Computing the Gluon Momentum Fraction of Nucleons on the Lattice through the Gradient Flow

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HadStruc

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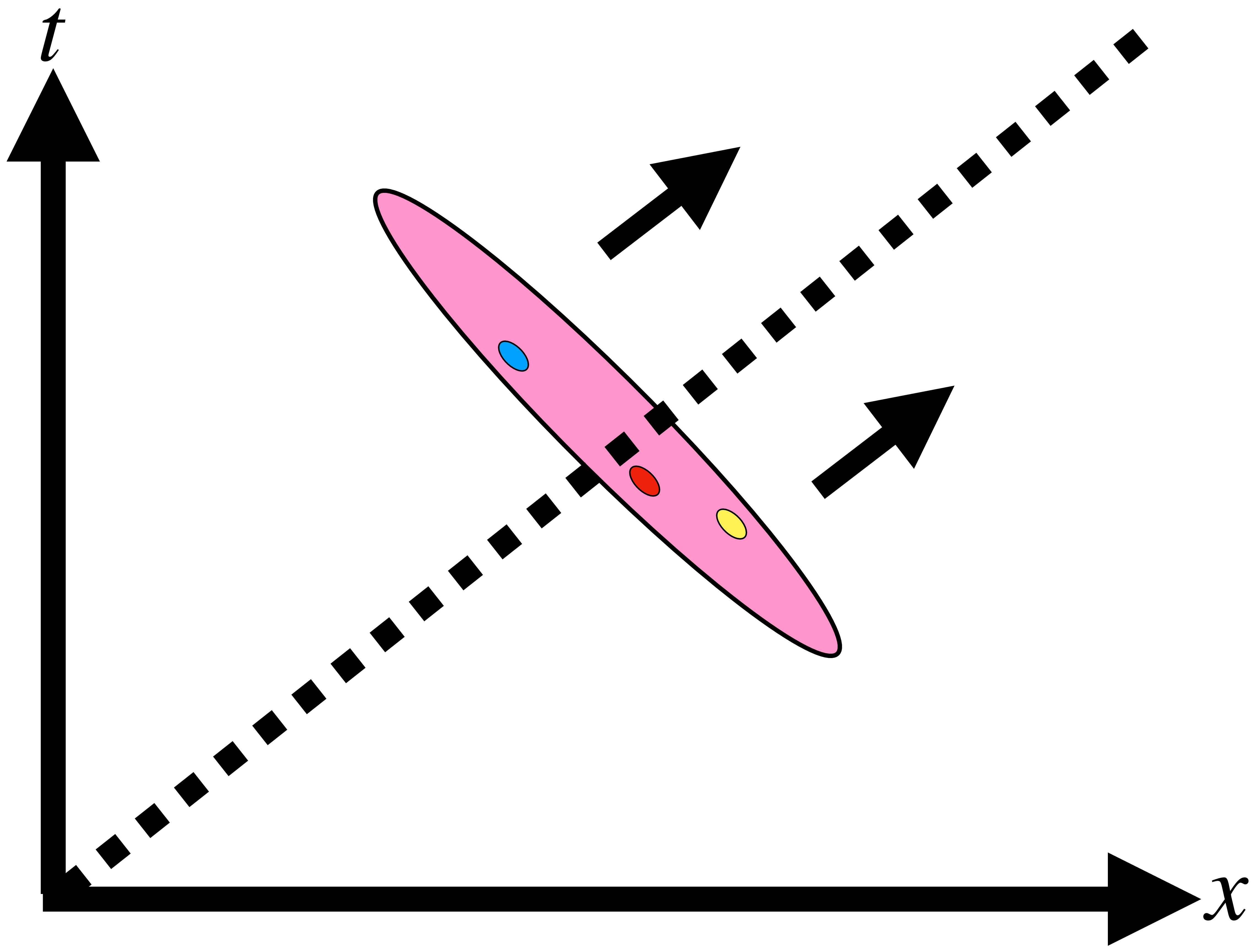
Hadronic Structure



$$M = \langle T^{00} \rangle = \langle \overline{T}^{00} \rangle + \langle \hat{T}^{00} \rangle$$

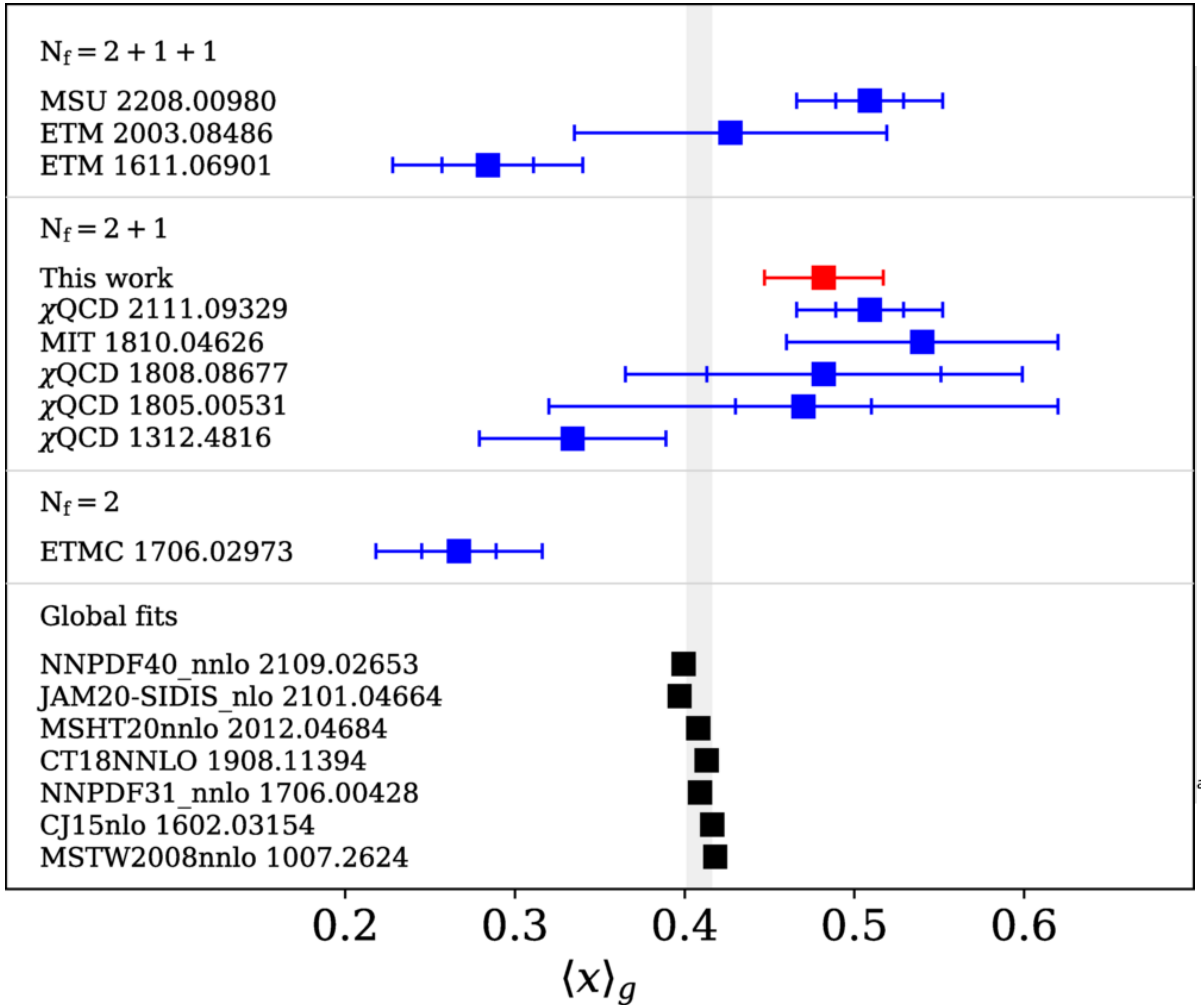
$$J = J_q + J_g = \frac{1}{2}\Delta\Sigma + L_q + \Delta G + L_g$$

$$1 = \sum_i \langle x \rangle_i$$

$$\langle x \rangle_{\mathrm{p}} = \int_0^1 dx\; x\, f_{\mathrm{p}}(x)$$

$$f_{q/H}(x,\mu) = \frac{1}{4\pi}\int dz^- \, e^{-ixP^+z^-} \langle H(P)\,|\,\bar{\psi}(z^-)\,\gamma^+\,W[z^-,0]\,\psi(0)\,|\,H(P)\rangle\Big|_{z^+=0,\; z_\perp=0}$$

Momentum Fractions



Group	(a [fm], M_π [MeV])	N_f	Fermion	Renorm.	$\langle x \rangle_g$
MSU 23 [24]	(0.09, 220)	$2+1+1$	Clover/HISQ	RI/MOM	0.509(20)(23)
ETMC 20 [22]	(0.08, 139)	$2+1+1$	T.M.	RI/MOM	$0.427(92)_{\rm St}$
ETMC 17 [35]	(0.08, 370)	$2+1+1$	T.M.	NLO PT	$0.284(27)_{\rm St}(17)_{\rm Ex}(24)_{\rm Sy}$
This work	(0.09, 358)	$2+1$	Clover	Gradient-Flow	$0.482(35)_{\rm St}$
MIT 24 [36]	(0.09, 170)	$2+1$	Clover	RI/MOM	$0.501(27)_{\rm St}$, $0.526(31)_{\rm St}$
χQCD 21 [23]	(0.143, 171)	$2+1$	DW	RI/MOM	$0.509(20)_{\rm St}(23)_{\rm Sy}$
MIT 19 [37]	(0.12, 450)	$2+1$	Clover	RI/MOM	$0.54(8)_{\rm St}$
χQCD 18 [20]	(0.11, {135, 372})	$2+1$	Overlap	RI/MOM	$0.47(4)_{\rm St}(11)_{\rm Sy}$
	(0.14, 171)				
χQCD 18 [21]	(0.11, 330)	$2+1$	DW	RI/MOM	$0.482(69)_{\rm St}(48)_{\rm Sy}$
	(0.082, 300)				
ETMC 17 [19]	(0.09, 130)	2	T.M.	NLO PT	$0.267(22)_{\rm St}(27)_{\rm Sy}$
χQCD 15 [17]	(0.11, {650, 538, 478})	0	Wilson	NLO PT	$0.334(55)_{\rm St}$
QCDSF/UKQCD	(0.09, 665–1170)	0	Clover	RI/MOM	$0.43(7)_{\rm St}(5)_{\rm Sy}$
Göckeler *et al.* [15]	(0.1, 585–993)	0	Wilson	NLO PT	0.53(23) [a]

[a] The results reported are at a renormalization scale $\mu = 2$ GeV except for the indicated entry which is at $\mu = 5$ GeV.

Gluon momentum fraction

Can be extracted from matrix elements of the form

$$z = 0$$

$$M(z,P) = Z(z)\langle P \,|\, G(z) W[z,0] \Gamma G(0) \,|\, P \rangle$$

$$M^j(z,P) = P^j \mathscr{M}(\nu, z^2) + z^j \mathscr{M}_z(\nu, z^2)$$

$$z = 0$$

Necessary for reduced Ioffe-time distributions

$$\mathfrak{M}(\nu, z^2) = \left(\frac{\mathscr{M}(\nu, z^2)}{\mathscr{M}(\nu, 0)\,|_{z=0}} \right) \Big/ \left(\frac{\mathscr{M}(0, z^2)\,|_{p=0}}{\mathscr{M}(0,0)\,|_{p=0,\, z=0}} \right)$$

T. Khan et. al. 2021

Monahan, Orginos 2014

Energy-Momentum Tensor

$$
T^{\mu\nu} = \begin{array}{c}
\begin{array}{cc} c^{-2}\cdot\left(\begin{array}{c}\text{energy}\\\text{density}\end{array}\right) & \begin{array}{c}\text{momentum}\\\text{density}\end{array} \end{array}\\
\begin{bmatrix}
T^{00} & T^{01} & T^{02} & T^{03}\\
T^{10} & T^{11} & T^{12} & T^{13}\\
T^{20} & T^{21} & T^{22} & T^{23}\\
T^{30} & T^{31} & T^{32} & T^{33}
\end{bmatrix}
\begin{array}{l}\text{shear}\\\text{stress}\\\text{pressure}\end{array}\\
\begin{array}{cc} \begin{array}{c}\text{energy}\\\text{flux}\end{array} & \begin{array}{c}\text{momentum}\\\text{flux}\end{array} \end{array}
\end{array}
= T_q^{\mu\nu} + T_g^{\mu\nu}
$$

$$
T_g^{\mu\nu} = G^{\mu\rho}G_{\rho}{}^{\nu} + \frac{1}{4}\delta^{\mu\nu}G^{\rho\sigma}G_{\rho\sigma}
$$

$$
= \mathcal{O}_1^{\mu\nu} + \mathcal{O}_2^{\mu\nu}
$$

Gluon momentum fraction

$$\mathcal{O}_1^{\mu\nu} = G^{\mu\rho} G_\rho{}^\nu$$

$$\langle P | \mathcal{O}_1^{44} - \frac{1}{3}\mathcal{O}_1^{jj} | P \rangle = \left(\frac{2}{3}\vec{p}^2 + 2E^2 \right) \langle x \rangle_g$$

$$\langle x \rangle_g \Big|_{\vec{p}=0} = \frac{1}{2m_N} \frac{\langle P | \mathcal{O}_1^{44} - \frac{1}{3}\mathcal{O}_1^{jj} | P \rangle}{\langle P | P \rangle}$$

Göckeler, Horsley, Ilgenfritz, et. al. 1996
Alexandrou et. al. 2017

Lattice details:

- Isotropic
- 2+1 stout-link (1-iteration) smeared clover Wilson fermions
- Tree-level tadpole improved Symanzik gauge action
- Configurations generated by rational HMC
- 64 temporal sources

a (fm)	M_π (MeV)	$L^3 \times N_t$	N_{cfg}
0.094(1)	358(3)	$32^3 \times 64$	1121

JLAB/W&M Collaboration

Computed 2-pt functions and disconnected gluon loops using new 24s cluster at JLab

Gradient-flow

$$A_\mu(x) \to B_\mu(\tau, x) \qquad B(0, x) = A_\mu(x)$$

$$\dot{B}_\mu = D_\nu G_{\nu\mu} \quad , \qquad D_\mu = \partial_\mu + [B_\mu, \cdot\,]$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

$$\dot{B}_\mu(\tau, x) = -\, g_0^2 \left\{ \partial_{x,\mu}\, S_W(B) \right\} B_\mu(\tau, x)$$

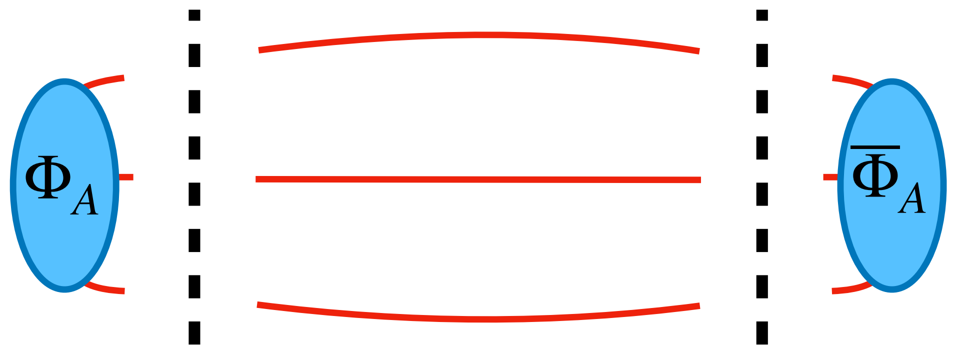
Lüscher, Weisz; 2011

Lüscher 2009, 2013, 2014

Distillation

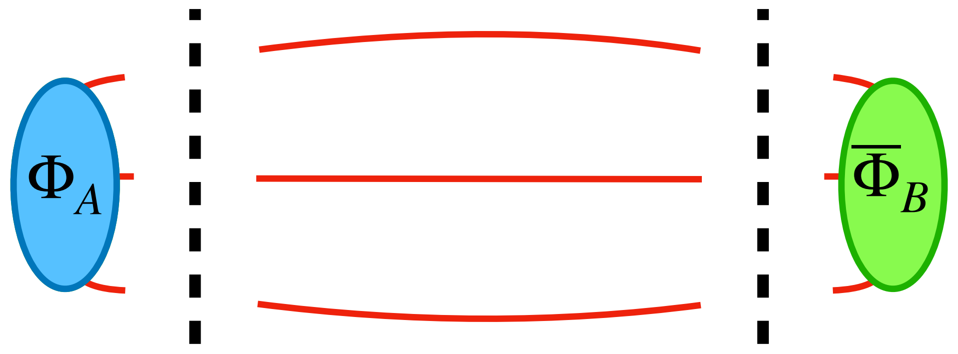
Low-rank approximation to Jacobi smearing kernel

$$-\nabla^2(t)\; v(t) = \lambda(t)\; v(t), \qquad \square(t) = \sum v(t) v^\dagger(t)$$



$$\Phi \sim \epsilon S \left(\mathscr{D} v\right) \left(\mathscr{D} v\right) \left(\mathscr{D} v\right)(t)$$

$$\tau\left(T_f, T_0\right) \sim v^\dagger(T_f)\, M^{-1}\left(T_f, T_0\right) v(T_0)$$



$$\langle \mathcal{O}_p(m) \overline{\mathcal{O}}_q(n) \rangle \sim \Phi_p(t_m)[\tau\tau\tau - \tau\tau\tau]\left(t_m, t_n\right) \Phi_q(t_n)$$

Peardon, et. al; 2009

Summed-GEVP

$$C_{ij}^{2\text{pt}}(t) = \langle \mathcal{O}_i(t)\overline{\mathcal{O}}_j(0)\rangle$$

$$C^{2\text{pt}}(t)\; u_n = \lambda_n(t, t_0) C^{2\text{pt}}(t_0) u_n$$

$$C_{ij}^{3\text{pt},\, s}(t) = \sum_{t_i=1}^{t-1} C_{ij}^{2\text{pt}}(t)\; \mathcal{O}(t_i)$$

$$\mathcal{M}_{nn}(t) = -\,\partial_t \left\{ \frac{u_n^\dagger \left[C^{3\text{pt},\, s}(t)\lambda_n^{-1}(t) - C^{3\text{pt},\, s}(t_0)\right] u_n}{u_n^\dagger C^{2\text{pt}}(t_0) u_n} \right\}$$

Blossier, et. al; 2009
Bulava, et. al; 2011

Bayesian Model-Averaging

$$pr(M|D) \approx \exp\left[-\frac{1}{2}(\chi^2_{\mathrm{aug}}(\mathbf{a}^\star) + 2k + 2N_{\mathrm{cut}})\right]$$

Model 1: $\left\{\boxed{C^{2\mathrm{pt}}(t_0 = 0),}\ C^{2\mathrm{pt}}(1),\ C^{2\mathrm{pt}}(2),\ C^{2\mathrm{pt}}(3),\ \ldots\right\}$

Model 2: $\left\{\boxed{C^{2\mathrm{pt}}(t_0 = 0),\ C^{2\mathrm{pt}}(1),}\ C^{2\mathrm{pt}}(2),\ C^{2\mathrm{pt}}(3),\ \ldots\right\}$

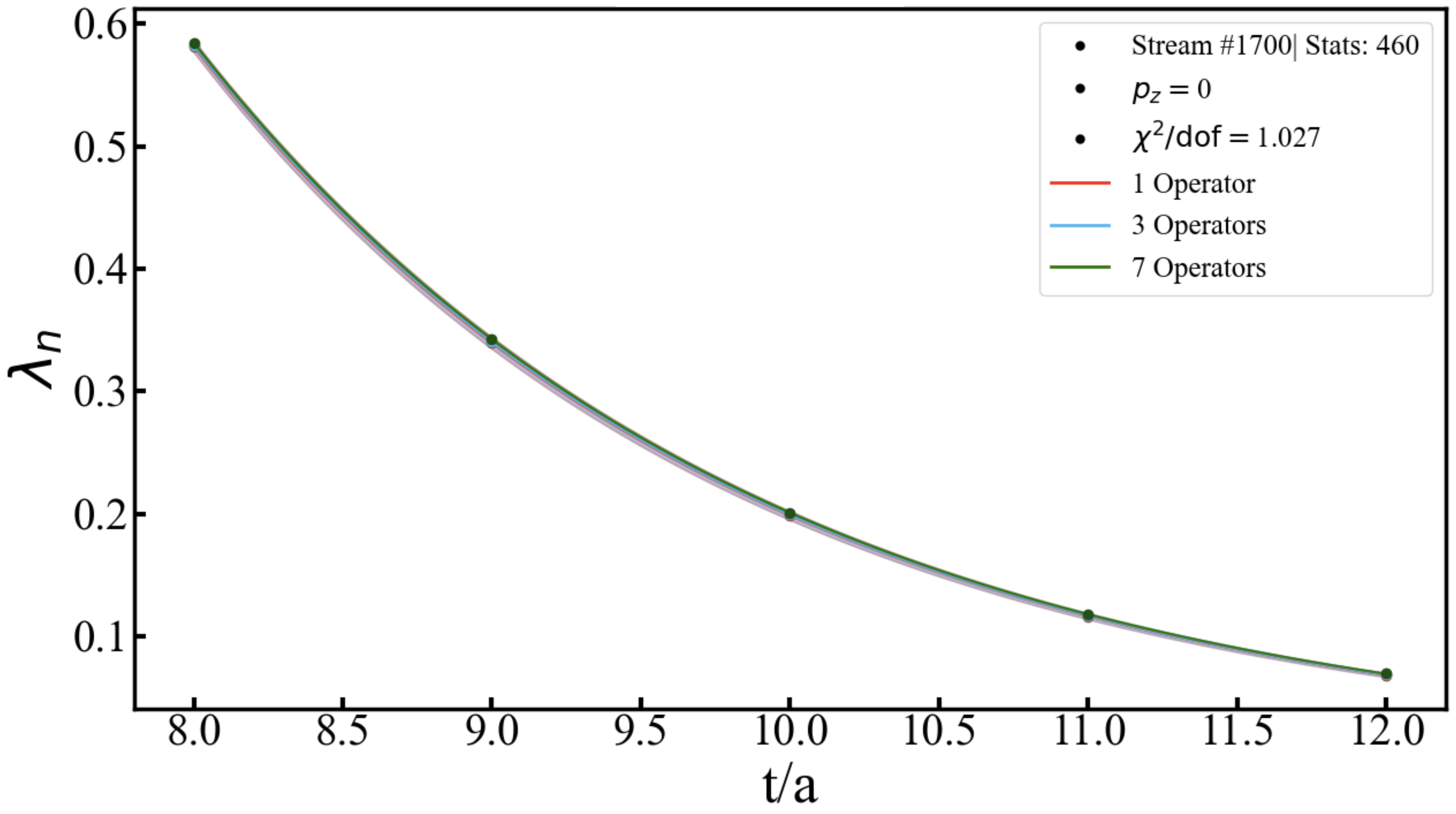
Model 3: $\left\{\boxed{C^{2\mathrm{pt}}(t_0 = 0),\ C^{2\mathrm{pt}}(1),\ C^{2\mathrm{pt}}(2),}\ C^{2\mathrm{pt}}(3),\ \ldots\right\}$

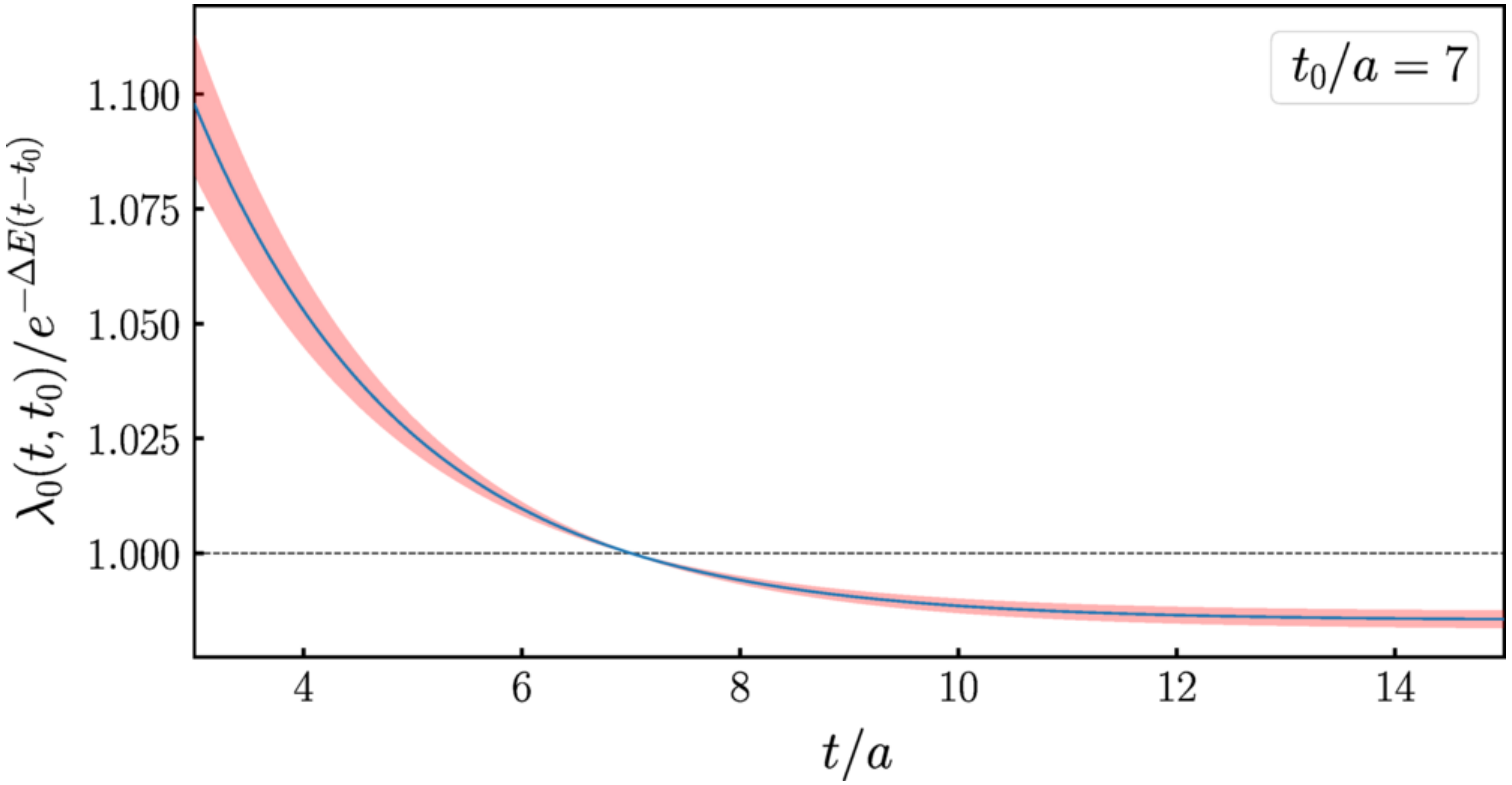
$\vdots$

Cut

Jay, Neil; 2019, 2022. 2023

Two-point correlation functions

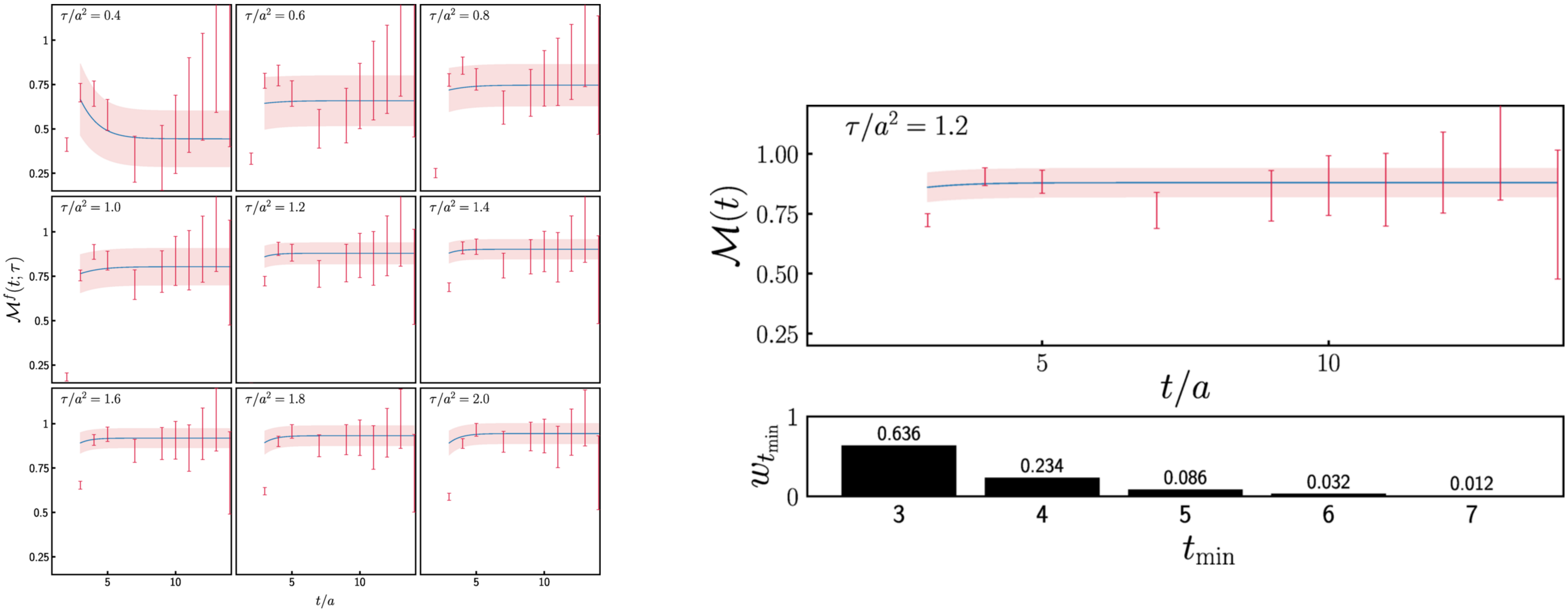




$$\lambda_n(t, t_0) = (1 - A)\, e^{-\Delta E(t-t_0)} + Ae^{-\Delta E'(t-t_0)}$$

Effective Matrix Elements Fits

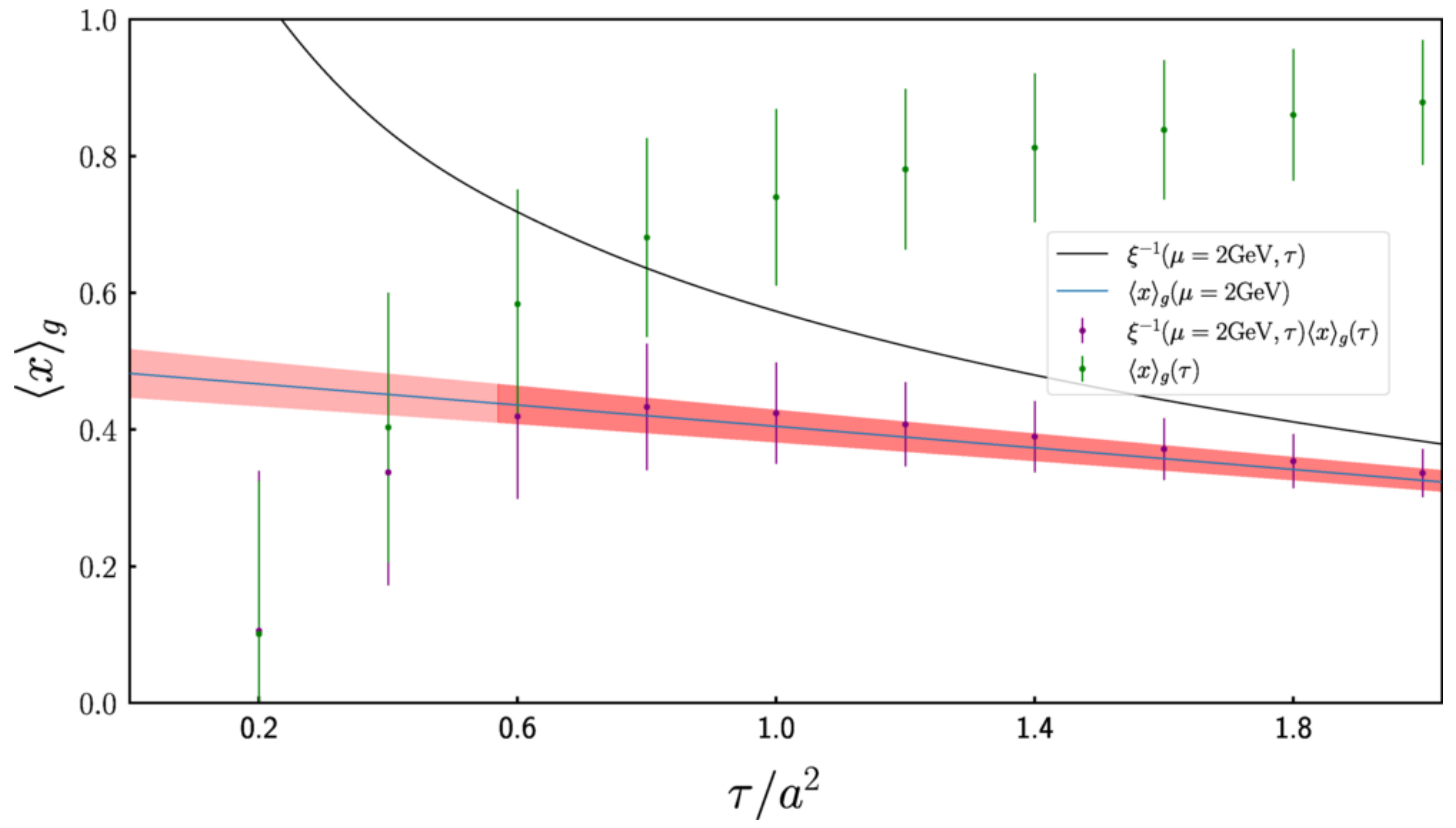
$$\mathscr{M}(t) = A + Bte^{-t\Delta E}$$



Gradient-flow Matching

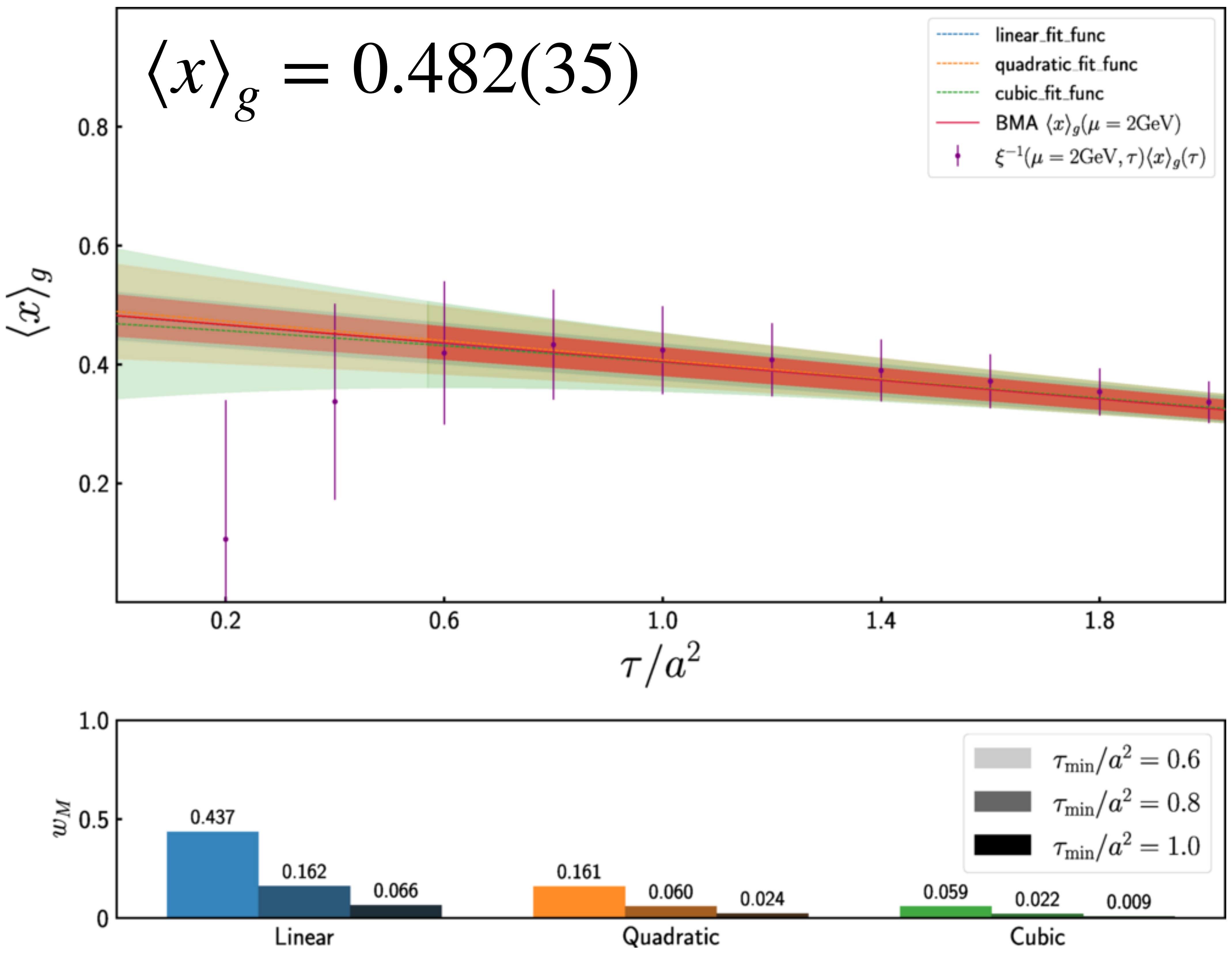
$$\tilde{\mathcal{O}}_i^{\mu\nu}(\tau, x) = \zeta_{i,j}(\tau)\mathcal{O}_j^{\mu\nu}(x) + \dots$$

$$\mathcal{O}_i^{\mu\nu}(x) = \zeta_{i,j}^{-1}(\tau)\tilde{\mathcal{O}}_j^{\mu\nu}(\tau, x) + \dots$$



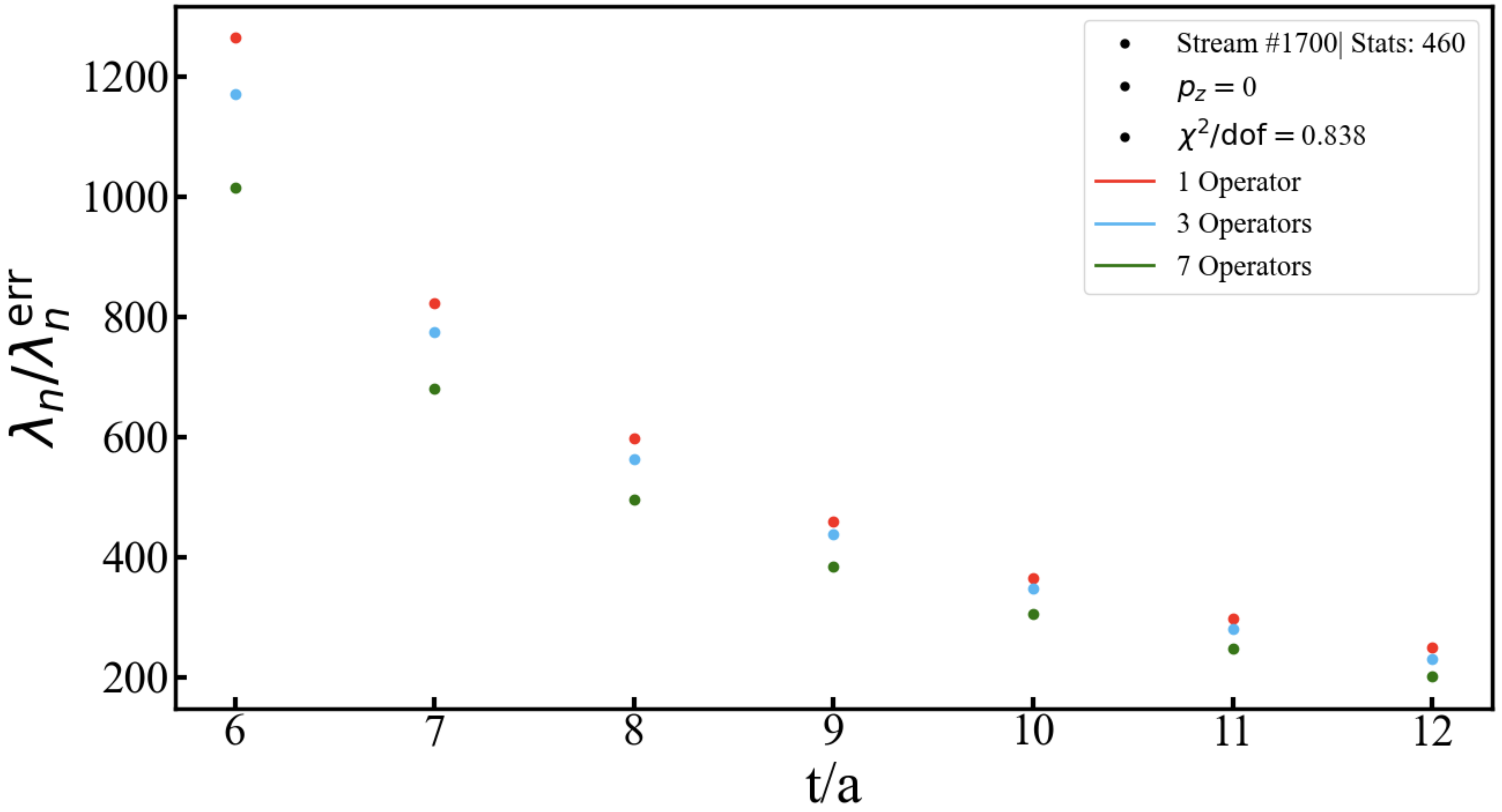
Harlander, Kluth, Lange; 2019
Makino, Suzuki; 2015

Gradient-flow Matching

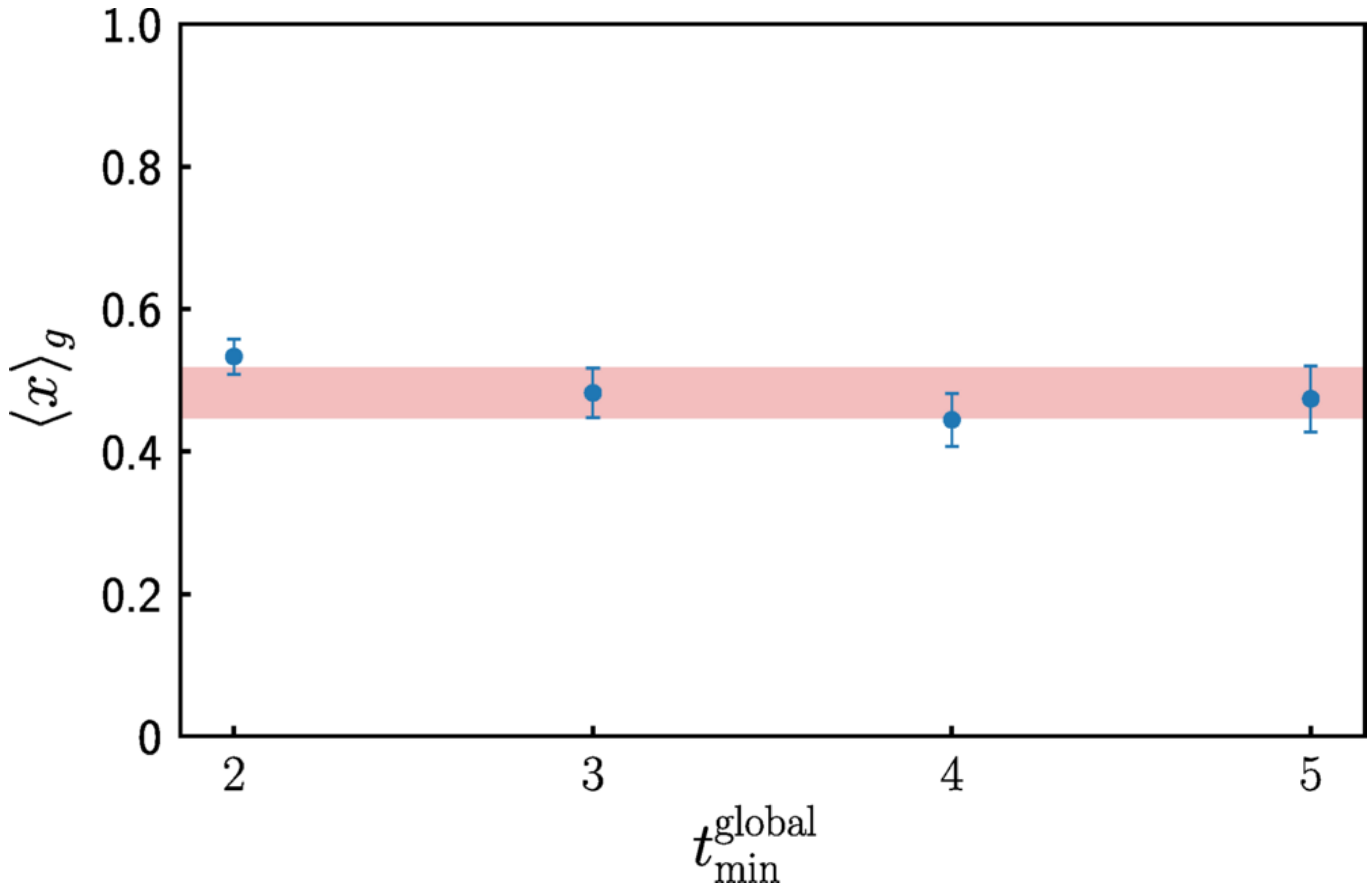


Summary

- Computed Two-Point Correlators
  - Using distillation
  - Implemented variational method to extract hadron masses
- Computed Gluon-Loops
  - Smoothed configurations with Gradient-Flow
- Combined data to generate Three-Point Correlators
  - Used sGEVP to extract three-point correlators
- Extracted gluon momentum fraction from combined data
  - Used BAIC method to fit correlator ratios
  - Recovered MS-bar results using matching coefficients



$$\lambda_n(t, t_0) = (1 - A)\, e^{-E(t-t_0)} + A e^{-E'(t-t_0)}$$


1.0
0.8
0.6
0.4
0.2
0
$\langle x \rangle_g$
2
3
4
5
$t_{\mathrm{min}}^{\mathrm{global}}$

Gluon momentum fraction

Can be extracted from matrix elements of the form

$$M^j_{\alpha\beta;\,\gamma\delta}(z,P) = Z(z)\langle P\,|\,G_{\alpha\beta}(z)W[z,0]\Gamma^j G_{\gamma\delta}(0)\,|\,P\rangle$$

$$z = 0$$

$$M^j(z,P) = P^j\mathcal{M}(\nu,z^2) + z^j\mathcal{M}_z(\nu,z^2)$$

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Necessary for reduced Ioffe-time distributions

$$\mathfrak{M}(\nu,z^2) = \left(\frac{\mathcal{M}(\nu,z^2)}{\mathcal{M}(\nu,0)\,|_{z=0}}\right) \Big/ \left(\frac{\mathcal{M}(0,z^2)\,|_{p=0}}{\mathcal{M}(0,0)\,|_{p=0,\,z=0}}\right)$$

T. Khan et. al. 2021

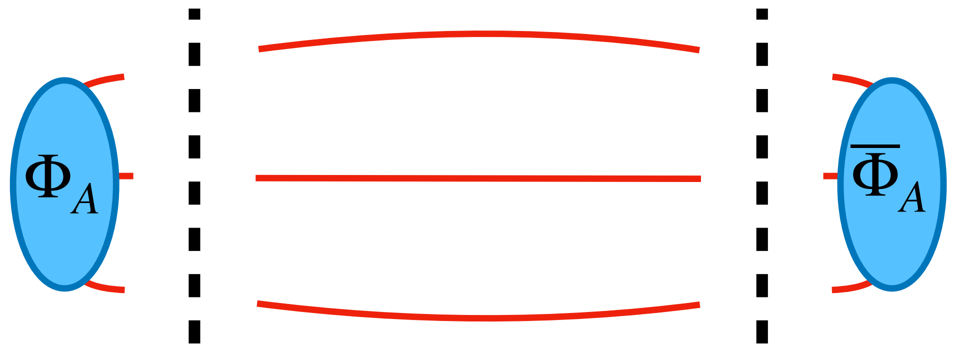
Monahan, Orginos 2014

Distillation

Low-rank approximation to Jacobi smearing kernel

$$-\nabla^2(t)\, v^{(k)}(t) = \lambda^{(k)}(t)\, v^{(k)}(t), \qquad \Box_{xy}(t) = \sum_{k=1}^{N_D} v_x^{(k)}(t) v_y^{(k)\dagger}(t)$$

$$\Phi^{(i,j,k)}_{r,\,\alpha\beta\gamma}(t) = \epsilon^{abc} S_{r,\,\alpha\beta\gamma} \left(\mathscr{D}_{1r} v^{(i)}\right)^a \left(\mathscr{D}_{2r} v^{(j)}\right)^b \left(\mathscr{D}_{3r} v^{(k)}\right)^c(t)$$



$$\tau^{(l,k)}_{\alpha\beta}\left(T_f, T_0\right) = v^{(l)\dagger}(T_f)\, M^{-1}_{\alpha\beta}\left(T_f, T_0\right) v^{(k)}(T_0)$$

$$\langle \mathcal{O}_p(m) \overline{\mathcal{O}}_q(n) \rangle = \Phi^{(i,j,k)}_{p,\,\alpha\beta\gamma}(t_m) \left[\tau^{(i,\bar{i})}_{\alpha\bar{\alpha}} \tau^{(j,\bar{j})}_{\beta\bar{\beta}} \tau^{(k,\bar{k})}_{\gamma\bar{\gamma}} - \tau^{(i,\bar{i})}_{\alpha\bar{\alpha}} \tau^{(j,\bar{k})}_{\beta\bar{\gamma}} \tau^{(k,\bar{j})}_{\gamma\bar{\beta}} \right] \left(t_m, t_n\right) \Phi^{(\bar{i},\bar{j},\bar{k})}_{q,\,\bar{\alpha}\bar{\beta}\bar{\gamma}}(t_n)$$

Peardon, et. al; 2009