Fermionic relativistic continuous MPS: musings on the Schwinger model

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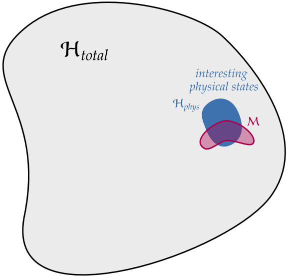
The variational method: compressing the wave function

Interesting states lie in *physical corner* of Hilbert space (exponential parametrization vastly inefficient)



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Guess submanifold $\mathcal{M} \subset \mathcal{H}$ (find a suitable wave function ansatz)

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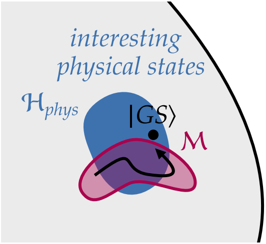

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Guess submanifold $\mathcal{M} \subset \mathcal{H}$ (find a suitable wave function ansatz)
Minimization:

$$|0\rangle \approx \mathrm{argmin}_{|\psi\rangle \in \mathcal{M}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

By definition one obtains an estimate for the GS energy that is lower bounded

$$\epsilon_0 \leq \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

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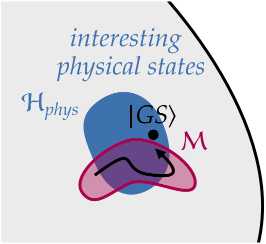

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Can we efficiently compress the wave function of quantum field theories?

How RCMPS meet Feynman's requirements

Feynman got frustrated by trying to apply the variational principle to quantum field theories:

I'm going to give away what I want to say, which is that I didn't get anywhere! I got very discouraged and I think I can see why the variational principle is not very useful. So I want to take, for the sake of argument, a very strong view - which is stronger than I really believe - and argue that it is no damn good at all! [Feynman '87]

He formulated three requirements on the submanifold $\mathcal{M}$:

Extensiveness: # params $\propto L$ not $\propto \exp(L)$

Computability: Efficiently evaluate $\langle\psi|H|\psi\rangle$ and $\langle\psi|\mathcal{O}(x)\mathcal{O}(y)|\psi\rangle$

Sensitivity to IR/UV: Approximation of IR dof should not degrade

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$$\left.\begin{array}{l}\checkmark\ \textbf{Extensiveness:}\\[1em] \checkmark\ \textbf{Computability:}\end{array}\right\} \to \text{solved by matrix product states}$$

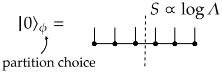
✓ **Sensitivity to IR/UV:** → solved by change of basis

Strategy: MPS $\xrightarrow[\text{continuum limit}]{}$ CMPS (2010) $\xrightarrow[\text{change of basis}]{}$ RCMPS (2021)

Relativistic difficulties

$$\hat{H}_{\text{Klein-Gordon}} := \hat{H}_0 = \int_{\mathbb{R}} \frac{\hat{\pi}(x)^2}{2} + \frac{(\nabla\hat{\phi}(x))^2}{2} + \frac{m^2}{2}\hat{\phi}(x)^2$$

1. Standard ϕ partition: $\mathcal{H} = \otimes_x \mathcal{H}_x^\phi$ (rigorously with discretisation), $S \propto \log(\Lambda)$



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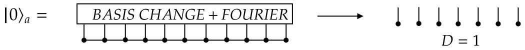
1. Standard ϕ partition: $\mathcal{H} = \otimes_x \mathcal{H}_x^\phi$ (rigorously with discretisation), $S \propto \log(\Lambda)$

2. Normal mode partition $\mathcal{H} = \otimes_x \mathcal{H}_x^a$: as in standard QFT textbooks, change basis

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\hat{\phi}(p) + \frac{\pi(p)}{\sqrt{\omega_p}}\right) \text{ with } \omega_p = \sqrt{p^2 + m^2}$$

$\rightarrow$ Go from $\phi(x)$ to $a(x) = \int \mathrm{d}p\; a(p) e^{ipx}$



Relativistic continuous MPS: bosons

$$|Q,R\rangle_{RCMPS} = \mathrm{tr}_{\mathbb{C}^D}\left\{\mathcal{P}\exp\left(\int_0^L \mathrm{d}x\; Q\otimes\mathbb{1} + R\otimes a^\dagger(x)\right)\right\}|0\rangle_a$$

- $a^\dagger(x)$ is the Fourier transform mode creation operator $a^\dagger(p)$
- $|0\rangle_a$ is the associated Fock vacuum, i.e. $a(x)\,|0\rangle_a = 0\ \forall x$
- Q, R are $D\times D$ matrices
- We take the trace over the auxiliary matrix space
- $[a(x), a^\dagger(y)] = \delta(x-y)$

[Tilloy '21]

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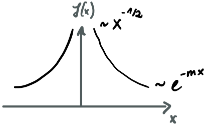
- Take $H_{\mathrm{QFT}} \equiv : H :_a$
- $|\text{free ground state}\rangle = |\text{vacuum}\rangle_a$

This solves the problematic free part exactly, and allows to define a finite interaction (in $1+1$)

Local observables under change in partition

$$\phi(x) = \int \mathrm{d}y J(x-y)\left[a(y) + a^\dagger(y)\right],$$

$$\text{with} \quad J(x) = \frac{1}{2\pi}\int \frac{\mathrm{d}k}{\sqrt{2\omega_k}} e^{-ikx} =$$



Trade (infinite) entanglement for (computable) non-locality!

[Tilloy '21]

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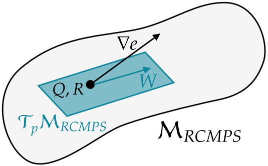

Compute $e_0 = \langle Q,R|h|Q,R\rangle$ and $\nabla_{Q,R}\, e_0$
Minimize e_0 with (geometric improvements of) gradient descent

minimum gives approximation of GS of interacting theory
$|Q,R\rangle = \mathrm{argmin}\,\langle Q,R|h|Q,R\rangle \simeq |\Omega\rangle$

$\langle Q,R|h|Q,R\rangle = e_0 + \epsilon$ minimum a true upper bound

non-trivial: can be done with cost $\propto D^3$

[Tilloy '21]

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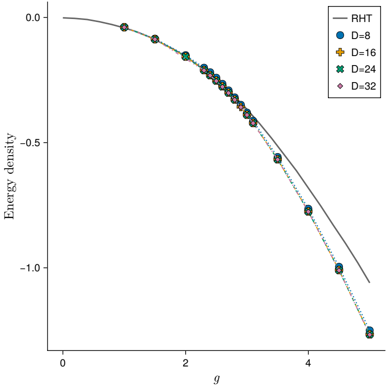
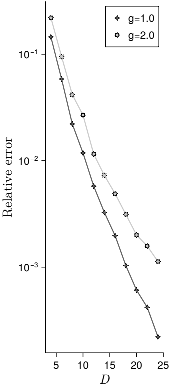
Example: renormalized ϕ_2^4 theory, $m = 1$

$$H = \int \mathrm{d}x \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{2}:\phi^2:_m + g:\phi^4:_m$$





[Tilloy '21]

Relativistic continuous MPS: bosons

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1. New ansatz for $1+1$ d bosonic relativistic QFT
2. No cutoff, UV or IR
3. UV is captured exactly even at $\mathrm{D}=0$
4. Efficient (cost poly D, error superpoly $1/\mathrm{D}$)
5. Rigorous (variational)

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The world is also made of fermions! In fact it is also made of gauge fields!

[Tilloy '21]

The massive Schwinger model

QED in 1+1 d:

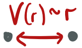
$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i\partial\!\!\!/ - eA\!\!\!/ - m)\,\psi + \frac{e\theta}{4\pi}\,\epsilon^{\mu\nu} F_{\mu\nu}$$

Gauge $A^1 = 0$: the photon is non-dynamical

Gauss's law $-\partial_1^2 A^0 = e\, j^0 \Rightarrow$ solve and substitute, with physical density $j^0 = {:}\psi^\dagger\psi{:}$

$$H = \int \mathrm{d}x\, \underbrace{\bar{\psi}\,(i\gamma^1\partial_1 + m)\,\psi}_{h_{Dirac}} - \frac{e^2}{4}\int \mathrm{d}x\, \underbrace{\mathrm{d}y\, j^0(x)\,|x-y|\, j^0(y)}_{h_{Coulomb}} - \underbrace{\frac{e^2\theta}{2\pi}\int \mathrm{d}x\, x\, j^0(x)}_{\text{set } \theta=0 \text{ for now}}$$

Many similarities with QCD: *a* $1+1$ *toy model that keeps the hard parts*



Confinement: linear potential $V(r) \sim e^2\, r$

Non-trivial vacuum: θ-dependence (sign problem for $\theta \neq 0$)

$\langle \bar{\psi}\psi \rangle \neq 0$ Chiral condensate

[Schwinger '62, Coleman '76]

Fermionic RCMPS

$$|Q, R_1, R_2\rangle_{fRCMPS} = \mathrm{tr}_{\mathbb{C}^D}\left\{\mathcal{P}\exp\left(\int_0^L \mathrm{d}x \; Q\otimes\mathbb{1} + R_1\otimes b_1^\dagger(x) + R_2\otimes b_2^\dagger(x)\right)\right\}|0\rangle_{b_i}$$

where $b_i^\dagger(x)$ are fermionic creation operators, verifying the canonical anticommutation relations

$$\{b_i(x), b_j^\dagger(y)\} = \delta_{ij}\delta(x-y) \quad \text{and} \quad \{b_i(x), b_j(y)\} = 0\,.$$

- $b_i^\dagger(x)$ are the Fourier transform mode creation operators $b_i^\dagger(p)$
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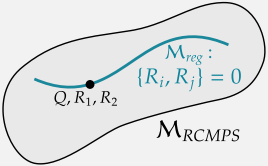
Regularity condition

$$\langle : h_{\mathrm{Dirac}} : \rangle_{\mathrm{fRCMPS}} = \textbf{finite} \qquad \text{if} \qquad \boxed{\{R_i, R_j\} = 0}$$



Schwinger model with fRCMPS: U(1) symmetry

Disconnected terms give IR divergence

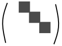
$$\left\langle : \hat{h}_{\text{Coulomb, disc}}(0) : \right\rangle = \int_{-\infty}^{\infty} \mathrm{d}y |y| \underbrace{\left(\langle l_0 | \left(R_1 \otimes \bar{R}_1 \right) | r_0 \rangle - \langle l_0 | \left(R_2 \otimes \bar{R}_2 \right) | r_0 \rangle \right)^2}_{\stackrel{!}{=} 0}$$

Further constrains $\mathcal{M}_{fRCMPS}$ → non-trivial, non-linear constraint on R_i!

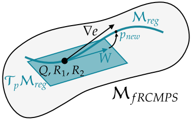
But physically corresponds to vanishing charge density $\langle j_0(x) \rangle = 0$

Need $U(1)$ **symmetric** fRCMPS: $\boxed{e^{i\theta} |Q, R_1, R_2\rangle_{fRCMPS} = |Q, R_1, R_2\rangle_{fRCMPS}}$

$Q =$ 

$R_1 =$ 

$R_2 =$ 



on $\mathcal{M}_{reg}$:
- $\{R_i, R_j\} = 0$
- $U(1)$ symmetric

Q is **block-diagonal**, $R_{1,2}$ are **block-off-diagonal**

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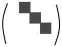
$R_1 =$ 

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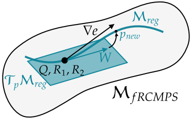

on $\mathcal{M}_{reg}$:

$\cdot \{R_i, R_j\} = 0$

$\cdot\, U(1)$ symmetric

For now : choose canonical matrices

$\cdot \{R_{1,c}, R_{2,c}\} = 0,\ R_{i,c}^2 = 0$

$\cdot\, e^{i\theta}|Q, R_{1,c}, R_{2,c}\rangle = |Q, R_{1,c}, R_{2,c}\rangle$ $\xrightarrow{\text{change of basis}}$ $R_i = U R_{i,c} U^{-1}$

contains variational parameters

$U =$ 

Q is **block-diagonal**, $R_{1,2}$ are **block-off-diagonal**

Computing expectation values

$$\mathcal{Z}_{j',j} = \left\langle Q, R \right| e^{\int_I j'_\alpha(x)\,\hat{b}^\dagger_\alpha(x)\,\mathrm{d}x}\, e^{\int_I j_\beta(y)\,\hat{b}_\beta(y)\,\mathrm{d}y} \left| Q, R \right\rangle$$

Functional derivatives bring down operators:

$$\frac{\delta}{\delta j'_\alpha(x)} \frac{\delta}{\delta j_\beta(y)}\, \mathcal{Z}\Big|_{j=j'=0} = \left\langle \hat{b}^\dagger_\alpha(x)\, \hat{b}_\beta(y) \right\rangle$$

Computing expectation values

$$\mathcal{Z}_{j,j'} := \mathrm{tr}\left[\mathcal{P}\exp\left(\int_0^L f\big(Q, R_1, R_2; j, j'\big)\,\mathrm{d}x\right)\right]$$

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1. Compute $\mathcal{Z}_{j,j'}$ by solving the ODE and taking the trace at $x = \infty$
2. Take derivatives in j, j' $\Rightarrow$ a coupled *linear* ODE hierarchy
3. In superoperator form: $\propto D^3$

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✓ $\langle{:}h_{\mathrm{Dirac}}{:}\rangle \to 0$ at $R_i = 0$ (exact)

✓ two independent implementations at $R_i \neq 0$ agree

Conclusion

- New ansatz to solve 1+1 d fermionic theories directly in the continuum and thermodynamic limit
- Have identified non-trivial UV and IR regularity conditions for Schwinger model that need to be imposed on the manifold

Next steps:

- Have preliminary results at $D = 4$ and $D = 8$ but need to be cross-checked
- $h_{\mathrm{C}oulomb}$ for now prohibitively slow (via ODE asymptotically $\propto D^3$ but with a huge prefactor) → maybe some clever trick?
- Calculate gradients in terms of Q, R_1, R_2 (for $h_{\mathrm{C}oulomb}$ a nightmare...)
- Generalize to $\theta \neq 0$ (possible via chiral rotation?)

Stay tuned for results on the Schwinger model!

Backup Slides

Coulomb interaction

$$\boxed{:j_0(x): = \int ds\, dt \Big[P(x-s, x-t)\big(b_1^\dagger(s) b_1(t) - b_2^\dagger(s) b_2(t)\big) + D(x-s, x-t)\big(b_1^\dagger(s) b_2^\dagger(t) - b_2(s) b_1(t)\big)\Big]}$$

Gives 16 channels for $: j_0(x) :: j_0(0) :$

$N_1 N_1$	$+P_y P_0$	$b_1^\dagger(s) b_1(t)\, b_1^\dagger(u) b_1(v)$
$N_1 N_2$	$-P_y P_0$	$b_1^\dagger(s) b_1(t)\, b_2^\dagger(u) b_2(v)$
$\vdots$	$\vdots$	$\vdots$
$\bar{\Pi}\bar{\Pi}$	$+D_y D_0$	$b_2(s) b_1(t)\, b_2(u) b_1(v)$

$N_1 = P\, b_1^\dagger b_1,\ N_2 = -P\, b_2^\dagger b_2,\ \Pi = D\, b_1^\dagger b_2^\dagger,\ \bar{\Pi} = -D\, b_2 b_1,$

where each term is

$\int ds\, dt\, du\, dv$ of (coeff) $\times$ (kernels) $\times$ (four-leg expectation)

$\rightarrow$ still possible via graded generating functional at cost $\propto D^3$, but a lot of bookkeeping

fRCMPS calculus

Via graded generating functional

$$\mathcal{Z}_{j,j'} := \mathrm{tr}\Big[\big(\langle 0| + \langle 1|\big)\mathcal{P}\exp\Big(\int \mathrm{d}u\,\big(T_+ \otimes |0\rangle\langle 0| + T_- \otimes |1\rangle\langle 1| + \sum_\alpha \big(j_\alpha(u) R_\alpha \otimes I_D + j'_\alpha(u) I_D \otimes \bar{R}_\alpha\big) \otimes \sigma_x\Big)|0\rangle\Big],$$

with $T_\pm = Q \otimes 1 + 1 \otimes \bar{Q} \pm \big(R_1 \otimes \bar{R}_1 + R_2 \otimes \bar{R}_2\big)$

parity switches at each fermion: even # of legs to the left $\Rightarrow T_+$, odd $\Rightarrow T_-$

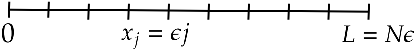
evolve across the line: linear ODE $\frac{d}{dx}|\rho\rangle = T_\pm|\rho\rangle$

From MPS to CMPS: taking the continuum limit

For bosons on a line:

$$|Q,R\rangle_{CMPS} = \mathrm{tr}_{\mathbb{C}^D}\left\{\mathcal{P}\exp\left(\int_0^L \mathrm{d}x\; Q\otimes\mathbb{1} + R\otimes\psi^\dagger(x)\right)\right\}|0\rangle_\psi$$

CMPS are a **continuum limit of MPS:**



$$|Q,R\rangle = \sum_{n_1,n_2,\ldots n_N\in\mathbb{N}} \mathrm{tr}\left[A_{n_1}A_{n_2}\cdots A_{n_N}\right]|n_1,n_2,\ldots,n_N\rangle + \mathcal{O}(\epsilon)$$

with

- $A_0 = \mathbb{1} + \epsilon Q\,,\qquad A_1 = \sqrt{\epsilon}R\,,\qquad A_2 = \epsilon R^2/2\,,\qquad A_{n\geq 3} = \mathcal{O}(\epsilon)$
- $|n_1,n_2,\ldots,n_N\rangle = b_1^{\dagger n_1} b_2^{\dagger n_2}\ldots b_N^{\dagger n_N}|0\rangle_\psi$
- $b_j^\dagger = \int_{j\epsilon}^{(j+1)\epsilon}\psi^\dagger(x)/\sqrt{\epsilon}$, such that $[b_j, b_k^\dagger] = \delta_{jk}$

Verstraete, Cirac '2010