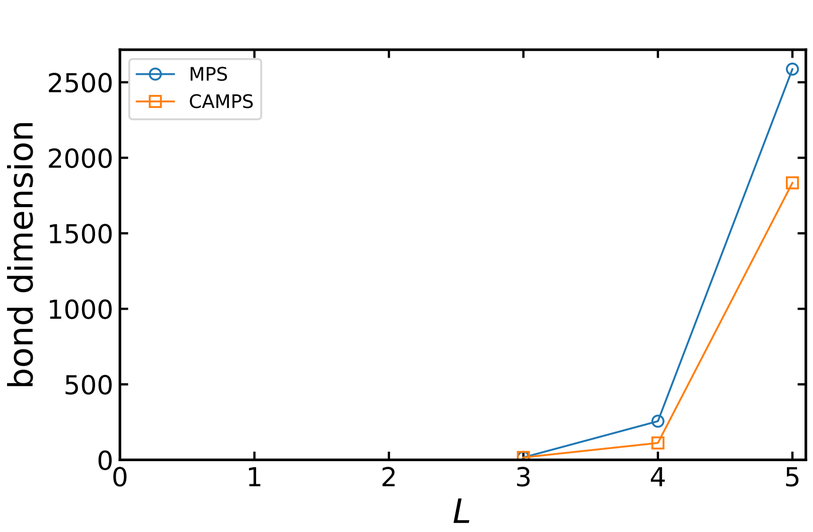
2d Heisenberg model

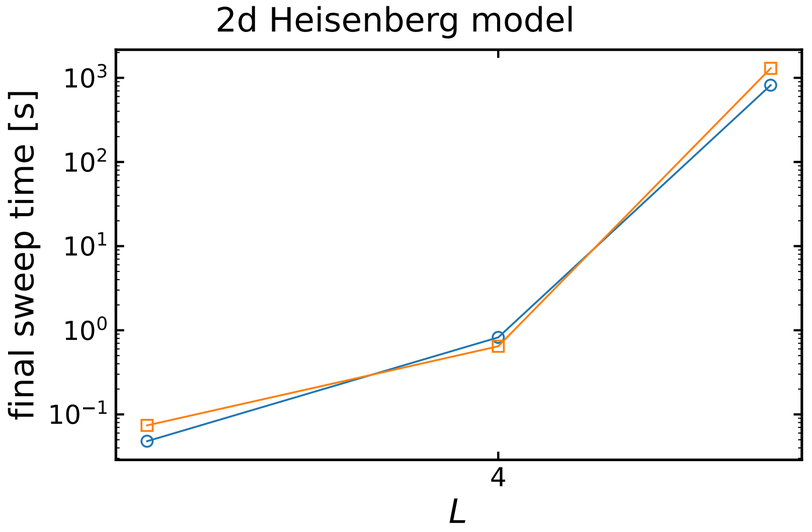
preliminary

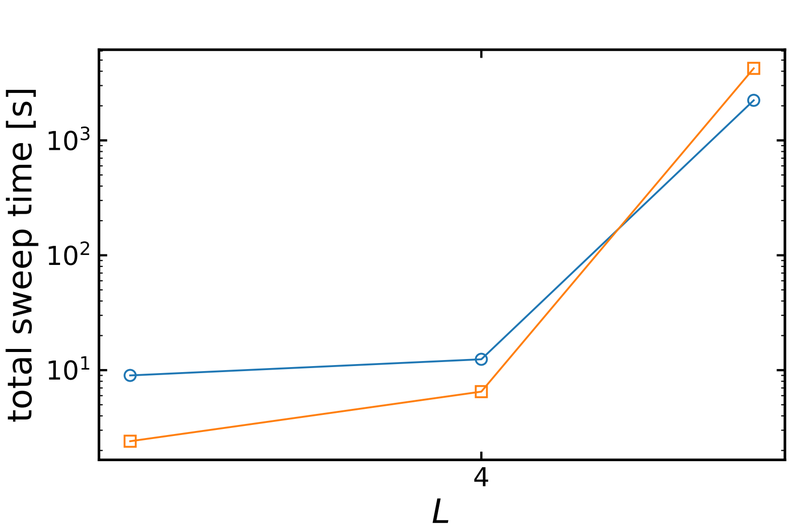








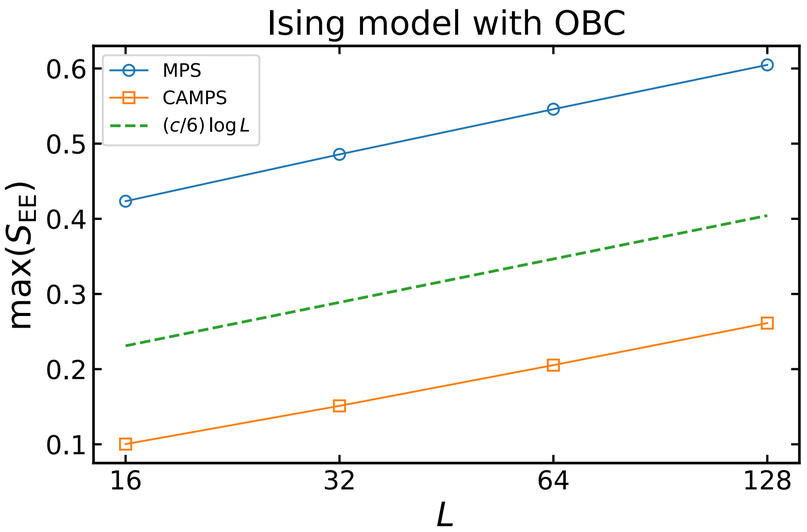


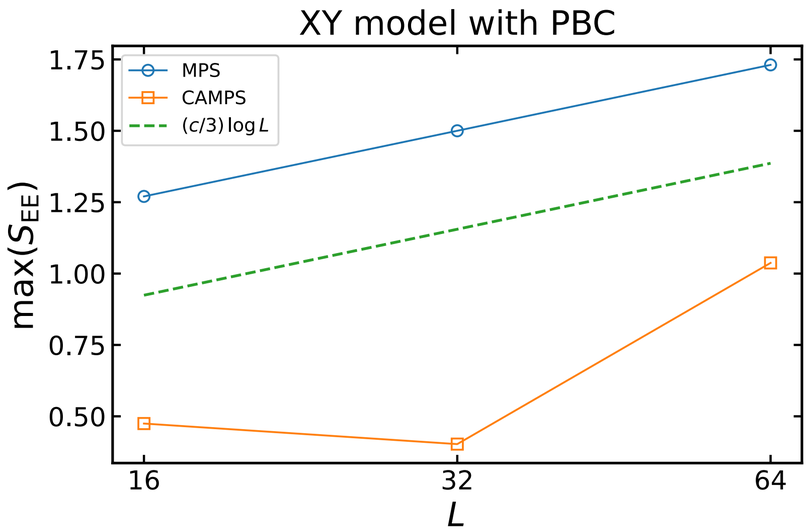


Thank you for listening.

Scaling of EE

- OBC: $S_{\mathrm{EE}} \sim \frac{c}{6} \log L + b$
- PBC: $S_{\mathrm{EE}} \sim \frac{c}{3} \log L$





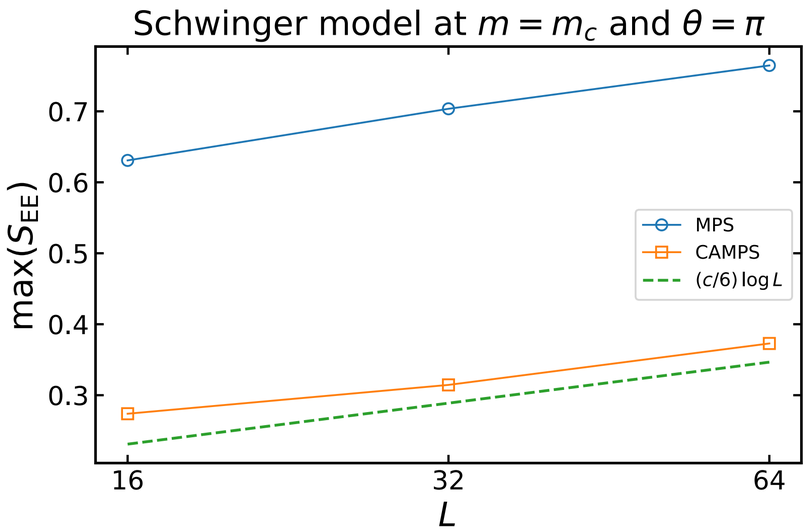


Tableau algorithm

[Aaronson & Gottesman (2004)]

- Pauli matrix: $p = X^x Z^z$ <—> two binary numbers: $x,\ z \in \{0,1\}$

  $I \leftrightarrow (0,0)$ $X \leftrightarrow (1,0)$ $Y \leftrightarrow (1,1)$ $Z \leftrightarrow (0,1)$ ($-i$ factor of Y is implicit)

- N_{PS} Pauli strings of length L with the sign factor: $P_\alpha = (-1)^{s_\alpha} \bigotimes_{n=1}^{L} X^{x_{\alpha,n}} Z^{z_{\alpha,n}}$

  <—> $(N_{PS} \times 2L) + N_{PS}$ binary numbers: $x_{\alpha,n},\ z_{\alpha,n}\ s_\alpha \in \{0,1\}$

  <—> “tableau” representation

$$T = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,L} & z_{1,1} & z_{1,2} & \cdots & z_{1,L} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,L} & z_{2,1} & z_{2,2} & \cdots & z_{2,L} \\ \vdots & & & & \vdots & & & \\ x_{N_{PS},1} & x_{N_{PS},2} & \cdots & x_{N_{PS},L} & z_{N_{PS},1} & z_{N_{PS},2} & \cdots & z_{N_{PS},L} \end{pmatrix} \qquad r = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{N_{PS}} \end{pmatrix}$$

Tableau algorithm

[Aaronson & Gottesman (2004)]

- L-site Clifford circuit: U <—> $2L \times 2L$ symplectic binary matrix: S $\quad S_{ab} \in \{0,1\}$

$$S \Lambda S^{\mathrm{T}} = \Lambda \quad \text{for} \quad \Lambda = \begin{pmatrix} 0_{L\times L} & I_{L\times L} \\ I_{L\times L} & 0_{L\times L} \end{pmatrix}$$ to conserve commutation relation

[Koenig & Smolin (2014)]

- 720 independent 2-site Clifford circuits <—> 720 4×4 symplectic matrices

- transformation of the Pauli strings by Clifford circuit: $UP_\alpha U^\dagger = P'_\alpha$

  <—> matrix multiplication of the tableau $T_{N_{PS}\times 4}$ and $S_{4\times 4}$

$$T' = TS = \begin{pmatrix} x_{1,1} & x_{1,2} & z_{1,1} & z_{1,2} \\ x_{2,1} & x_{2,2} & z_{2,1} & z_{2,2} \\ \vdots & & \vdots & \\ x_{N_{PS},1} & x_{N_{PS},2} & z_{N_{PS},1} & z_{N_{PS},2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

binary addition := XOR

XOR	0	1
0	0	1
1	1	0

Tableau algorithm

[Aaronson & Gottesman (2004)]

bases (generators)
of the 2-site Pauli string
$g \in \{X \otimes I,\ Z \otimes I,\ I \otimes X,\ I \otimes Z\}$

$=$

identity tableau

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\cdots U_2 U_1 g U_1^\dagger U_2^\dagger \cdots$ ⇕ Clifford gates

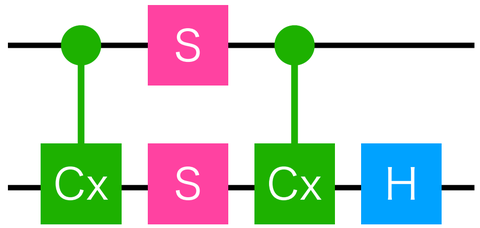
$TR_1R_2\cdots$ ⇕ diagonalization by elementary column operations

transformed bases
$g' \in \{X \otimes X,\ Z \otimes Y,\ Z \otimes I,\ I \otimes X\}$

$=$

nontrivial tableau

$$T' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



Reduction of Clifford group

Only 20 of 720 Clifford circuits are enough to reduce the EE [E. Fux et al. (2025)]

- 1-site Clifford gate = local permutation of $\{X, Y, Z\}$
  —> does not change the EE

$|\mathbf{C}_2/\mathbf{P}_2| = 720 \quad |\mathbf{C}_1/\mathbf{P}_1| = 6$

※$\mathbf{C}_L$ is divided by the Pauli group $\mathbf{P}_L$ in the tableau algorithm

- The 2-site Clifford group $\mathbf{C}_2$ is classified into (right) cosets by the 1-site Clifford subgroup $\mathbf{C}_1 \otimes \mathbf{C}_1$

$$\tilde{\mathbf{C}}_2 = \left\{ (\mathbf{C}_1 \otimes \mathbf{C}_1) U \,\middle|\, U \in \mathbf{C}_2 \right\}$$

$$|\tilde{\mathbf{C}}_2| = \frac{|\mathbf{C}_2|}{|\mathbf{C}_1 \otimes \mathbf{C}_1|} = \frac{720 \times 4^2}{6^2 \times 4^2} = 20$$