The pion mass splitting from lattice QCD

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Reporting on 2605.29902: Alessandro de Santis, Dominik Erb, HM

Pion mass splitting

$$\text{PDG}: \qquad M_{\pi^\pm} - M_{\pi^0} = 4.5936(5)\ \text{MeV}.$$

In the chiral limit ($M_{\pi^0} = 0$), the pion mass splitting can be expressed as$^{(\star)}$

$$M_{\pi^\pm}^2 - M_{\pi^0}^2 = -\frac{3\alpha}{4\pi F_\pi^2} \int_0^\infty ds\, s\, \ln(s/s_0)\, (\rho_V(s) - \rho_A(s))$$

with $\rho_{V,A}(s)$ the spectral functions of the (isovector) vector and axial-vector currents.

Chiral perturbation theory with dynamical photons:
developed by R. Urech, Nucl.Phys. B433 (1995) 234.

- of purely electromagnetic origin, up to O($(m_u - m_d)^2$) effects; no counterterm needed in calculation at O(α);
- here: benchmark for testing the methods we apply to computing electromagnetic corrections to hadronic vacuum polarisation.

(*) T. Das et al., PRL18, 759 (1967). By the 2nd Weinberg sum rule, the RHS is independent of s_0. See recent extension by Bijnens, Hermansson-Truedsson, Rodriguez-Sanchez 2602.05672.

Dynamical photons in lattice QCD

Two main aspects of how we handle 'dynamical' photons in lattice QCD:

I. use coordinate-space methods

$\star$ motivation: avoid power-law corrections in the box size L;
keep the observable *local*, not spread over the entire volume.

II. where needed, use a Pauli-Villars-type UV cutoff $\Lambda \ll \pi/a$ on the photon virtuality.

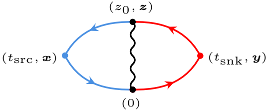
$\star$ motivation: simplifies renormalisation;
bare e.m. correction is independent of the lattice action;
after renormalization, unlikely that high-virtuality photons are quantitatively important in a low-energy observable such as $a_\mu^{\rm hvp}$.

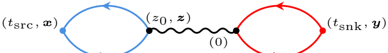
(Doubly) Pauli-Villars regulated propagator (finite at $x = 0$):

$$\mathcal{G}_\Lambda(x) = G_0(x) - 2G_{\frac{\Lambda}{\sqrt{2}}}(x) + G_\Lambda(x), \qquad G_m(x) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{p^2 + m^2}$$

Quark-level diagrams

To order $(\alpha, m_u - m_d)$, the pion mass splitting is a purely electromagnetic effect. In lattice QCD, two quark-level diagrams contribute:





Result obtained with infinite volume, continuum photon propagator:

$$M_{\pi^\pm} - M_{\pi^0} = 4.62(16)_{\rm stat}(12)_{\rm syst}\,\text{MeV} \qquad (\text{cf. } [M_{\pi^\pm} - M_{\pi^0}]_{\rm PDG} = 4.5936(5)\ \text{MeV})$$

NB. the contribution of elastic $(\gamma^*\pi)$ intermediate states is corrected for finite-volume effects.

Alessandro De Santis, Erb, HM, 2605.29902;

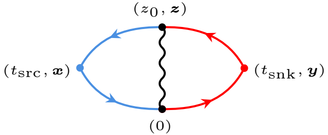
Previous calculations: Feng, Jin, Riberdy 2108.05311 (PRL); Frezzotti et al 2202.11970 (PRD); ...

Table of CLS ensembles used

- O(a) improved Wilson fermions with treelevel O(a^2) improved gauge action
- local current at the origin, conserved current at z (integrated over)

ID	$L^3 \times T$	a [fm]	β	L [fm]	M_π [MeV]	$M_\pi L$	M_{VMD} [MeV]	N_{config}
H102	$32^3 \times 96$	0.085	3.40	2.7	358	4.9	856	1000
H105	$32^3 \times 96$	0.085	3.40	2.7	283	3.9	832	450
N101	$48^3 \times 128$	0.085	3.40	4.1	282	5.8	811	400
C101	$48^3 \times 96$	0.085	3.40	4.1	222	4.6	784	2000
B450	$32^3 \times 64$	0.075	3.46	2.4	422	5.1	902	200
S400	$32^3 \times 128$	0.075	3.46	2.4	355	4.3	843	2000
N452	$48^3 \times 128$	0.075	3.46	3.6	356	3.6	858	1000
N451	$48^3 \times 128$	0.075	3.46	3.6	291	5.3	832	1900
D450	$64^3 \times 128$	0.075	3.46	4.8	219	5.3	794	450
D452	$64^3 \times 128$	0.075	3.46	4.8	156	3.8	770	700
N203	$48^3 \times 128$	0.064	3.55	3.0	349	5.4	861	1500
N200	$48^3 \times 128$	0.064	3.55	3.0	286	4.4	831	800
D251	$64^3 \times 128$	0.064	3.55	4.1	286	5.9	830	500
D200	$64^3 \times 128$	0.064	3.55	4.1	202	4.2	767	600
N302	$48^3 \times 128$	0.049	3.70	2.4	350	4.2	896	2000
J303	$64^3 \times 192$	0.049	3.70	3.1	260	4.1	834	600

Computing the connected diagram



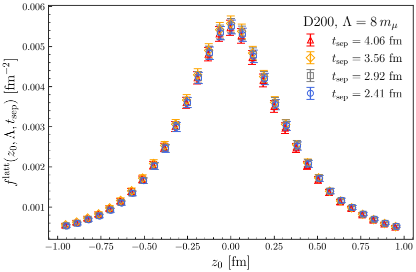
$$\Delta M_\pi(\Lambda) = \frac{e^2}{2}(q_u - q_d)^2 \lim_{t_{\rm sep}\to\infty} \sum_{z_0=-\infty}^{+\infty} f^{\rm latt}(z_0, \Lambda, t_{\rm sep}),$$

$$f^{\rm latt}(z_0, \Lambda, t_{\rm sep}) = \frac{\sum_{\vec{x},\vec{y},\vec{z}} \delta_{\mu\nu} \mathcal{G}_\Lambda(z)\, C^{\rm conn}_{\mu\nu}(z_0, t_{\rm sep}, \vec{x}, \vec{y}, z)}{\sum_{\vec{x},\vec{y}} C^{\rm 2pt}(t_{\rm sep}, \vec{x}, \vec{y})}$$

$$\begin{aligned} C^{\rm conn}_{\mu\nu}(z_0, t_{\rm sep}, \vec{x}, \vec{y}, \vec{z}) &= -\langle {\rm Tr}\{\gamma_\mu S(0,x)\gamma_5 S(x,z)\gamma_\nu S(z,y)\gamma_5 S(y,0)\}\rangle \\ C^{\rm 2pt}(x_0 - y_0, \vec{x}, \vec{y}) &= \langle {\rm Tr}\{\gamma_5 S(x,y)\gamma_5 S(y,x)\}\rangle \end{aligned}$$

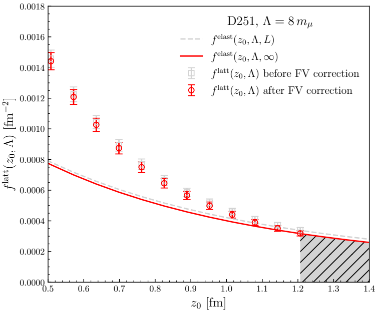
- on a given gauge configuration, a point source was placed at the origin, chosen randomly within the middle timeslice (open b.c. in time)
- sequential propagators are computed over the source and sink, which are in timeslices $\pm t_{\rm sep}/2$
- the two sequential propagators are 'tied' together at the vectorial vertex z.

Aspects of the pion mass splitting calculation



Integrand on ensemble D200

$64^3\times128,\ a = 0.064\,\text{fm},\ M_\pi = 202\,\text{MeV}$



Tail of integrand on ensemble D251

$64^3\times128,\ a = 0.064\,\text{fm},\ M_\pi = 286\,\text{MeV}$

Spectral representation: the elastic contribution $\pi(\vec{0}) \xrightarrow{j_\mu} \pi(\vec{p}) \xrightarrow{j_\nu} \pi(\vec{0})$

$$f^{\rm elast}(z_0, \Lambda, L) = \frac{1}{2L^3} \int d^3z\, \mathcal{G}_\Lambda(z) \sum_{\vec{p}} e^{(M_\pi - E_{\vec{p}})|z_0| + i\vec{p}\cdot\vec{z}}\, \frac{E_{\vec{p}} + M_\pi}{E_{\vec{p}}} F(-(p'-p)^2)^2.$$

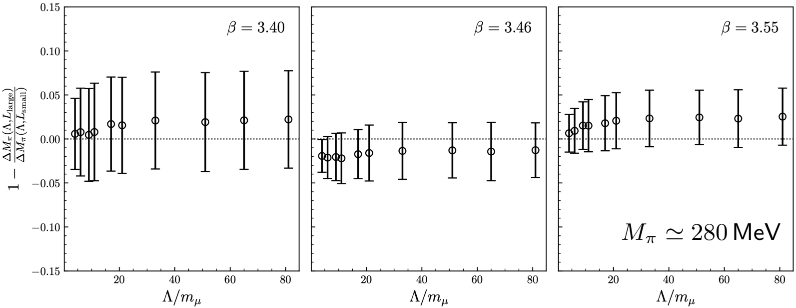
In practice, we have used a VMD form factor, $F(q^2) = M_{\rm VMD}^2/(Q^2 + M_{\rm VMD}^2)$, which already describes the lattice data at $|z_0| \gtrsim 1.2$ fm well.

Correcting the elastic part of the integrand for finite-volume effects

Improved representation of the z_0-integrand:

$$\Delta M_\pi(\Lambda) = \frac{e^2}{2}\Big[a \sum_{z_0=-t_c}^{t_c} f^{\rm corr}(z_0,\Lambda) + 2\int_{t_c}^{\infty} dz_0\, f^{\rm elast}(z_0,\Lambda,\infty)\Big],$$

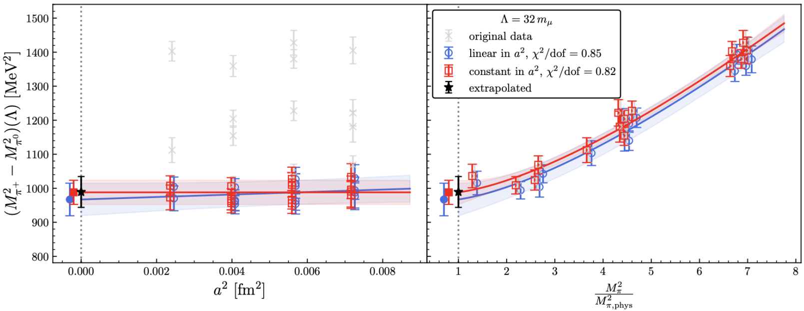
$$f^{\rm corr}(z_0,\Lambda) = \lim_{t_{\rm sep}\to\infty} f^{\rm latt}(z_0,\Lambda,t_{\rm sep}) + \Big[f^{\rm elast}(z_0,\Lambda,\infty) - f^{\rm elast}(z_0,\Lambda,L)\Big].$$



Chiral and continuum extrapolation (at fixed Λ)

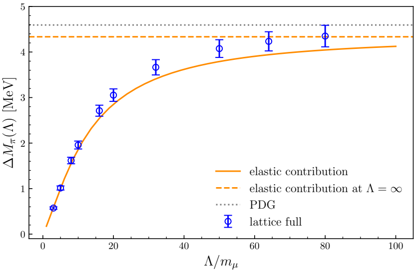
Our most general ansatz:

$$[M^2_{\pi^+} - M^2_{\pi^0}](\Lambda, a, M_\pi) = \frac{e^2}{2}\left[c_0 + c_a a^2 + c_\pi^{(1)} M_\pi^2 + c_\pi^{(2)} M_\pi^2 \log\frac{M_\pi^2}{\mu^2} + c_{a\pi}(aM_\pi)^2\right].$$

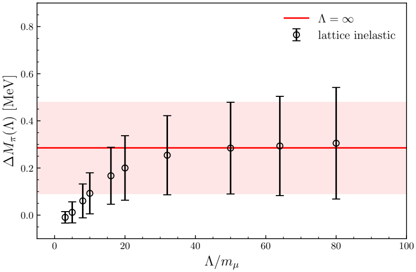


Prediction for $c_\pi^{(2)}$ at $\Lambda = \infty$: see R. Urech, Nucl.Phys. B433 (1995)234, Eq. (95)

Λ dependence of the pion mass splitting (post chiral/continuum extrapolation)



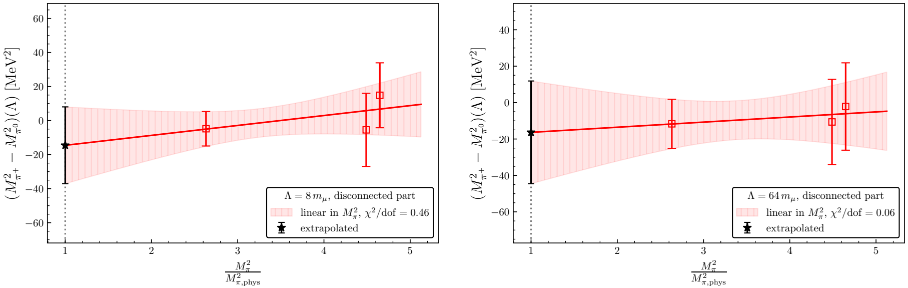
Full splitting



Inelastic part only

- Elastic part of the forward Compton amplitude accounts for most of the Λ dependence (computable knowing the pion form factor);
- its contribution remains UV-finite for any e.m. mass shift (δM_K, δM_p,...);
- to expedite the approach to $\Lambda = \infty$, subtract from the lattice data the elastic contribution computed at finite Λ, and add it back at $\Lambda = \infty$;
- for the inelastic contribution, the $\Lambda = \infty$ limit is reached around $\Lambda \simeq 3\,$GeV.

Disconnected diagram



- the disconnected diagram vanishes in the chiral limit, i.e. it is O$(\alpha(m_u + m_d))$ (de Divitiis et al. 1303.4896 (PRD))
- irrespective of $(m_u + m_d)$, it does not receive any elastic contribution (2605.29902).
- our lattice calculation yields $\delta M_\pi = -0.06(10)$ MeV after a linear extrapolation in M_π^2, not enforcing the value zero in the chiral limit.

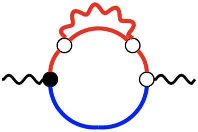
Computational aspects: factorizing sums

Inserting $a^4 \sum_y a^4 \sum_x G_m(x-y) j_\mu(x) j_\nu(y)$ naively leads to a volume^2 sum.

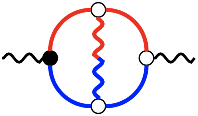
The sums over x and y can be disentangled using integral representations of $G_m(x-y)$, for instance:

$$G_m(x-y) = 4 \int d^4w \, G_m^{5d}(x-w) G_m^{5d}(w-y)$$

(5d propagators are elementary functions: $G_m^{5d}(x) = \frac{m|x|+1}{8\pi^2|x|^3} \, e^{-m|x|}$)



(a) (4)a



(b) (4)b

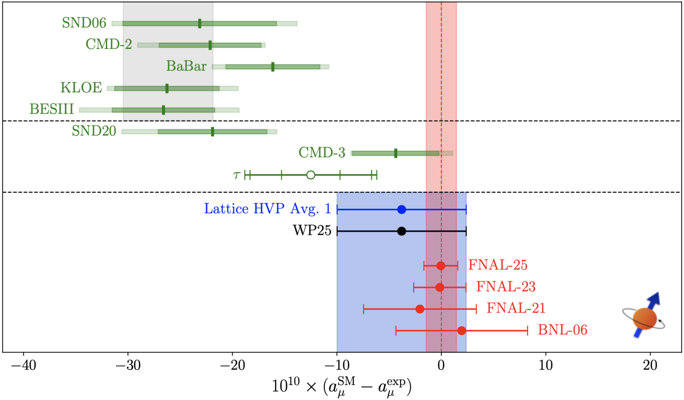
We found this method to be advantageous for diagram (4)b, but not for (4)a. See talk by Dominik Erb (Wed).

2603.13086 Erb, HM, Ottnad (JHEP).

Conclusion

- we have computed the pion mass splitting with about 4% uncertainty;
- testing ground for the photon propagator, PV-regulated at scale Λ;
- the inelastic contribution saturates within uncertainties at $\Lambda \approx$3 GeV;
- this makes us optimistic that reaching sufficiently large Λ will not limit the precision of our calculations of the e.m. corrections to HVP.

Status of a_μ [White Paper of Muon g-2 Theory Initiative 2505.21476]



- Experimental precision: $\delta_{\mathrm{exp}} a_\mu = 14.5 \times 10^{-11}$.
- The experimental effort was recently awarded the 2026 Breakthrough Prize in Fundamental Physics.
- Current bottleneck of the SM prediction: hadronic vacuum polarization!
- uncertainties: $\delta_{\mathrm{LQCD}} a_\mu^{\mathrm{hvp}} = 61 \times 10^{-11}$, $\delta_{\mathrm{IB}} a_\mu^{\mathrm{hvp}} = 34 \times 10^{-11}$.

Elastic contribution to forward Compton amplitude on the pion

$$\begin{aligned}
\delta[M^2_{\pi^+}]^{\text{elast}}(\Lambda) \quad &= \quad \frac{ie^2}{2}\int \frac{d^4k}{(2\pi)^4}\, g^{\mu\nu}T^{\text{elast}}_{\mu\nu}(k)\, \left[\frac{1}{k^2+i\epsilon} - \frac{2}{k^2 - M^2_\Lambda/2 + i\epsilon} + \frac{1}{k^2 - M^2_\Lambda + i\epsilon}\right] \\
&= \quad \frac{\alpha}{8\pi}\int_0^\infty ds\; F_\pi(-s)^2 \left[4W(s) + \frac{s}{M^2_\pi}(W(s)-1)\right]\left[\frac{1}{s} - \frac{2}{s+M^2_\Lambda/2} + \frac{1}{s+M^2_\Lambda}\right].
\end{aligned}$$

with $W(s) = \sqrt{1+4M^2_\pi/s}$.

D. Stamen et al 2202.11106 (EPJC).

The role of isospin symmetry

Renormalized variables: $(\Delta M_K;\ \bar{M}_\pi,\ \bar{M}_K,\ \bar{M}_B)$ and $\alpha \simeq 1/137$
($\Delta M_K = M_{K+} - M_{K0}$ and $\bar{M} =$ average over an isospin multiplet).

Let $j^3_\mu = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)$ be the isospin 1
and $j^8_\mu = \frac{1}{6}(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d - 2\bar{s}\gamma_\mu s)$ the isospin 0 part of the e.m. current:

$$\overline{\Pi} = \left(\overline{\Pi}^{33} + \overline{\Pi}^{88}\right) + 2\underbrace{\overline{\Pi}^{38}}_{\text{isospin-violating}}\ .$$

$$\overline{\Pi}^{38}(Q^2) = \lim_{\Lambda\to\infty}\left[\overline{\Pi}^{38,\gamma}(Q^2,\Lambda) - \frac{\partial\overline{\Pi}^{38}(Q^2)}{\partial\Delta M_K}\cdot\Delta M_K^\gamma(\Lambda)\right] + \frac{\partial\overline{\Pi}^{38}(Q^2)}{\partial\Delta M_K}\cdot\Delta M_K^{\text{phys}}\,.$$

For $a =$ either 3 or 8:

$$\begin{aligned}\delta\overline{\Pi}^{aa}(Q^2) \quad &= \quad \lim_{\Lambda\to\infty}\left[\overline{\Pi}^{aa,\gamma}(Q^2,\Lambda) - \frac{\partial\overline{\Pi}^{aa}(Q^2)}{\partial\bar{M}_K}\cdot\bar{M}_K^\gamma(\Lambda)\right.\\ &\qquad\left. - \frac{\partial\overline{\Pi}^{aa}(Q^2)}{\partial\bar{M}_\pi}\cdot\bar{M}_\pi^\gamma(\Lambda) - \frac{\partial\overline{\Pi}^{aa}(Q^2)}{\partial\bar{M}_B}\cdot\bar{M}_B^\gamma(\Lambda)\right].\end{aligned}$$