A Novel Quark Smearing Technique and its Application to Distillation

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arXiv:2608(9).xxxxx

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Why do we need a new method?

State-of-the-art method: Distillation (Peardon et al (2009))

It (and its variations like Lap-H) works quite well, made outstanding contribution to our field.

However,

* Computing and storage intensive for baryons, particularly for multi-baryons
* Prohibitively computing resource demanding for nuclei with large number of operators in a GEVP setup.
* Main issues: large number of eigenvectors, inversions, Wick-contractions for local multi-mesons (tetraquarks), and multi-baryons

Can we address these bottlenecks?

Localized Basis: Motivation

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- Multi-hadron studies are quite contraction dominated.

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- "Distillation" as a method for all-to-all quark propagator estimation at low-energy subspaces. [1]

- An efficient way to removing high frequency modes towards a goal of obtaining all-to-all quark propagators with minimal resources.
- Basically projects the quark fields to the lowest n_v number of eigenvectors of lattice Laplacian,

$$\Delta(t)v_k(t) = \lambda_k(t)v_k(t), \text{ for } \quad k \in \{1, \cdots, Nv\}$$

- Quark propagators,

$$\tau_{\alpha\beta}(t_f, t_i) = v_\alpha^\dagger(t_f)D^{-1}v_\beta(t_i), \quad \alpha, \beta \in \{1,2,\cdots, N_v\}$$

[1] M. Peardon et al PRD 80 (2009) 054506.

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- Multigrid like "sparsening" algorithms targeting contractions cost [2,3]

Full correlator

$$C_{full}(\vec{p},t) = \sum_{\vec{x}\in\Lambda_3} \langle\Omega|e^{-i\vec{p}\cdot\vec{x}}\mathcal{O}(\vec{x},t)\mathcal{O}^\dagger(\vec{0},0)|\Omega\rangle,$$

$$\Lambda_3 \equiv \{(n_1,n_2,n_3)|0\le n_i < L\}$$

Sparse correlator

$$C_{sparse}(\vec{p},t) = \sum_{\vec{x}\in\tilde{\Lambda}_3(N)} \langle\Omega|e^{-i\vec{p}\cdot\vec{x}}\mathcal{O}(\vec{x},t)\mathcal{O}^\dagger(\vec{0},0)|\Omega\rangle,$$

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[1] M. Peardon et al PRD 80 (2009) 054506.
[2] W. Detmold et al PRD 104 (2021),
[3] A. Stump et al. PRD 113 (2026)

Localized Basis: **Wavelets**

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Proposition:

- Use **Wavelet**-type localised wave-packets as a basis to project quark fields to low energy subspace ("Distillation")
- In the localised bases, leverage sparsity to address contraction cost ("sparsening")

Another Localised basis: Lang et al: Phys. Rev. D **111**, 114510 (2025)

Background

- **Wilson's Pioneering Work:**
  - The use of wavelet like functions, for nonperturbative analysis of quantum field theories (QFTs), first emerged in a **1965** work by K.G Wilson, titled "**Model Hamiltonians for Local Quantum Field Theory**".[KG Wilson, Phys. Rev. **140**, B445, (1965)]
  - He wanted to construct a set of functions, which he called "Wave-Packet" functions, that rapidly goes to zero as one move away from the assigned region in both position and momentum space.
  - He constructed a wave-packet which is either fully localized in position and have a long tail in the momentum space and vice-versa.

- **Work of Daubechies:**
  - More than two decades later (**1988)** Daubechies developed a orthonormal bases which has a strikingly similar properties to the Wilson's **wave-packet** functions.[I. Daubechies, Comm. Pure Appl. Math. 41, 909 (1988)]
- **Wavelet in QFT:**
  - With this wavelet,, in **1994** Wilson emphasized the importance of wavelets for non-perturbative analysis of quantum chromodynamics (QCD) in 1994.[KG Wilson et al. ,Phys. Rev. D **49**, 6720,(1994)]
  - In QFT: Phys. Rev. D **87**, 116011 (2013), Bulut, Polyzou
  - Scalar field theory: Basak, Chakraborty, **NM**

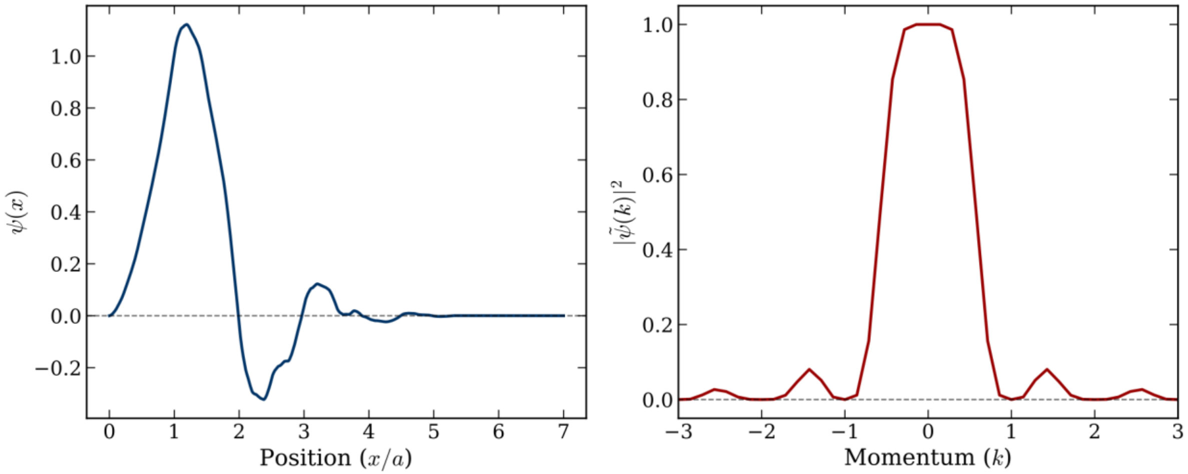
Phys. Rev. D 113 (2026) 8, 085021;
Phys. Rev. D 114 (2026) 1, 014502

What is a Wavelet?

Core concept:

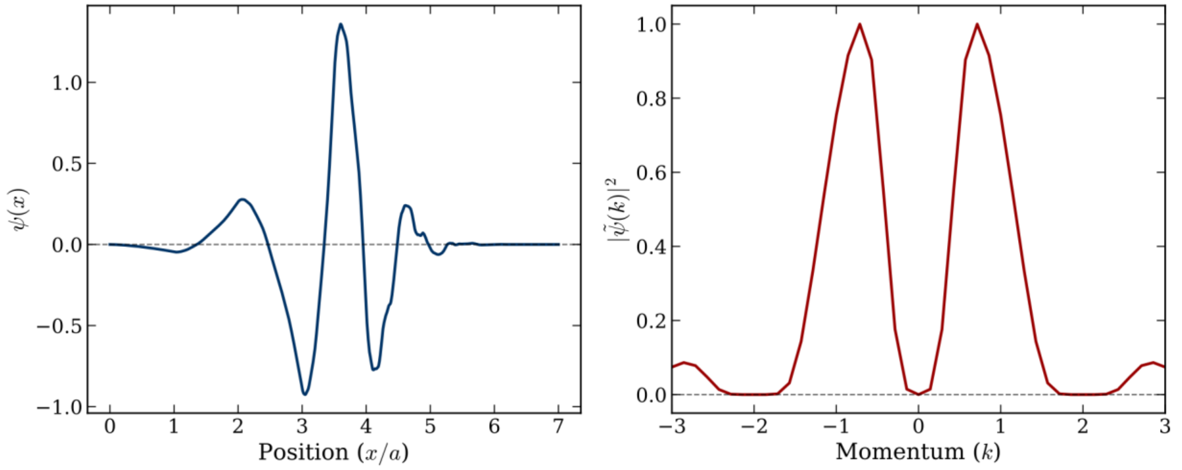
- A wavelet is a wave-like oscillation with an amplitude that starts at zero, increases, and strictly decays back to zero (compact support).
- The Crucial Difference: Traditional Fourier bases (sine/cosine waves) are perfectly localized in momentum but entirely delocalized in space. Wavelets are localized in both space and momentum.

Low Frequency Projectors : Scaling Function



- Allows one to perform multi-resolution analysis (MRA)[1,2] by decomposing the functions in approximate and detail components; extensively employed in signal processing.

Add higher frequencies gradually : Wavelet Function



- Perfect reconstruction of the signal from the components similar to inverse Fourier transfrom.

[1] I. Daubechies, Orthonormal bases of compactly supported wavelets, Communications on Pure and Applied Mathematics 41, 909 (1988)

[2] S. Mallat, A theory for multiresolution signal decomposition: the wavelet representation (1989)

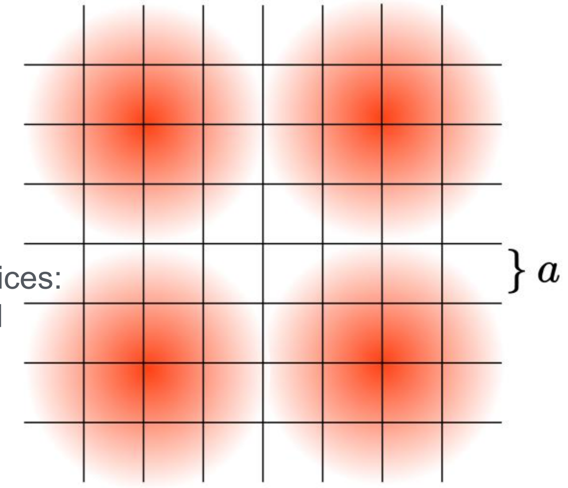
Localized Wavelet Bases: Construction

- We use resolvent based spectral wavelets basis:

$$v_i(\vec{x}) = (\mathrm{I} - \tau\Delta(t))^{-m}\delta^3(\vec{x} - \vec{x}_i)$$

$$\vec{x}_i \in \tilde{\Lambda}_3(N) \equiv \{(x_1, x_2, x_3) | x_i mod N = 0\}$$

N.J. Higham, Functions of Matrices: Theory and Computation, SIAM



Wavelet bases defined on coarse geometry spanning the entire lattice

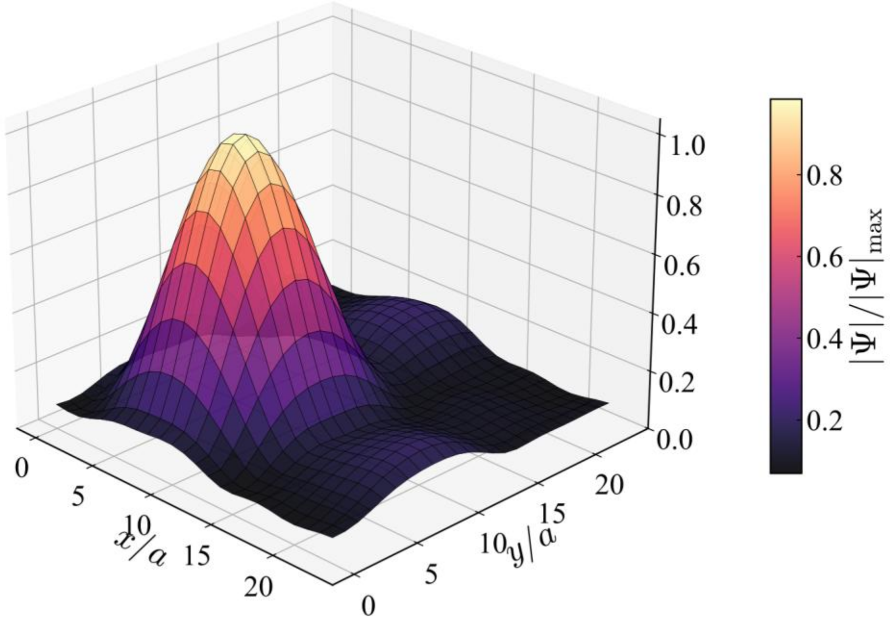
- $v_j(\vec{x})$'s are localised by construction, but not orthogonal!

- Orthogonalize using Lowdin orthogonalisation,

$$G_{ij} = \sum_{\vec{x}} v_i^\dagger(\vec{x}) v_j(\vec{x}), \qquad u_i(\vec{x}) = G_{ij}^{-1/2} v_j(\vec{x})$$

- $u_i(\vec{x})$ are now localised and orthogonal. Define, low subspace projection operator,

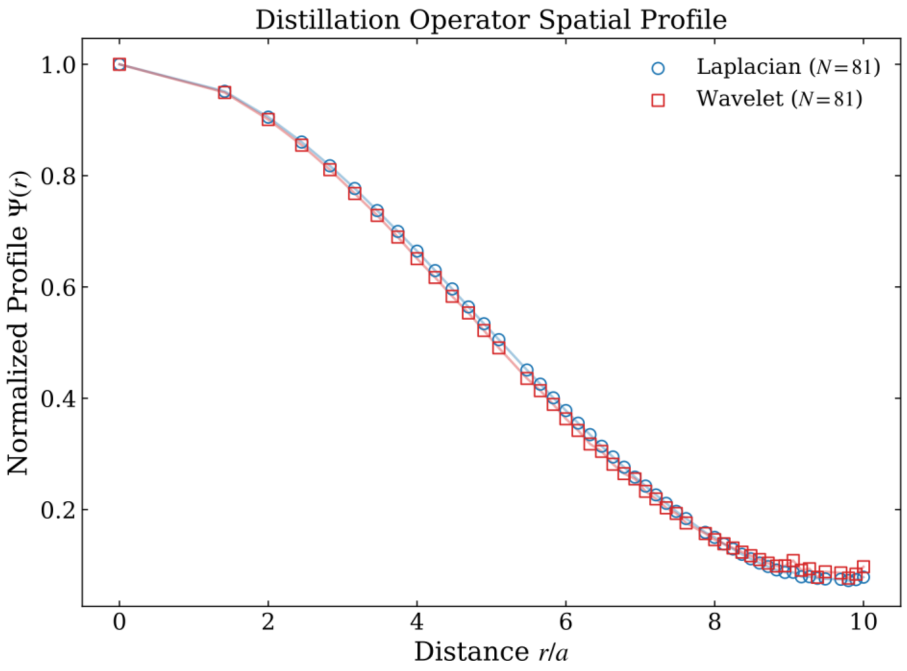
$$V(t) = u_i(\vec{x}, t) u_i^\dagger(\vec{y}, t)$$



projected quark fields $\tilde{q}(\mathbf{x}, t) = V(\mathbf{x}, \mathbf{y}; t) q(\mathbf{y}, t)$

Wavelet-Based Distillation

- As a test case, we use $24^3 \times 48$ lattice generated by CLS collaboration with $m_\pi = 429$ MeV and $a = 0.098$ fm.
- For standard distillation, we use $n_v = 81$.
- For wavelet distillation, we use 3^3 coarse geometry in 24^3 lattice with three color degrees of freedom on each site. $n_v = 81, m = 4, \tau = 8, N = 8$.
- Smearing profile matches well.
- Basis construction is 5 times cheaper, avoids eigenvalue calculation.



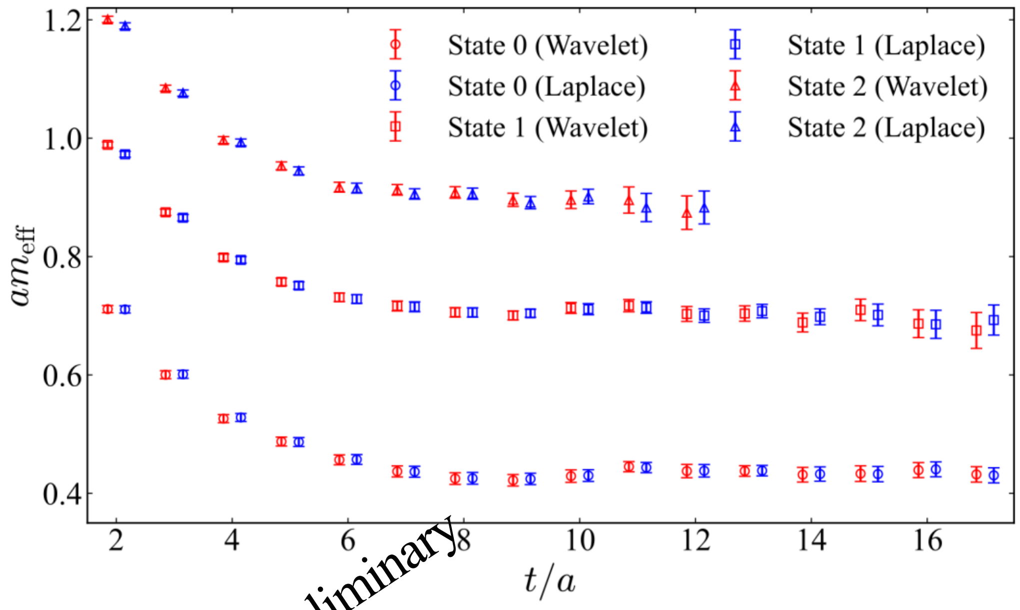
Comparison of smearing profile generated using standard distillation and wavelet-based distillation

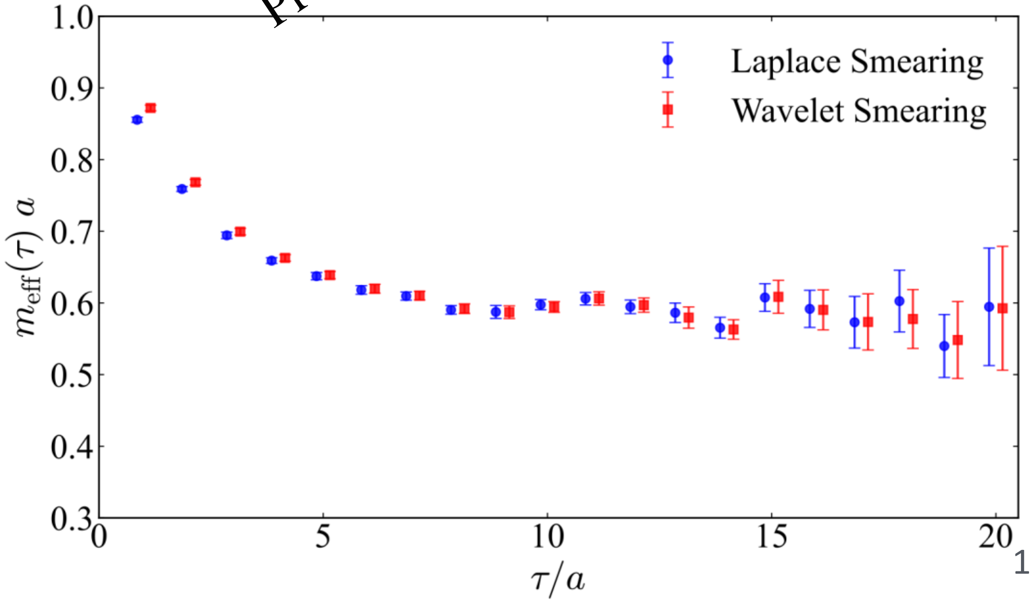
Comparison of FV Spectra

- We compare the FV spectrum of simple single and multi hadrons systems.
- One of the simplest multi-hadron system is $\pi\pi$ in isospin 2 channel. For which we consider bi-local operators of the form,

$$\mathcal{O}_{\pi\pi}^{I=2}(\vec{p}=0) = \mathcal{O}_{\pi^+}(\vec{p})\mathcal{O}_{\pi^+}(-\vec{p}), \text{ for } |\vec{p}^2| \leq 4$$

- Total number of operators is 5 and show the lowest three FV energy levels.
- For baryon sector, we compare the effective energy of single nucleon.
- In all cases the signal quality looks almost same within errorbars.



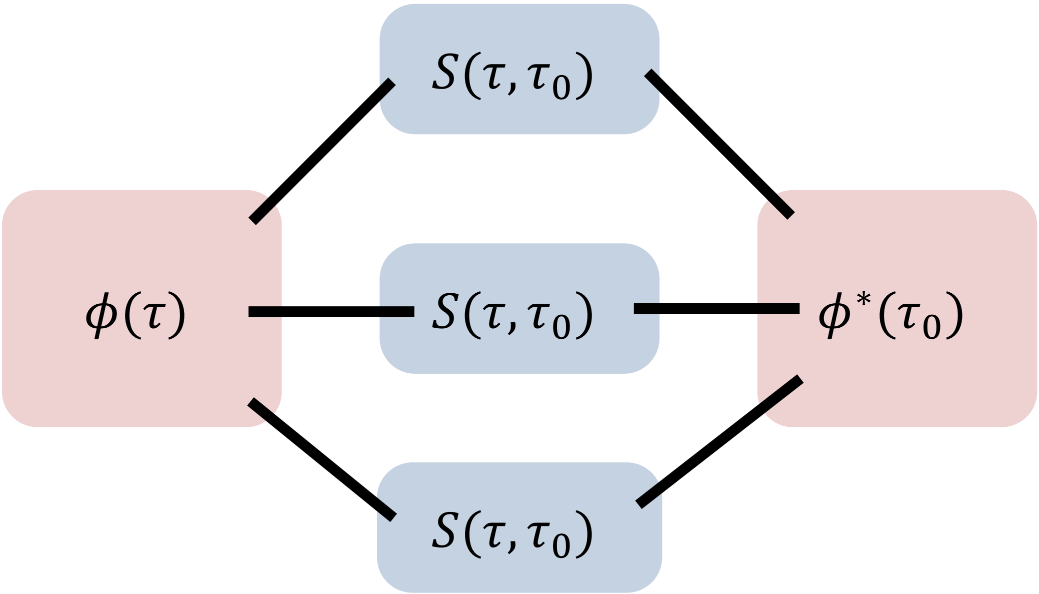


Stochastic Wavelet Distillation

Nucleon 2-pt correlation function:

$$C(\tau) = \phi_{ijk}^{\alpha\beta\gamma}(\tau) S_{ii'}^{\alpha\alpha'}(\tau,\tau_0) S_{jj'}^{\beta\beta'}(\tau,\tau_0) S_{kk'}^{\gamma\gamma'}(\tau,\tau_0) \left(\phi(\tau_0)_{i'j'k'}^{\alpha'\beta'\gamma'}\right)^*$$

$+ \text{crossterm}$



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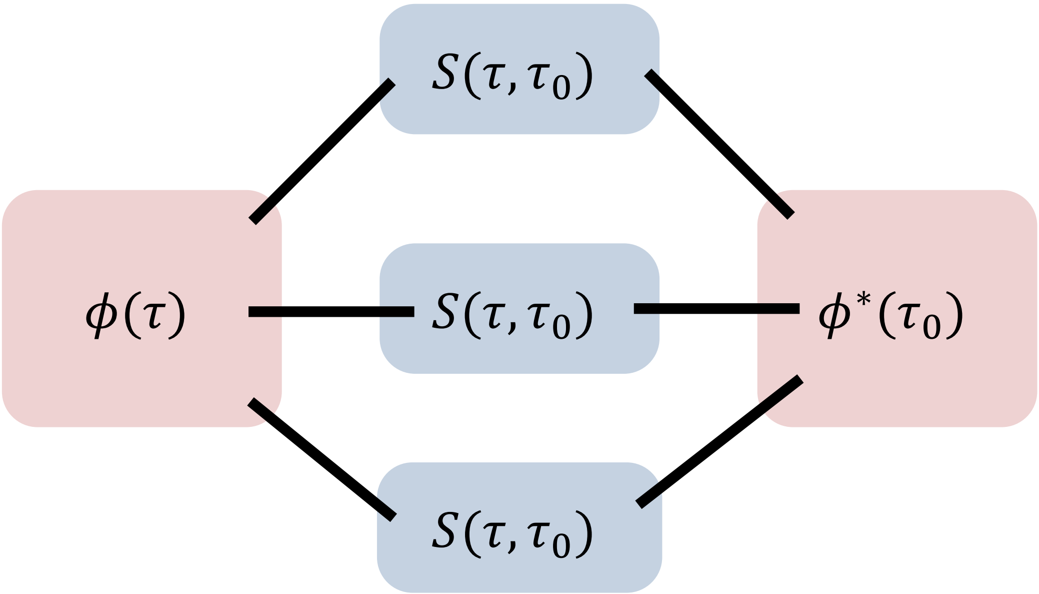
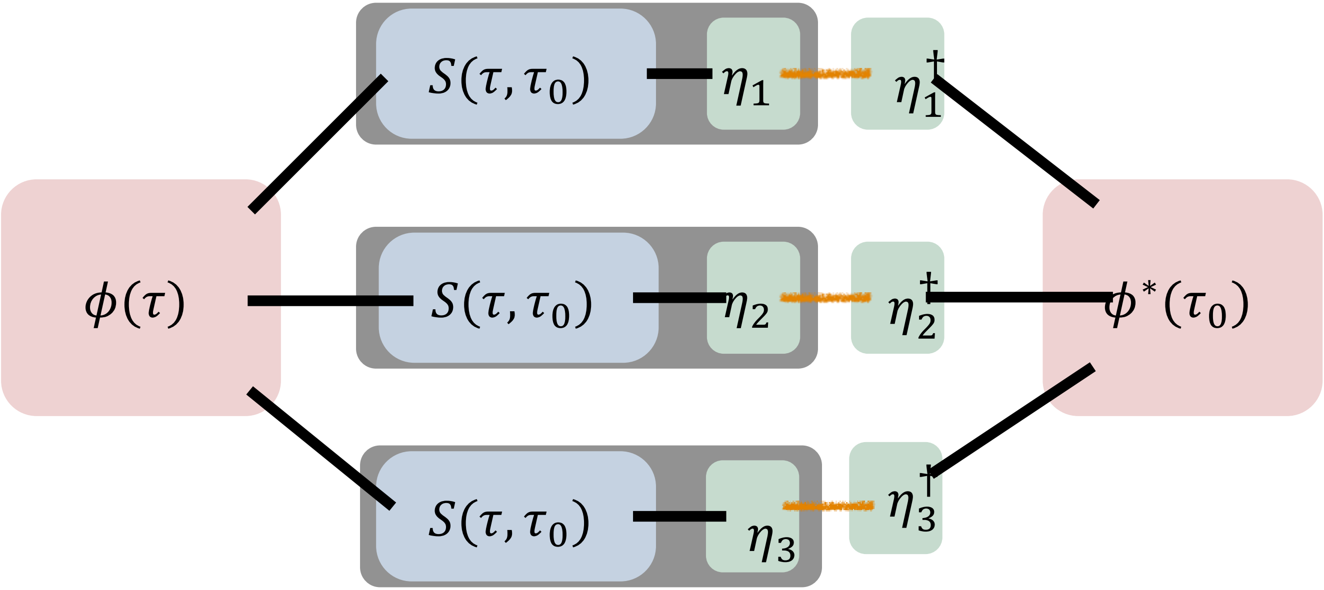
$n_v \times n_s$ $\qquad n_v = 81$

$n_d \times n_s$ $\qquad n_d = 6$





Stochastic Estimation at the source

Advantage: 1. Less number of Dirac inversion

2. Contraction costs are also reduced

η_i's are stochastic noise vector, a popular choice in lattice community is Z_4 noise $\{\pm 1, \pm i\}$

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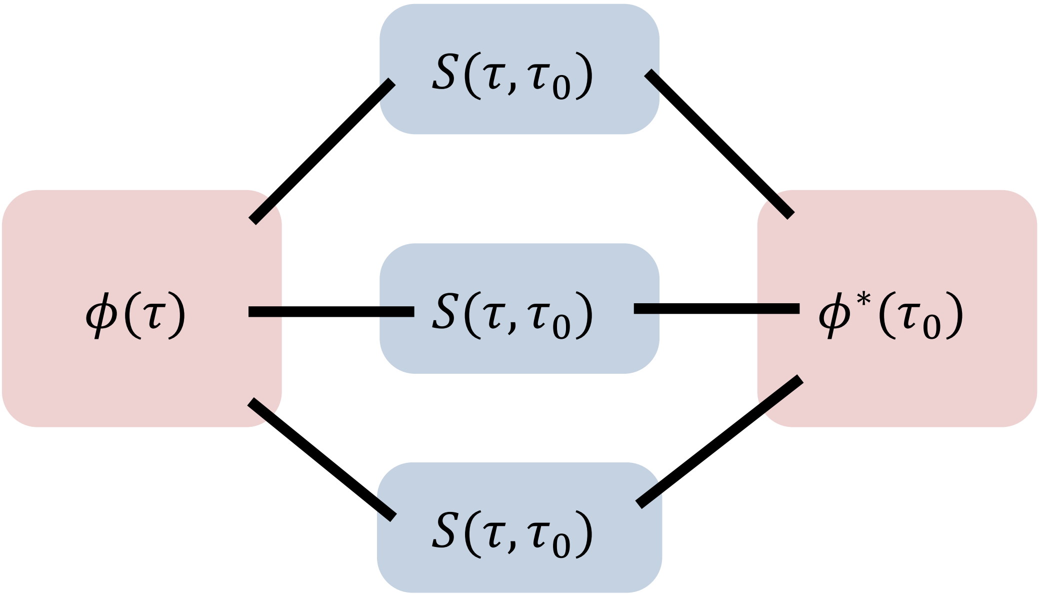

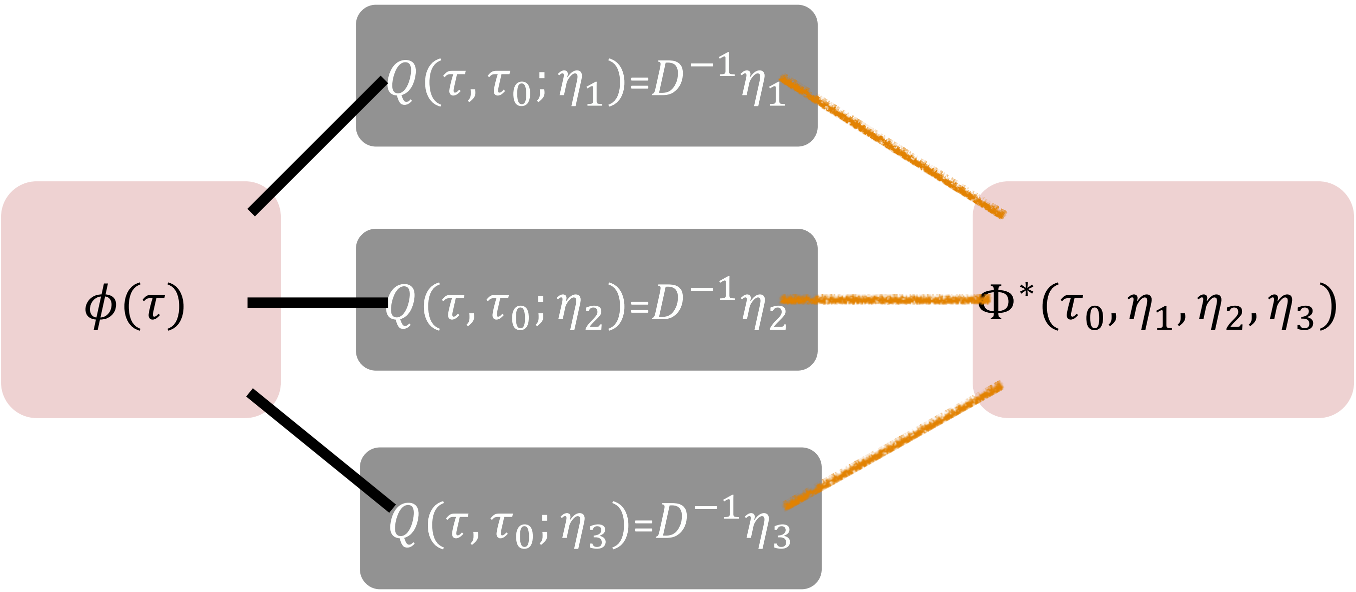

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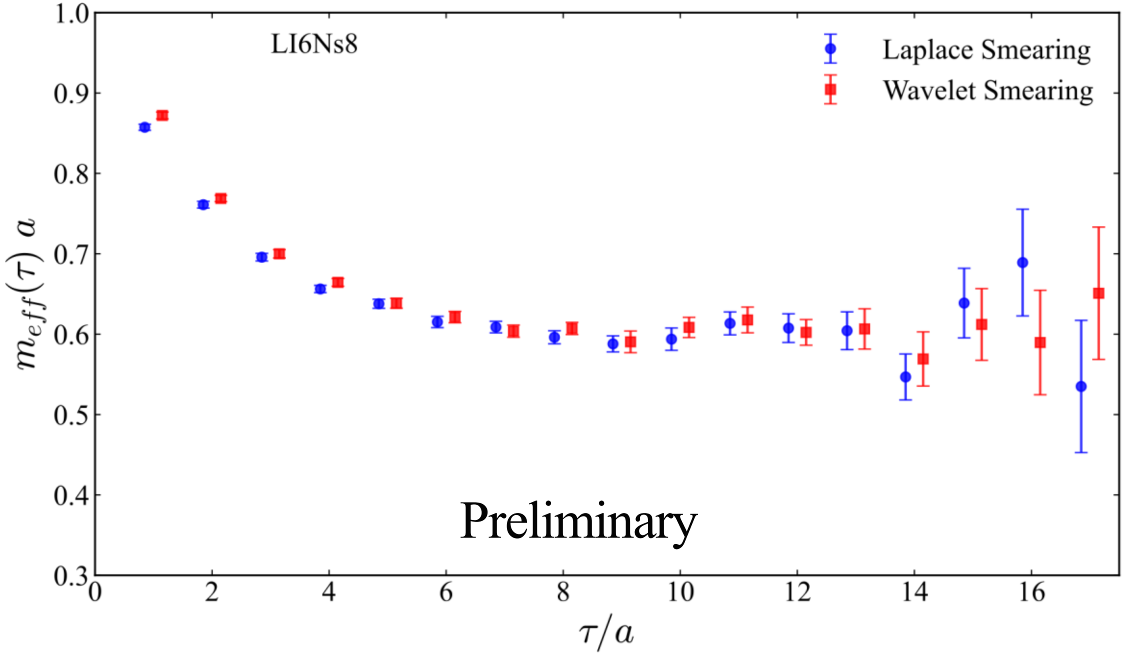
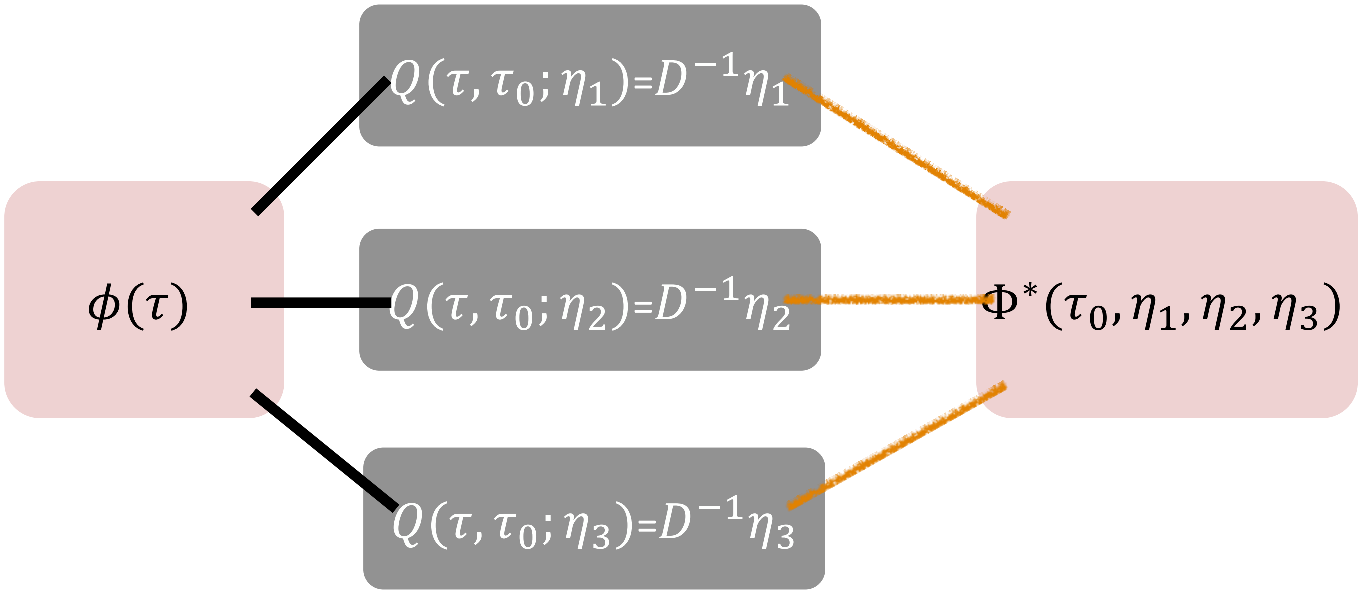


Comparison of effective mass of nucleon for Laplace and wavelet distillation with stochastic noise vectors as the source



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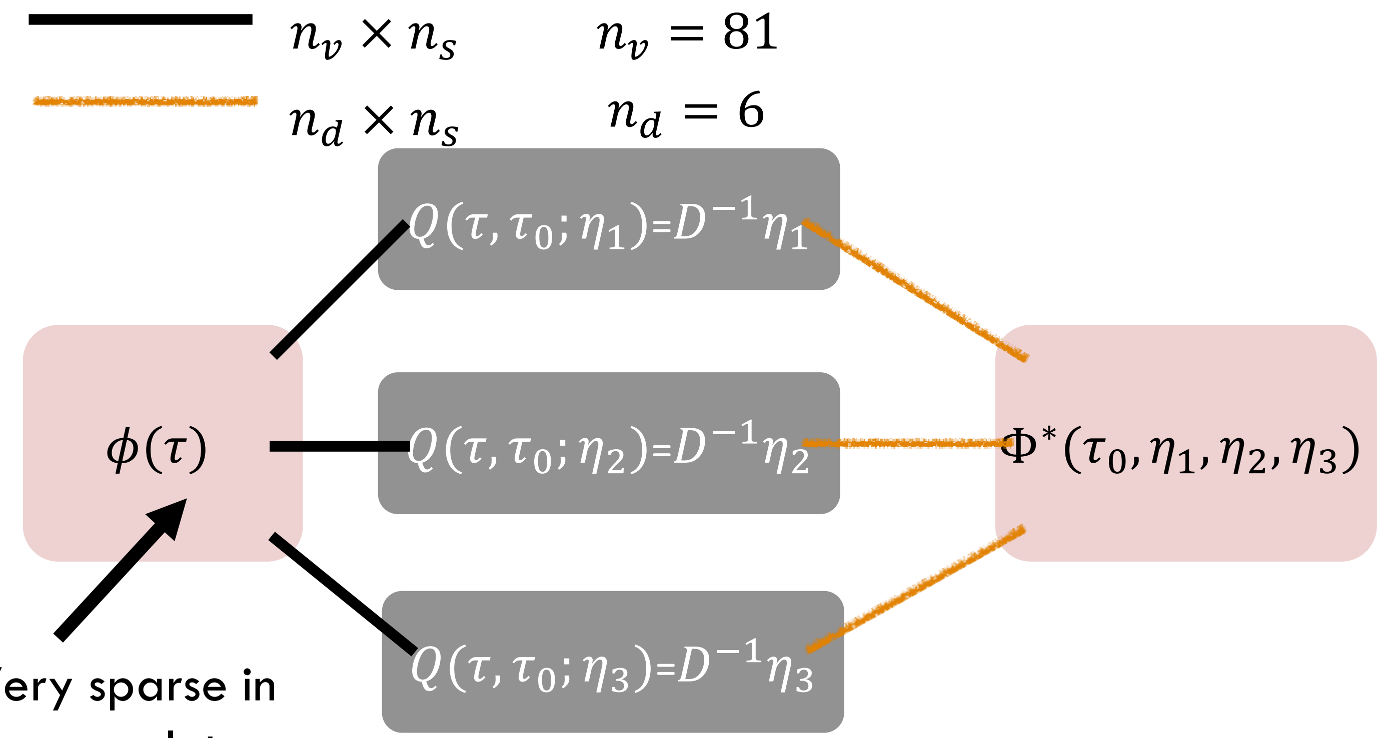
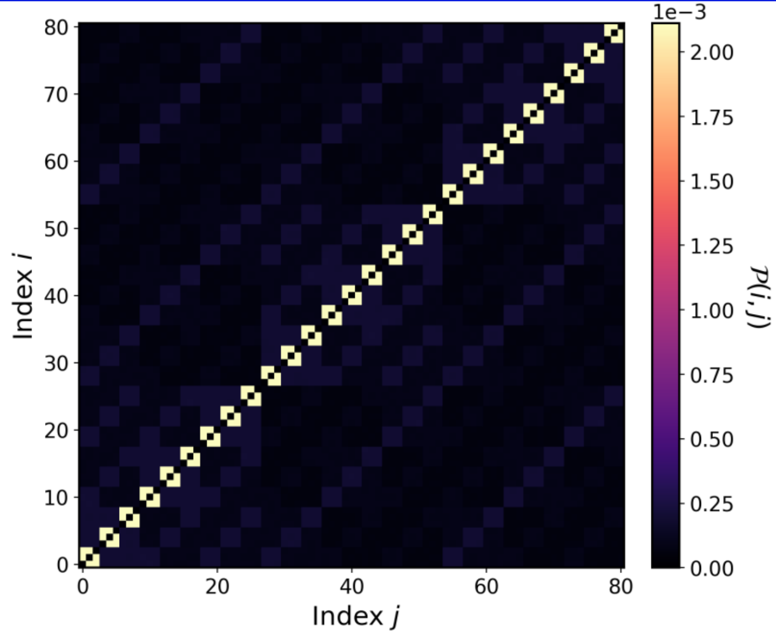
Sparse Acceleration (Hansen-Hurwitz)



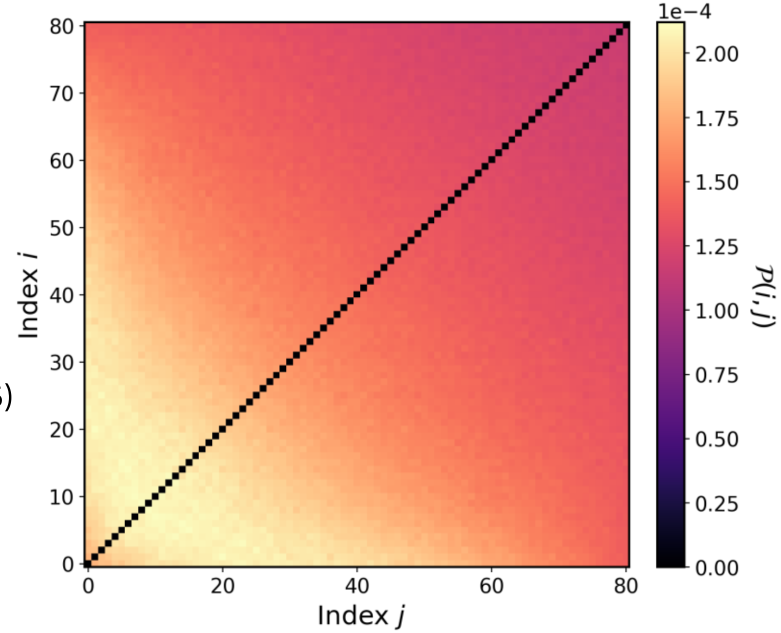
Very sparse in the wavelet bases

$$p_{ijk} = \frac{|\phi_{ijk}|}{\sum_{ijk}|\phi_{ijk}|};\ P_{ij} = \sum_k P_{ijk}$$

Wavelets



Laplace



Lang et al: Phys. Rev. D **111**, 114510 (2025)

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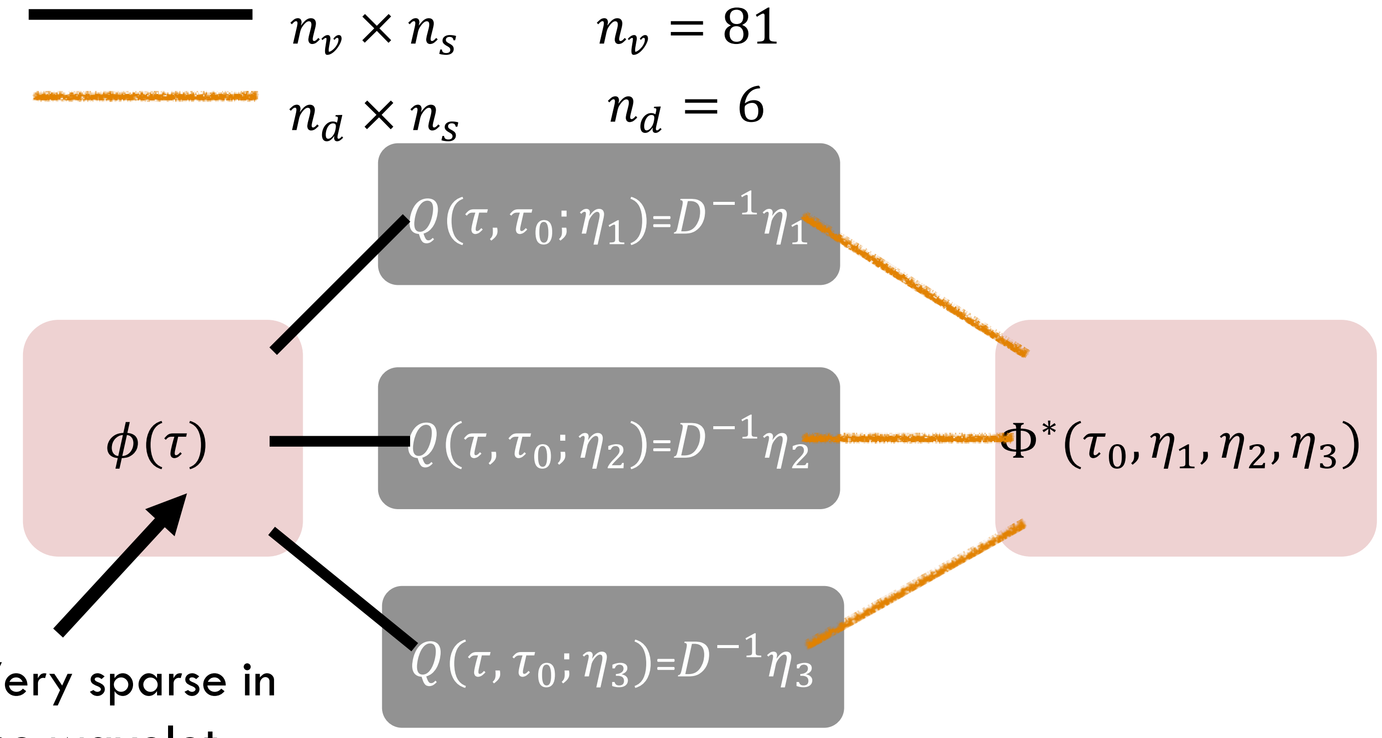

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To leverage the sparsity, we use Hansen-Hurwitz (HH) estimator [1] at the sink,

$$X = \phi_{ijk} A_i B_j C_k$$

Can be estimated as,

$$\hat{X} = \sum_{i,j,k \in \mathcal{Z}} \phi_{ijk} A_i B_j C_k / P_{ijk}$$

Where, $\mathcal{Z}$ is the set of samples generated using the probability density, P_{ijk}.

Using HH estimator at the sink and regular stochastic estimator at the source reduces the contraction time by $\mathcal{O}(n_v/(n_d \cdot n_s))$

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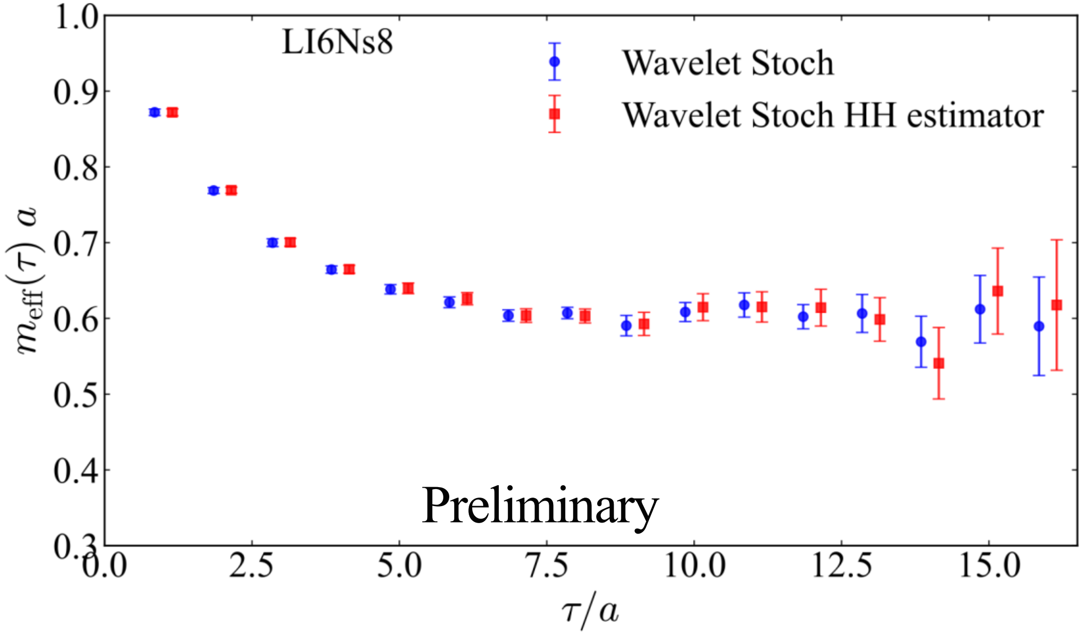

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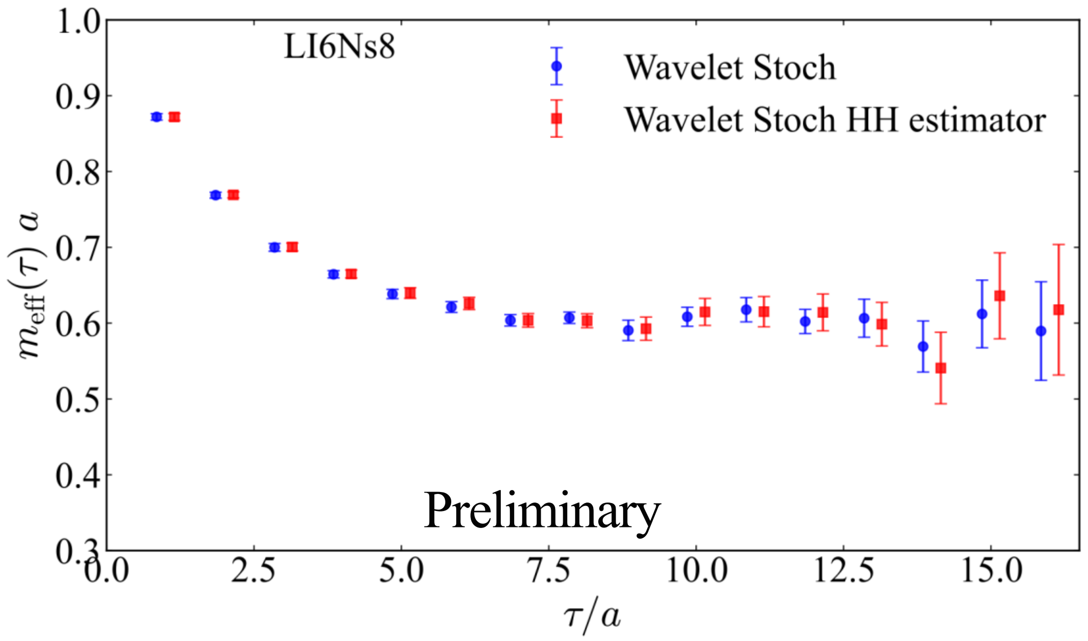

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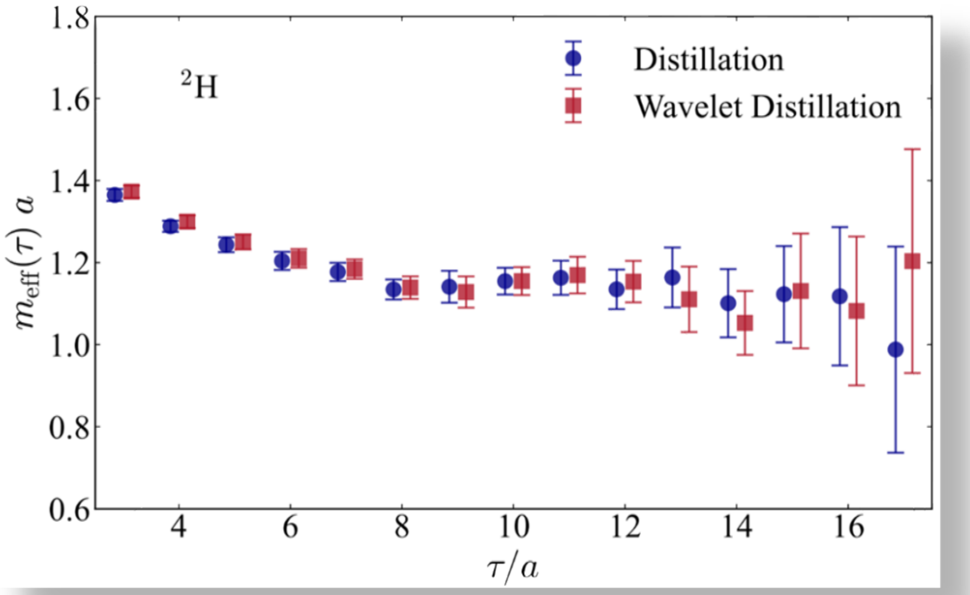
Cost comparison with the same SNR

Method	Basis Construction	Inversion	Wick Contraction	Storage
Regular Distillation	1	1	1	A bottleneck, specifically for multi-hadron systems
Wavelet Distillation (This work)	~ 0.2	1	~ 0.4	No eigenvalues, less storage

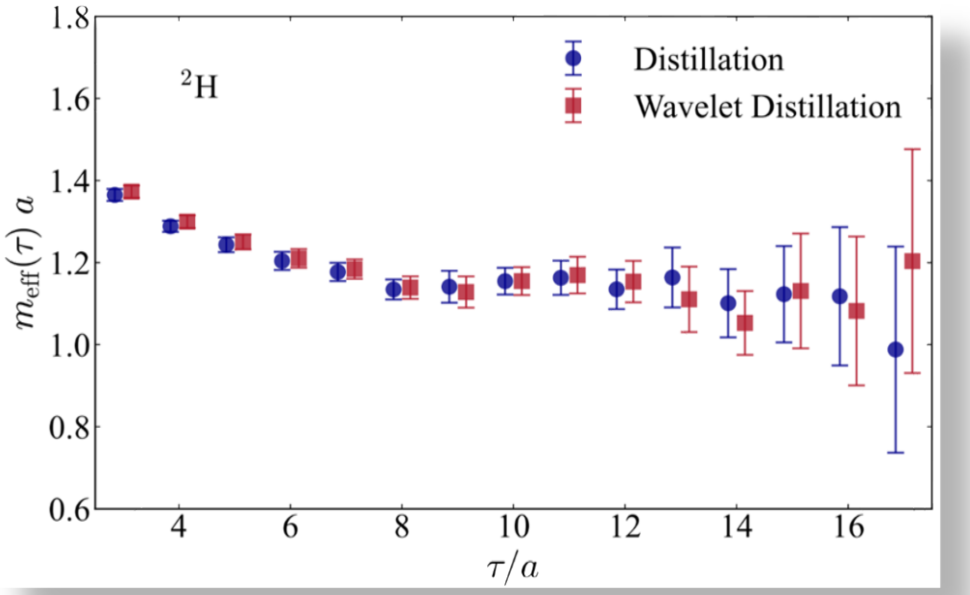
In practice, we observed 2.5 times speed up with HH estimator!

Multi-baryons



Vanilla Vs Wavelets

Multi-baryons



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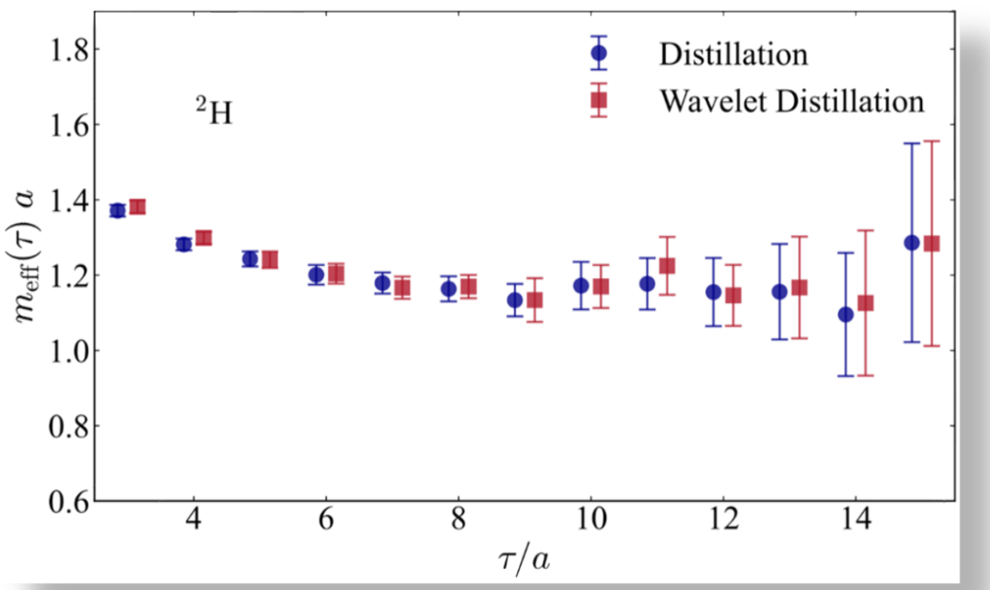
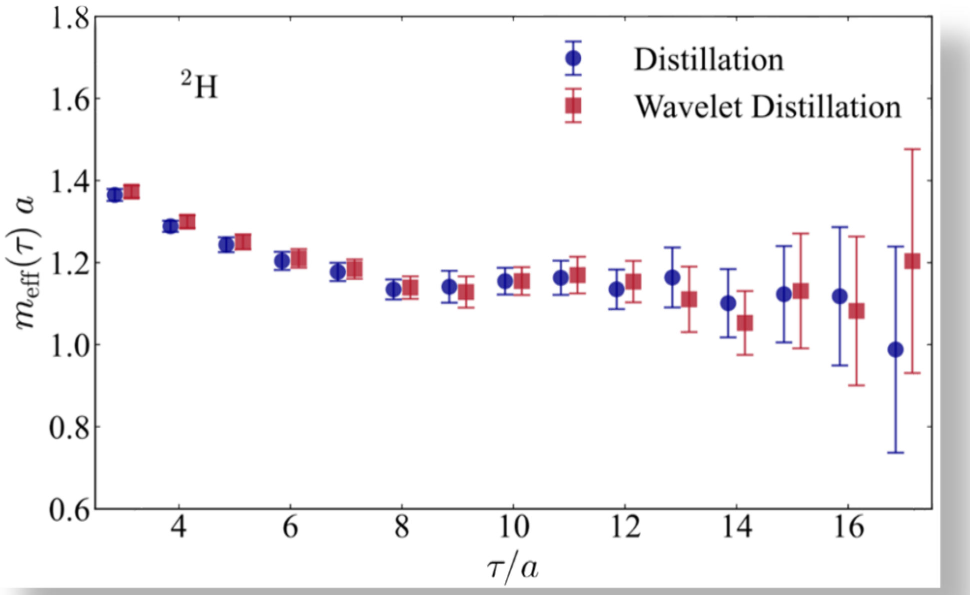


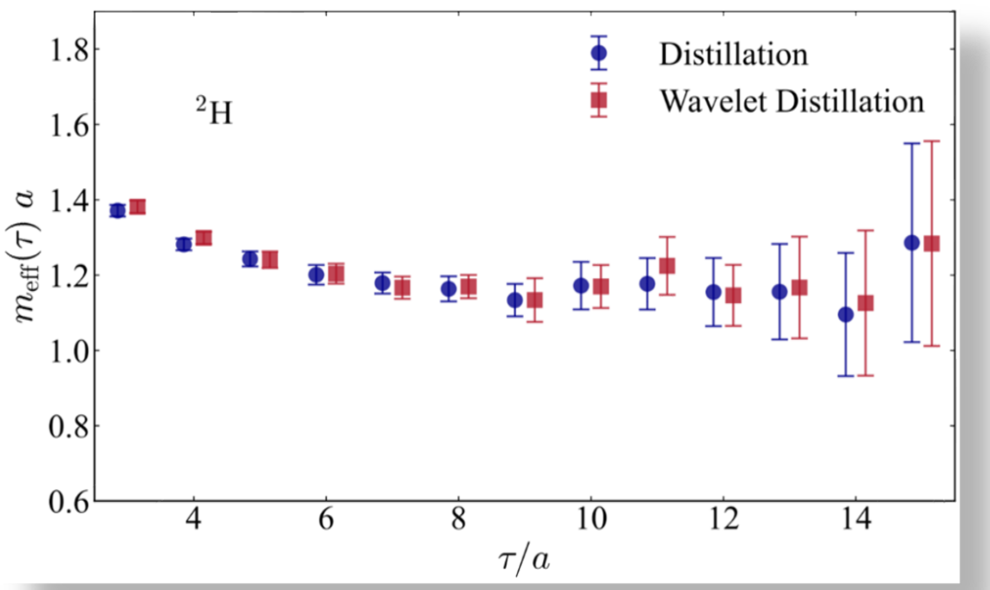
Stochastics: Vanilla Vs Wavelets

Multi-baryons



Vanilla Vs Wavelets

Preliminary



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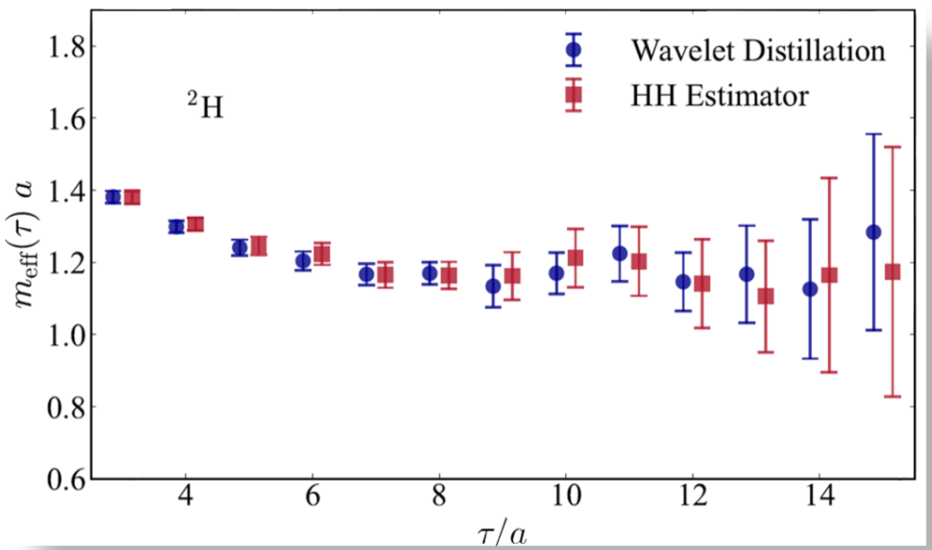
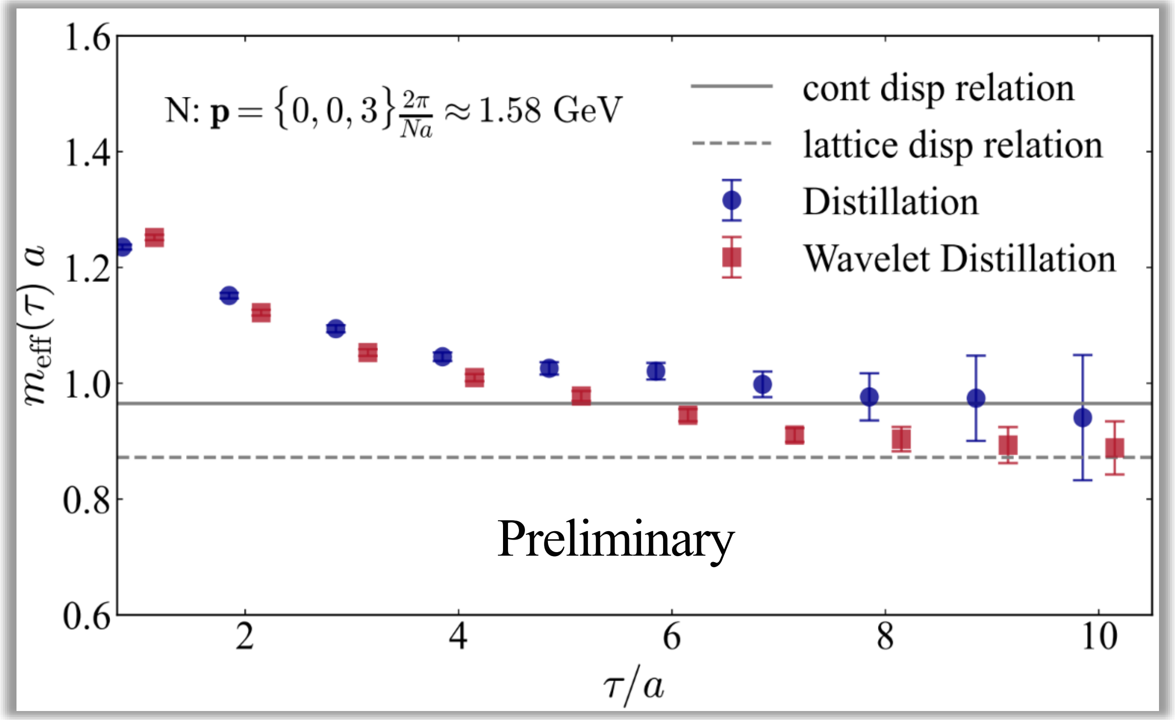


Stochastics: Wavelets (regular) Vs Wavelets (HH)

Momentum Correlators



Momentum phases (Bali'2016) to the gauge transporter $U_{\mathbf{x},\mu} \to U_{\mathbf{x},\mu}\, e^{i\mathbf{k}\cdot\hat{\mu}}$

+

Wavelet basis functions constructed from delta sources on a coarse lattice geometry

More tests are necessary to comprehend the benefits (ongoing)

Summary and Outlook

- A novel method, based on wavelets, is introduced.
- We have demonstrated the method's efficacy and efficiency:
  - For sparsening of quark propagators
  - Within distillation
  - For momentum correlators
  - For multi-hadrons
- Preliminary results show drastic reduction of storage as basis vectors are not necessary to store.
- Substantial reduction in computing timing for Wick-contractions
- More tests for nuclei with distillation is ongoing

Chakraborty, Basak, NM: arXiv:2608(9).xxxxx

Stay tuned!