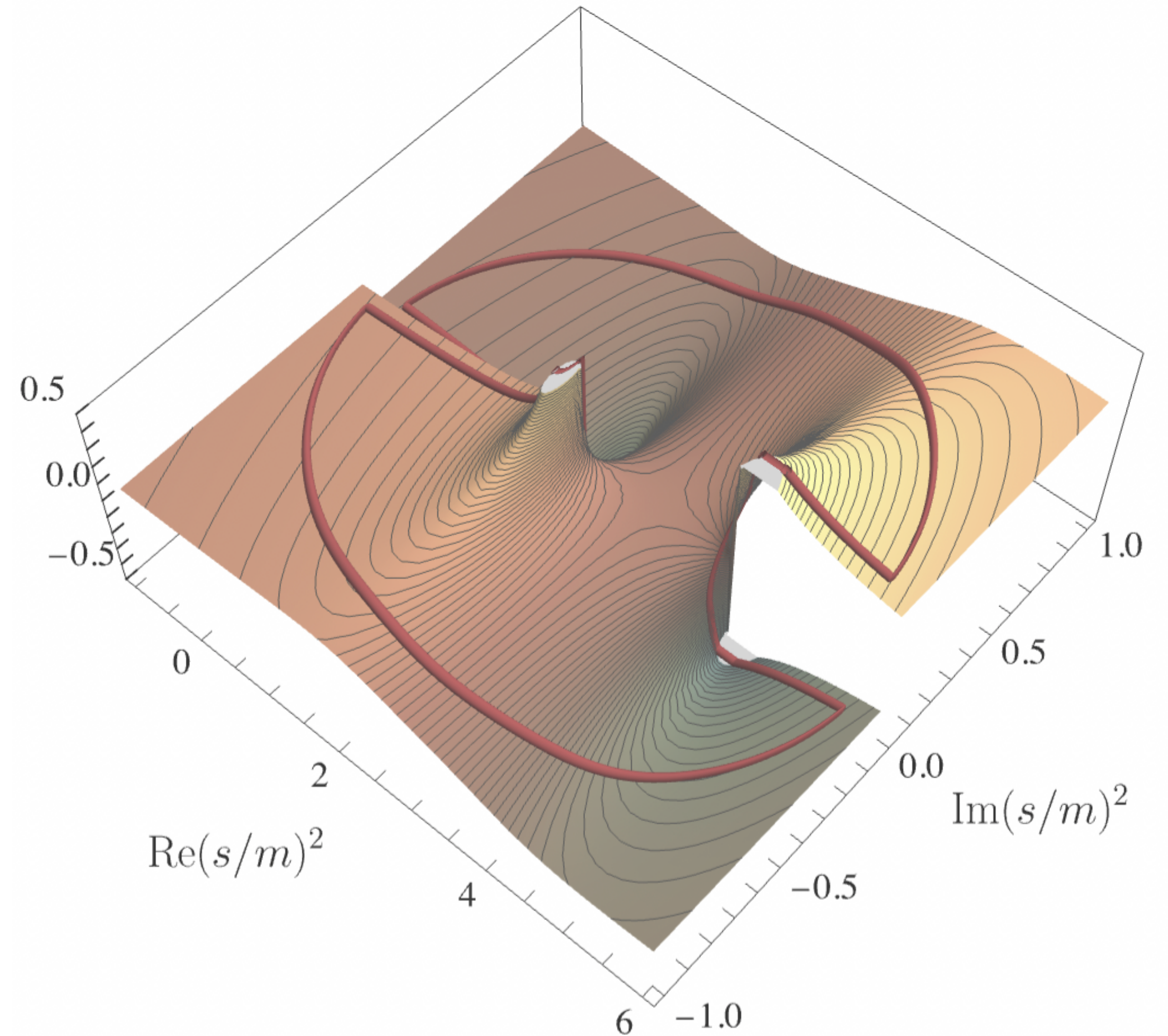


Sebastian M. Dawid

Finite-volume quantization condition from the N/D representation



INDIANA UNIVERSITY
BLOOMINGTON



Lattice 2026, University of Maryland, 07.29.2026

Summary

- 1) Novel two-body quantization condition derived from the dispersive representation of the amplitude
- 2) Applicable for energies below the nearest one- and multi-particle-exchange left-hand branch points
- 3) **Advantage:** conceptually simple separation of singularities, preserves unitarity & analyticity
- 4) **Current limitations:** modeling of the N function & generalization to coupled channels and spins

Relevant literature

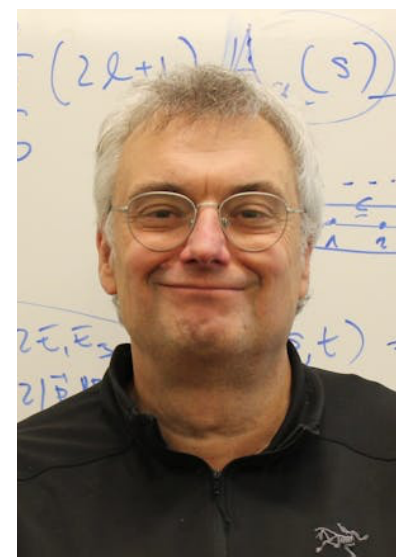
Dawid, Jackura, Szczepaniak
Rodas, Qiu, et al. (JPAC)

Finite-volume quantization condition from the N/D representation

(April 2025)

Finite-volume analysis of the H-dibaryon including left-hand cut effects

(May 2026)



Adam Szczepaniak (IU)



Andrew Jackura (W&M)



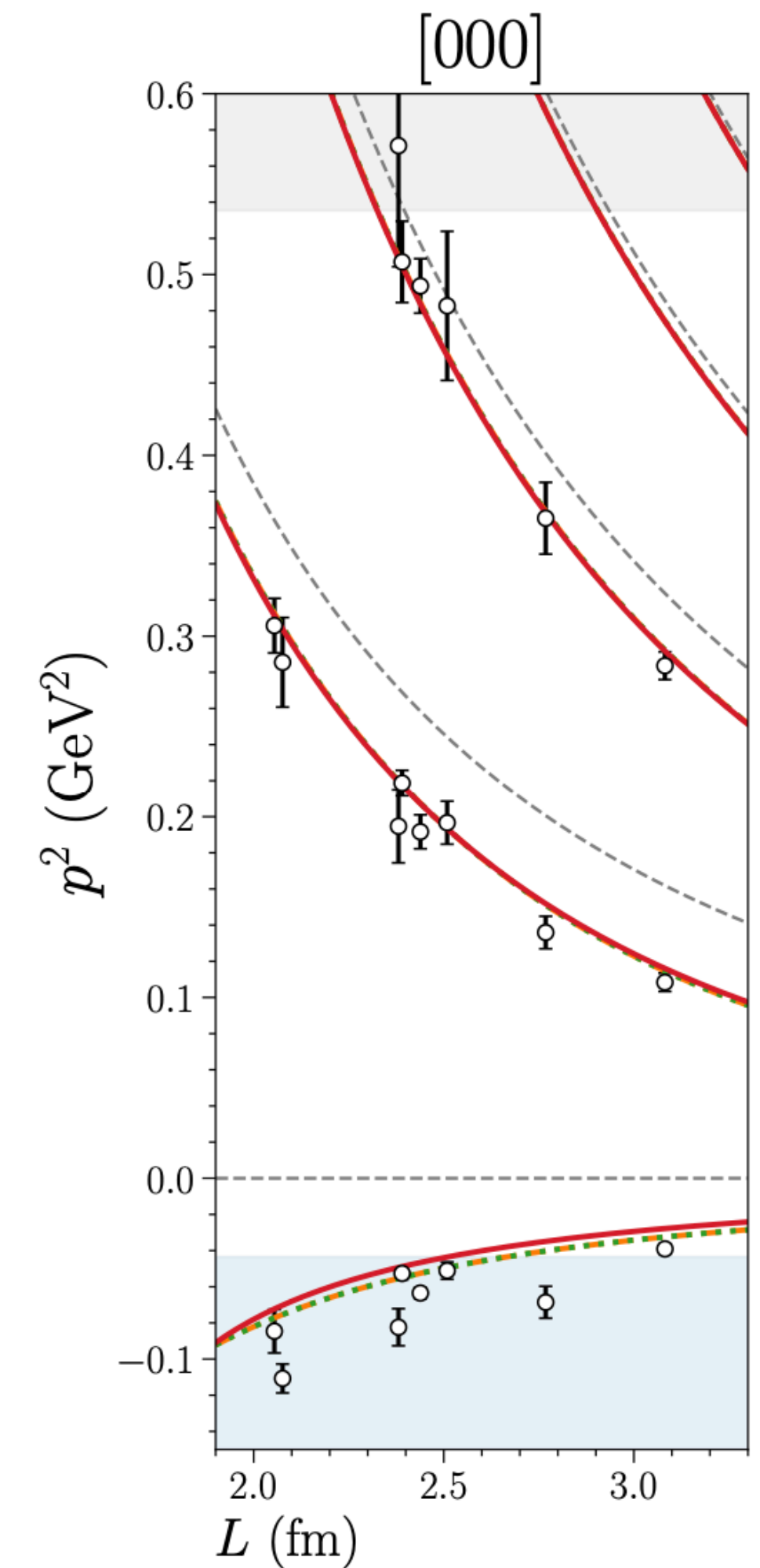
Arkaitz Rodas (ODU)



Lin Qiu (ODU)

Related talks

Lin Qiu on Friday, 11:30, Juan Ramon Jimenez room
Miguel Salg, today, in a moment, right here



The left-hand cut problem

Particle exchanges lead to an additional power-law dependence.
Lüscher-like formalisms break down at corresponding energies.

Related talk Stephen Sharpe, Th, 4:30 p.m., Juan Ramon Jimenez room

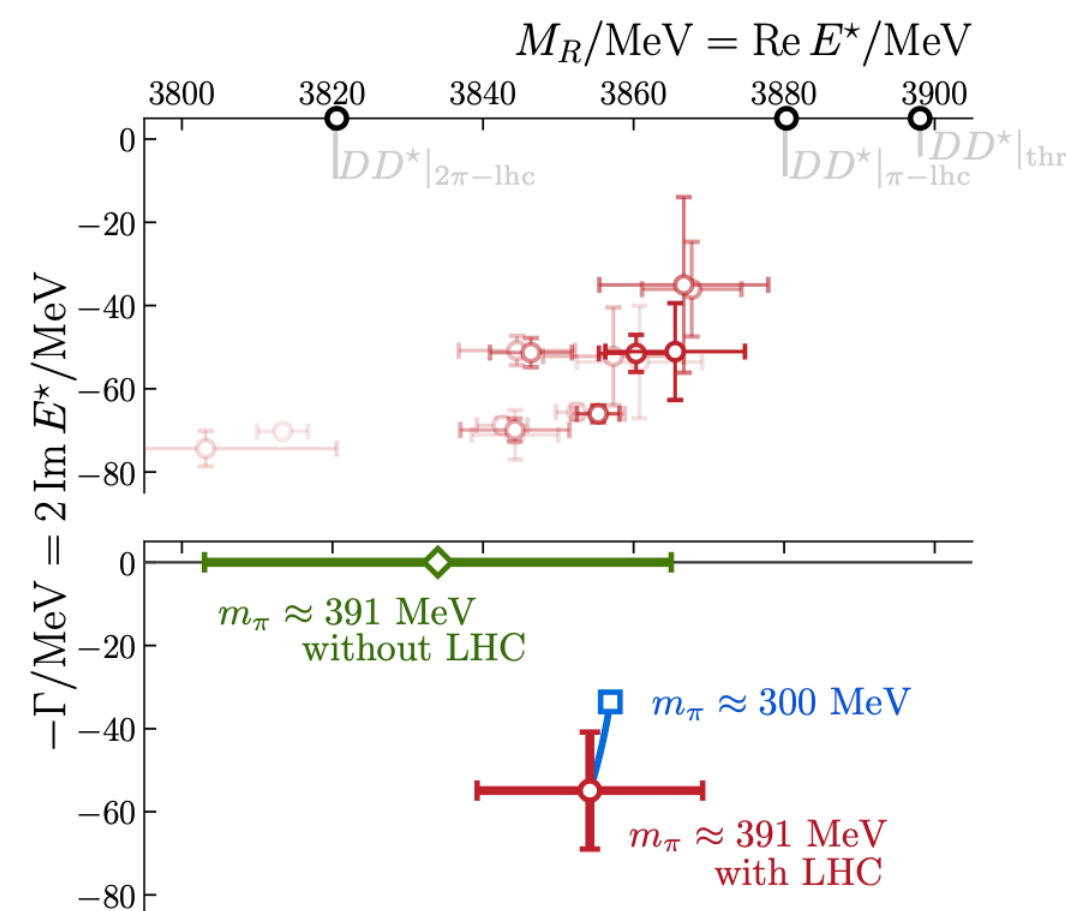
Quantization condition

$$\det_{\ell m_\ell} [\mathcal{K}_2^{-1}(E) + F(E, \mathbf{P}, L)] = 0$$

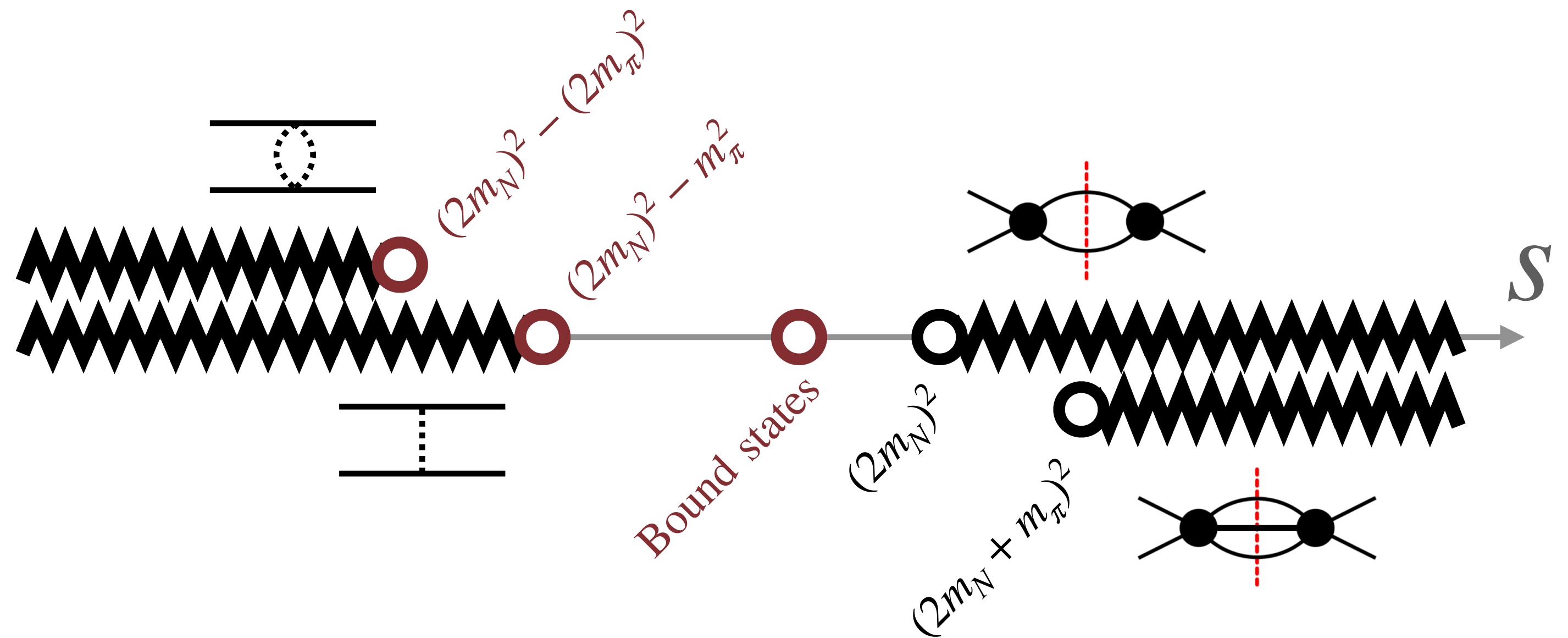
Real matrix in the angular momentum space

Real, known special function

DD* scattering



Whyte, Wilson, Thomas (HadSpec)
 Luchini, Briceño (HadSpec)



The left-hand cut problem

Particle exchanges lead to an additional power-law dependence.
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Related talk Stephen Sharpe, Th, 4:30 p.m., Juan Ramon Jimenez room

Are left-hand cuts important?



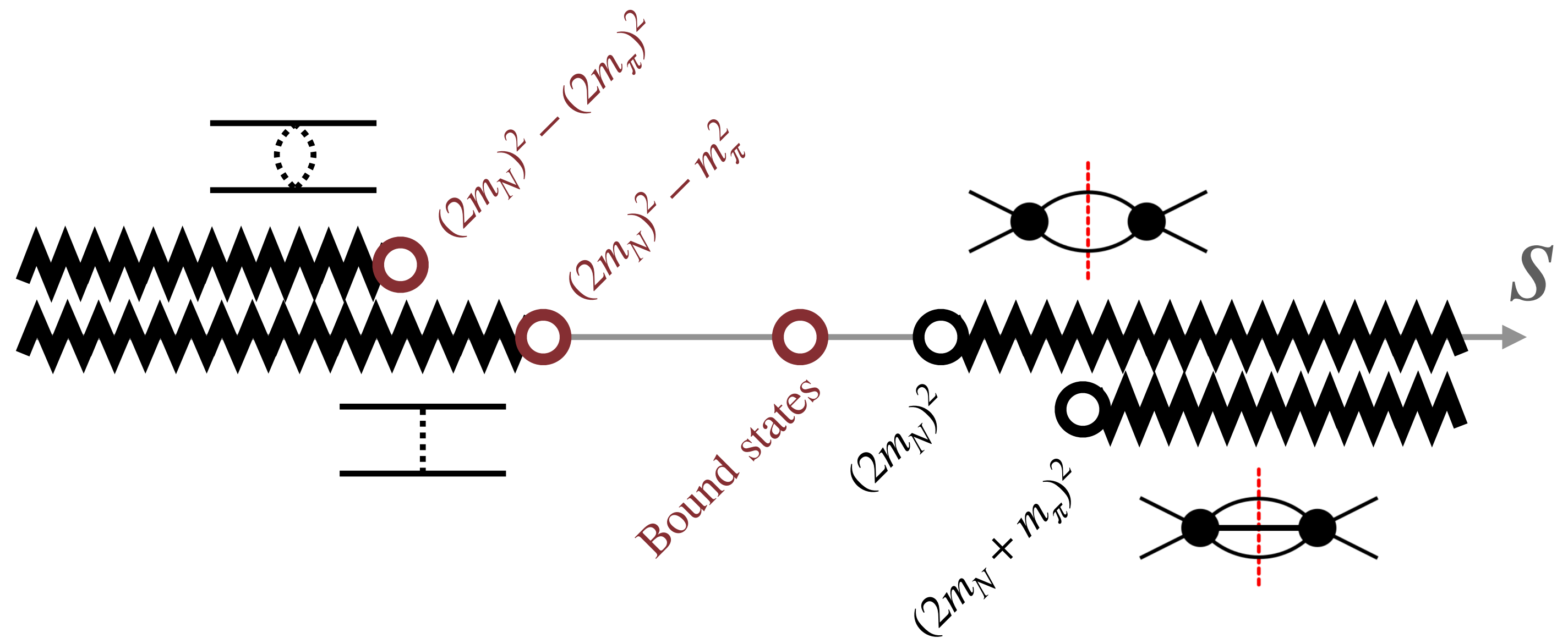
For extra motivation, see an excellent poster by Jorge Baeza Ballesteros

Quantization condition

$$\det_{\ell m_\ell} [\mathcal{K}_2^{-1}(E) + F(E, \mathbf{P}, L)] = 0$$

↑
Real matrix in the angular momentum space

↖
Real, known special function



N/D in infinite volume

Chew & Mandelstam, **Theory of low-energy pion-pion interactions**
Bjorken, **Construction of coupled scattering and production amplitudes...**

Elastic unitarity

$$\text{Im } \mathcal{M}_\ell(s) = \rho(s) |\mathcal{M}_\ell(s)|^2 \Theta(s - s_{\text{thr}}.)$$



$$\begin{aligned} \text{Im } \mathcal{M}_\ell(s)^{-1} &= -\rho(s) \Theta(s - s_{\text{thr}}) \\ \rho(s) &\propto \sqrt{s - s_{\text{thr}}} \end{aligned}$$

K matrix

$$\mathcal{M}_\ell^{-1}(s) = \mathcal{K}_\ell^{-1}(s) - i\rho(s)$$



finite-volume unitarity +

Sato & Bedaque, **Fitting two nucleons inside a box...**

Lüscher QC

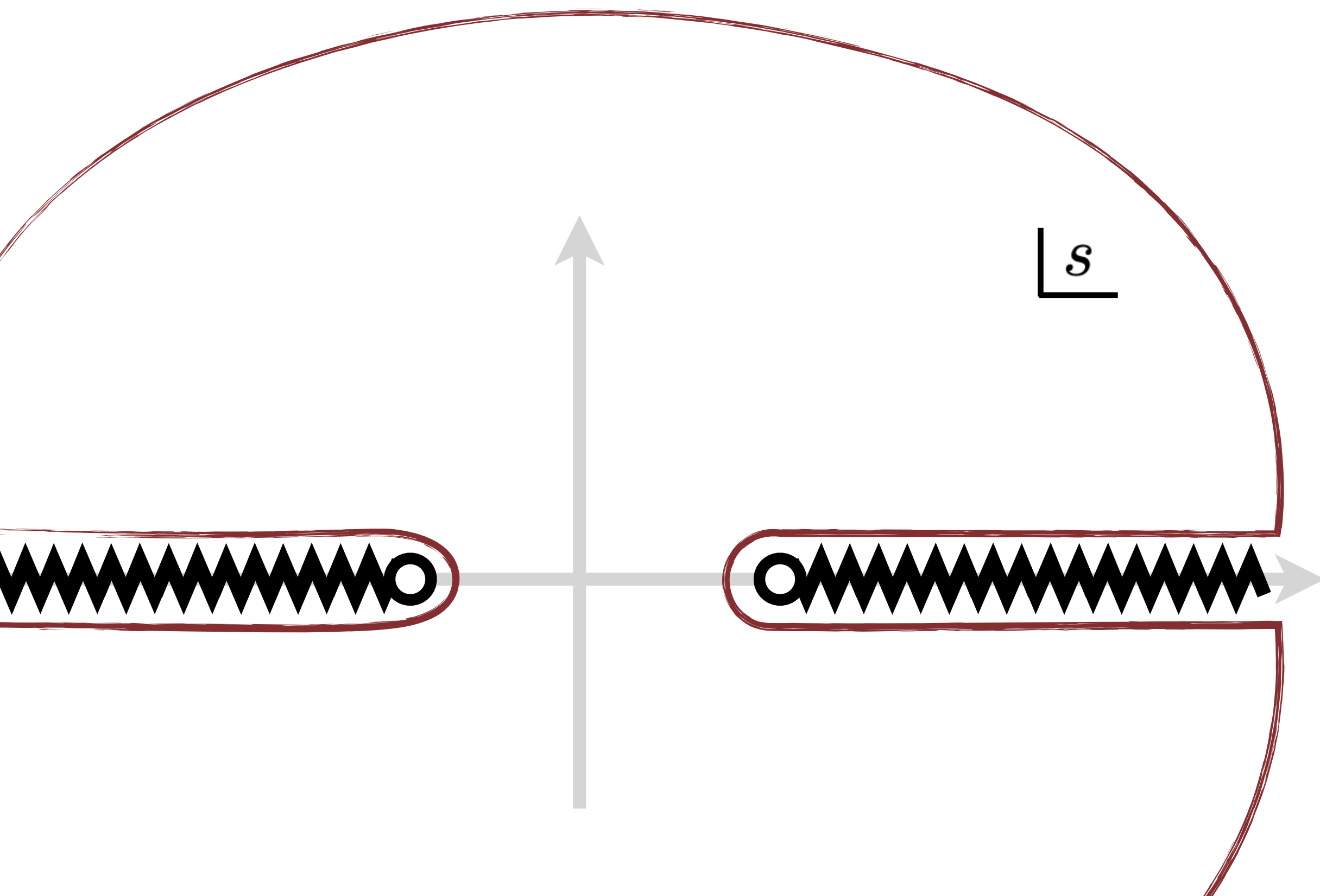
$$\det_{\ell m_\ell} [\mathcal{K}^{-1}(s) + F(s)] = 0$$

N/D in infinite volume

Chew & Mandelstam, **Theory of low-energy pion-pion interactions**
Bjorken, **Construction of coupled scattering and production amplitudes...**

Cauchy theorem
(Riemann-Hilbert problem)

$$\mathcal{M}_\ell(s) = \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{\text{Im } \mathcal{M}_\ell(s')}{s' - s} ds' + \frac{1}{\pi} \int_{s_{\text{thr.}}}^{\infty} \frac{\text{Im } \mathcal{M}_\ell(s')}{s' - s} ds'$$



Right- and left-hand cut inputs

- right-hand discontinuity is fixed by experimental data (unitarity)

$$\text{Im } \mathcal{M}_\ell(s) = \rho(s) |\mathcal{M}_\ell(s)|^2$$

- left-hand discontinuity is modeled as exchanges of particles

$$\mathcal{T}_\ell = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]}$$

The diagrams represent particle exchanges in the t-channel. Diagram 1 shows a vertical dashed line connecting two horizontal lines. Diagram 2 shows a vertical dashed line with a loop on the right. Diagram 3 shows a vertical dashed line with a loop on the left.

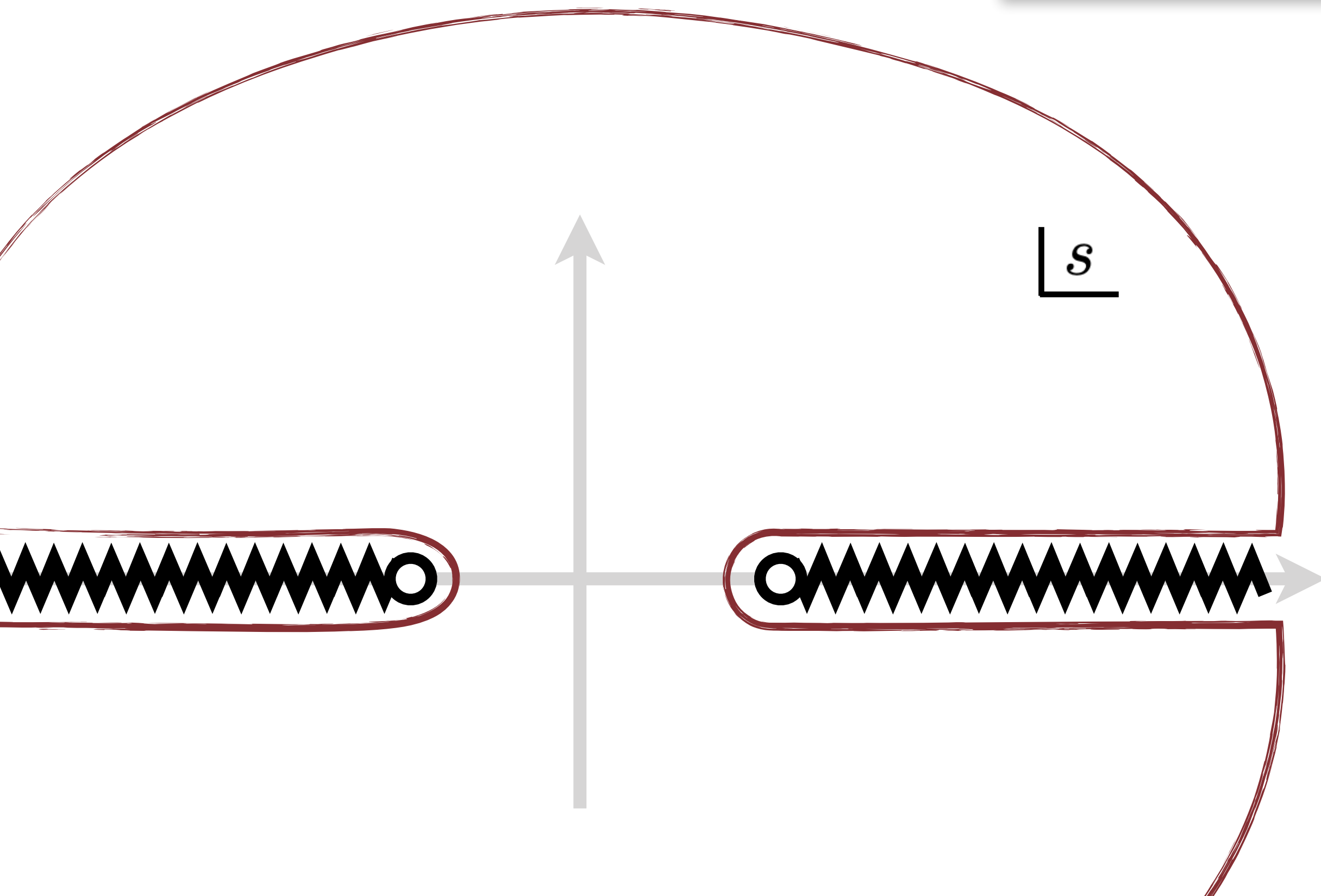
N/D in infinite volume

Chew & Mandelstam, **Theory of low-energy pion-pion interactions**
Bjorken, **Construction of coupled scattering and production amplitudes...**

Wiener-Hopf decomposition

$$\mathcal{M}_\ell(s) = \mathcal{N}_\ell(s) \mathcal{D}_\ell^{-1}(s) \quad \leftarrow \text{contains (only) the right-hand singularities}$$

$\uparrow$
 contains (only) the left-hand singularities



$$\text{Im } \mathcal{D}_\ell(s) = -\rho(s) \mathcal{N}(s) \Theta(s - s_{\text{thr}})$$

$$\mathcal{M}_\ell(s) = \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{\text{Im } \mathcal{M}_\ell(s')}{s' - s} ds' + \frac{1}{\pi} \int_{s_{\text{thr.}}}^{\infty} \frac{\text{Im } \mathcal{M}_\ell(s')}{s' - s} ds'$$

reduced to coupled linear integral equations

$$\mathcal{D}_\ell(s) = 1 - \int_{s_{\text{thr}}}^{\infty} \frac{ds'}{\pi} \frac{\rho(s')}{s' - s - i\epsilon} \mathcal{N}_\ell(s')$$

$$\mathcal{N}_\ell(s) = \int_{-\infty}^{s_{\text{lh.c}}} \frac{ds'}{\pi} \frac{\text{Im } \mathcal{M}_\ell(s')}{s' - s - i\epsilon} \mathcal{D}_\ell(s')$$

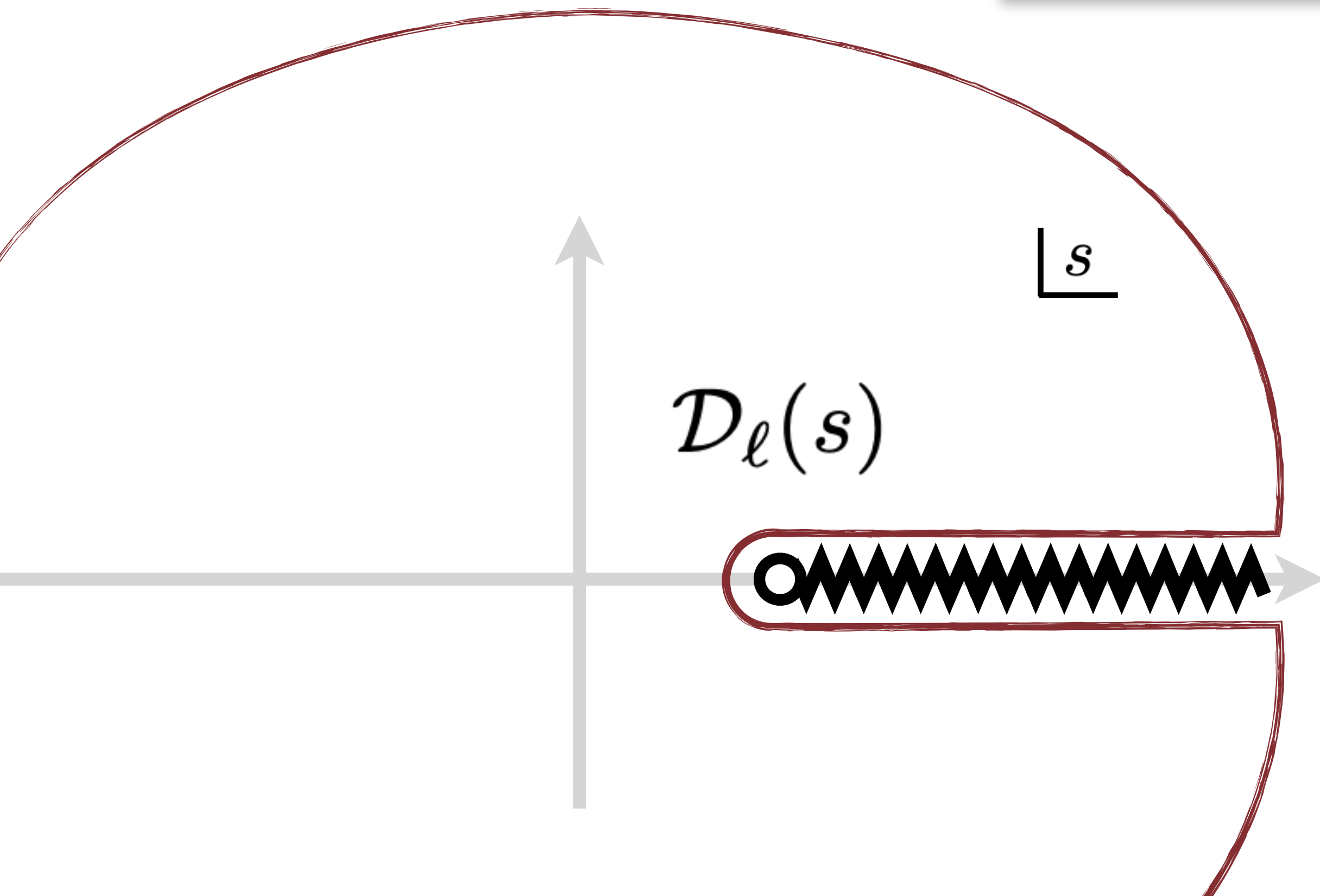
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N/D in finite volume

Dawid, Jackura, Szczepaniak,
Finite-volume quantization condition from the N/D representation

Transition operator in FV

$$T_L(z) = H_{I,L} + H_{I,L} G_L(z) H_{I,L}, \quad G_L(z) = \frac{1}{z - H_L}$$

$$H_L = H_{0,L} + H_{I,L}$$

first resolvent identity

$$G_L(z) - G_L(z^*) = (z - z^*) G_L(z) G_L(z^*)$$

Unitarity in the FV

$$T_L(z) - T_L(z^*) = T_L(z) 2\pi i \delta(z - H_{0,L}) T_L(z^*)$$

$$z = E + i\epsilon$$

unrelated, but note also that

$$T_L(z) - T_L(z^*) = H_{I,L} 2\pi i \delta(z - H_L) H_{I,L}$$

Right-hand singularities

$$2 \operatorname{Im} \mathcal{M}^{-1}(\mathbf{p}', \mathbf{p}; \mathbf{P}) = \frac{1}{L^3} \sum_{\mathbf{k}} \frac{\delta(E - \omega_1(\mathbf{k}) - \omega_2(\mathbf{P} - \mathbf{k}))}{2\omega_1(\mathbf{k})\omega_2(\mathbf{P} - \mathbf{k})}$$

N/D in finite volume

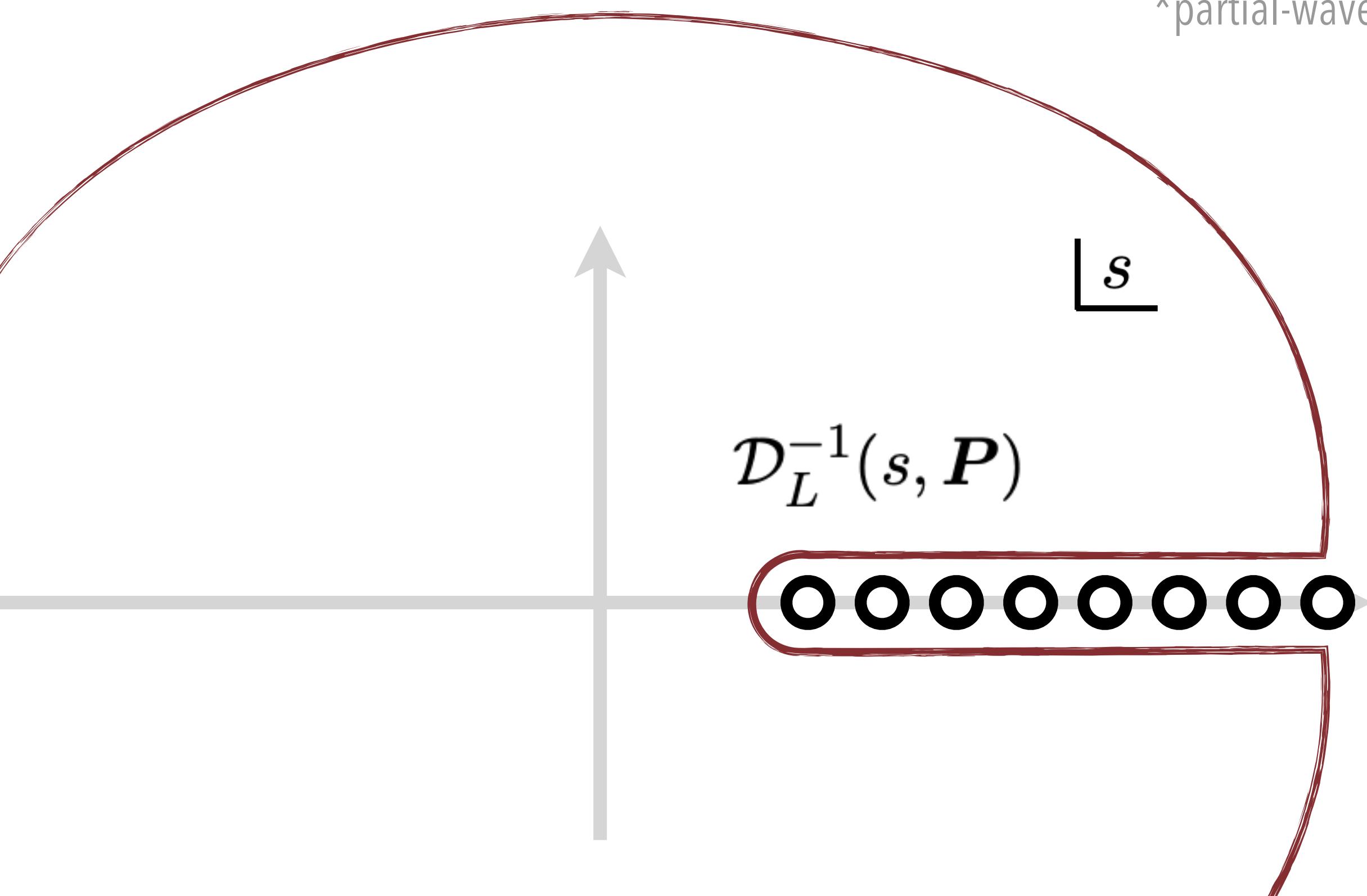
Dawid, Jackura, Szczepaniak,
Finite-volume quantization condition from the N/D representation

Wiener-Hopf decomposition

$$\mathcal{M}_L(s, \mathbf{P}) = \mathcal{N}_L(s, \mathbf{P}) \mathcal{D}_L^{-1}(s, \mathbf{P}) \quad \leftarrow \text{explicit } \mathbf{P} \text{ dependence}$$

↑
matrices in the $(\ell', m'_\ell; \ell, m_\ell)$ space of angular momenta* (box breaks rotational symmetry)

*partial-wave projection of the finite-volume on-shell amplitude has to be defined carefully



recall, in the infinite volume

$$\text{Im } \mathcal{D}_\ell(s) = -\rho(s) \mathcal{N}(s) \Theta(s - s_{\text{thr}})$$

In the finite volume analogous relation holds;
simplified S-wave case yields (full version in the article)

$$\text{Im } \mathcal{D}_L(s, \mathbf{P}) = -\frac{1}{L^3} \sum_{\mathbf{k}} \frac{2s}{E} \frac{\delta(s - E^*(\mathbf{k})^2)}{2\omega_1(\mathbf{k})2\omega_2(\mathbf{P} - \mathbf{k})} \mathcal{N}_L(s, \mathbf{P})$$

pick the Pac-Man contour and integrate over complex s !

N/D quantization condition

Dawid, Jackura, Szczepaniak,
Finite-volume quantization condition from the N/D representation

Finite-volume D function

$$\mathcal{D}_L(s, \mathbf{P}) = \mathbb{1} + \frac{1}{L^3} \sum_{\mathbf{k}} \frac{\overbrace{\mathcal{L}(\mathbf{k}, \mathbf{P})}^{\text{kinematic normalization factor}}}{s - \underbrace{E^*(\mathbf{k})^2}_{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}} \mathcal{N}_L(E^*(\mathbf{k})^2, \mathbf{P})$$

- Here I show the simplified S-wave case; full version is found in the article
- FV N function above can be replaced by the IV N up to exp-suppressed corrections
- This is the unsubtracted version of D ; subtractions might be necessary to ensure convergence
- The formalism may incorporate “bare” bound-state poles or CDD poles

Quantization condition

$$\det_{\ell m_\ell} \mathcal{D}_L = 0$$

Finite-volume energy levels

$$\{E_n\}$$

IV numerator function

$$\mathcal{N}_\ell(s)$$

N/D quantization condition

Dawid, Jackura, Szczepaniak,
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$$\mathcal{D}_L(s, \mathbf{P}) = \mathbb{1} + \frac{1}{L^3} \sum_{\mathbf{k}} \frac{\mathcal{L}(\mathbf{k}, \mathbf{P})}{s - E^*(\mathbf{k})^2} \mathcal{N}(E^*(\mathbf{k})^2)$$

kinematic normalization factor

$\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)$

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$\uparrow$ kinematic normalization factor
 $\xrightarrow{\hspace{1.5cm}}$ $\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)$

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Finite-volume energy levels

$$\{E_n\}$$

IV numerator function

$$\mathcal{N}_\ell(s)$$

$$\mathcal{M}_\ell(s) = \mathcal{N}_\ell(s) \mathcal{D}_\ell^{-1}(s)$$

$$\mathcal{D}_\ell(s) = 1 - \int_{s_{\text{thr}}}^{\infty} \frac{ds'}{\pi} \frac{\rho(s')}{s' - s - i\epsilon} \mathcal{N}_\ell(s')$$

Caveats

Dawid, Jackura, Szczepaniak,
Finite-volume quantization condition from the N/D representation

❖ What does it mean to model the N function?

$$\mathcal{N}_\ell(s) = \int_{-\infty}^{s_{\text{lh}}} \frac{ds'}{\pi} \frac{\text{Im } \mathcal{M}_\ell(s')}{s' - s - i\epsilon} \mathcal{D}_\ell(s')$$

Example

Left-hand singularity given by a left-hand pole at s_0

$$\mathcal{N}_\ell(s) = \frac{g\mathcal{D}_\ell(s_0)}{s - s_0} \Rightarrow \mathcal{M}_\ell^{-1}(s) = -\frac{s - s_0}{g} + \mathcal{I}_{\text{CM}}(s)$$

Example

N obtained from the dispersive integral equation

$$\mathcal{N}_\ell(s) = \mathcal{T}(s) + \int_{s_{\text{thr}}}^{\infty} \frac{ds'}{\pi} \frac{\rho_\ell(s') [\mathcal{T}_\ell(s') - \mathcal{T}_\ell(s)]}{s' - s - i\epsilon} \mathcal{N}_\ell(s')$$

❖ Can the FV N/D incorporate coupled channels and spins?

Currently not.

Bjorken, Construction of coupled scattering and production amplitudes...
Bjorken, Nauenberger, Symmetry of N/D solutions for coupled scattering...

$$\mathcal{M} = \mathcal{N} \mathcal{D}^{-1}$$

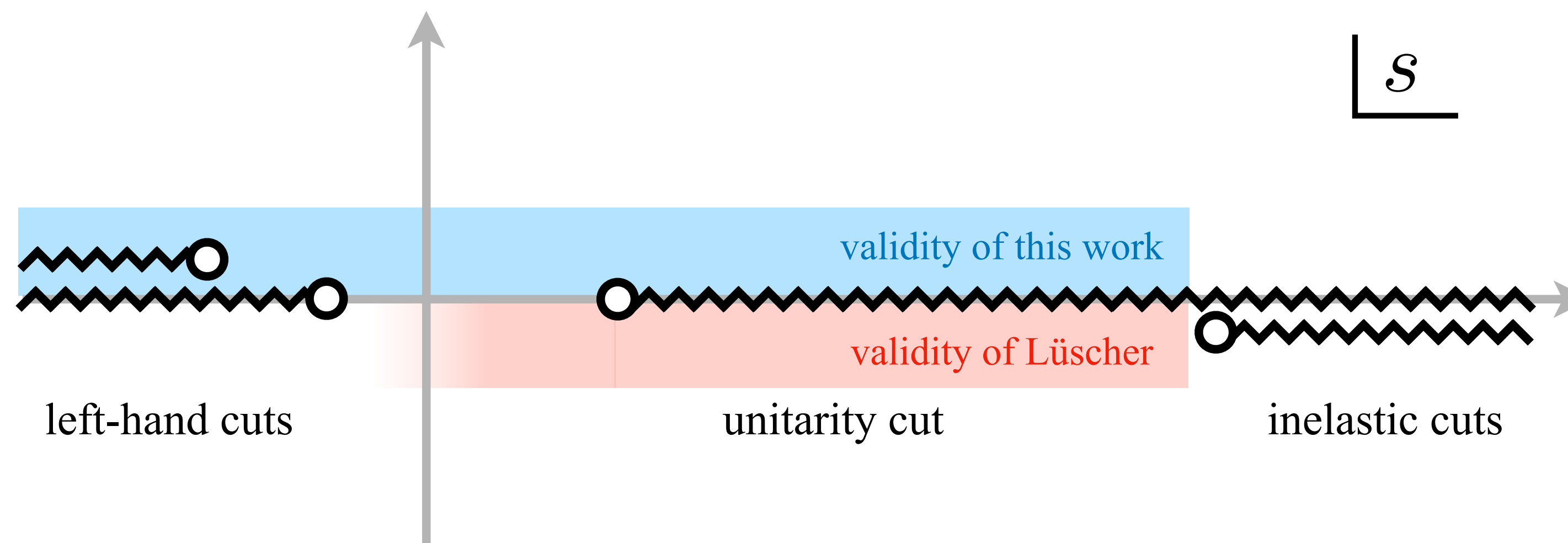
become matrices in the channel and partial-wave space

$$\text{sym. Im } \mathcal{M} \Rightarrow \text{sym. } \mathcal{M}$$

symmetric discontinuities imply symmetric N/D amplitude

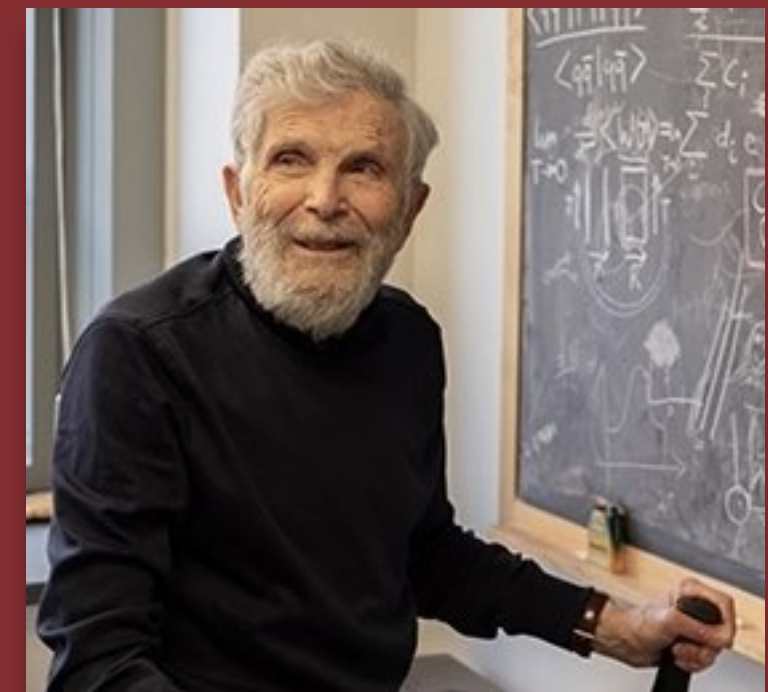
Summary

- N/D separates crossed-channel from direct-channel singularities, also in the finite volume
- Finite-volume energies are determined by the condition $\det[D_L] = 0$ and $N = N_L + O(\exp(-\Delta L))$
- Amplitude is reconstructed by solving the dispersive N/D equations
- Future work: tests & implementation of the formalism, generalization to coupled channels and spins



Thank you!

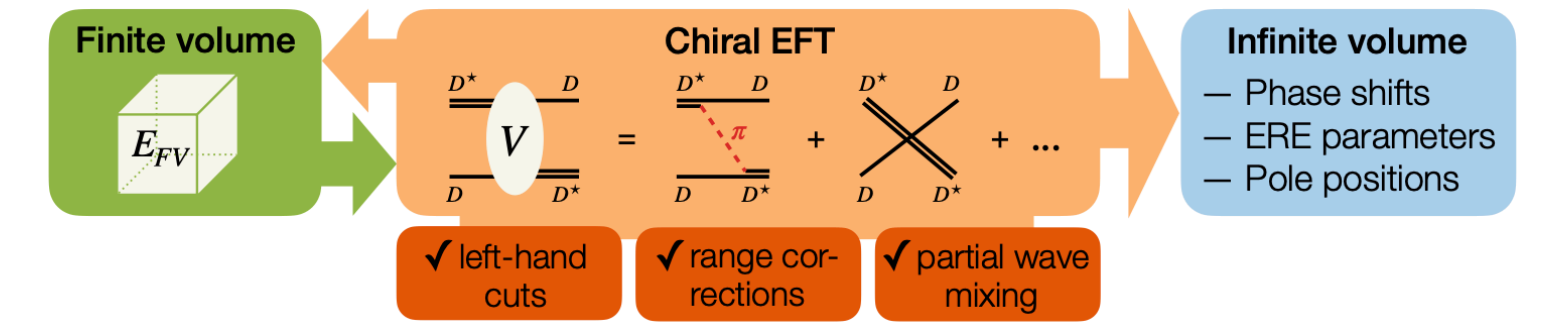
This talk is dedicated to the memory of Marshall Baker (1932-2026), who, together with Schwinger, wrote the first nonperturbative FV QC.



Available LHC approaches

- **Plane-wave basis approach** [[Meng & Eppelbaum \(2021\)](#) and [Meng et al. \(2024\)](#)]
 - ❖ Based on the Lippmann-Schwinger equation with chiral EFT potentials and their LECs
 - ❖ Evaluation of the determinant condition in the plane-wave basis
- **Two-body RFT approach** [[Baião Raposo & Hansen \(2024\)](#) and [Baião Raposo et al. \(2025\)](#)]
 - ❖ Based on diagrammatic expansion of the Bethe-Salpeter kernel with OPEs singled out
 - ❖ The amplitude is split into a ladder and a "short-range" part with an unknown K matrix
- **Two-body NREFT approach** [[Bubna et al. \(2024\)](#) and [Bubna et al. \(2025\)](#)]
 - ❖ Combines best features of the two above approaches in the NR setting
- **N/D quantization condition** [[Dawid, Jackura, Szczepaniak \(2025\)](#)]
 - ❖ Well, unless something went wrong, this talk was about this paper...
- **Three-body RFT approach** [[Hansen, Romero-López, Sharpe \(2024\)](#)]
 - ❖ Based on the LSZ reduction of the three-body amplitude fixed by three-body QC
 - ❖ Relies on analytic continuation of the physical one-particle exchange to the virtual region

$$\det_{k,\ell,m} [\mathbb{1} - \mathcal{K}_3(E^*) \mathbf{F}_3(E, \mathbf{P}, L)] = 0$$

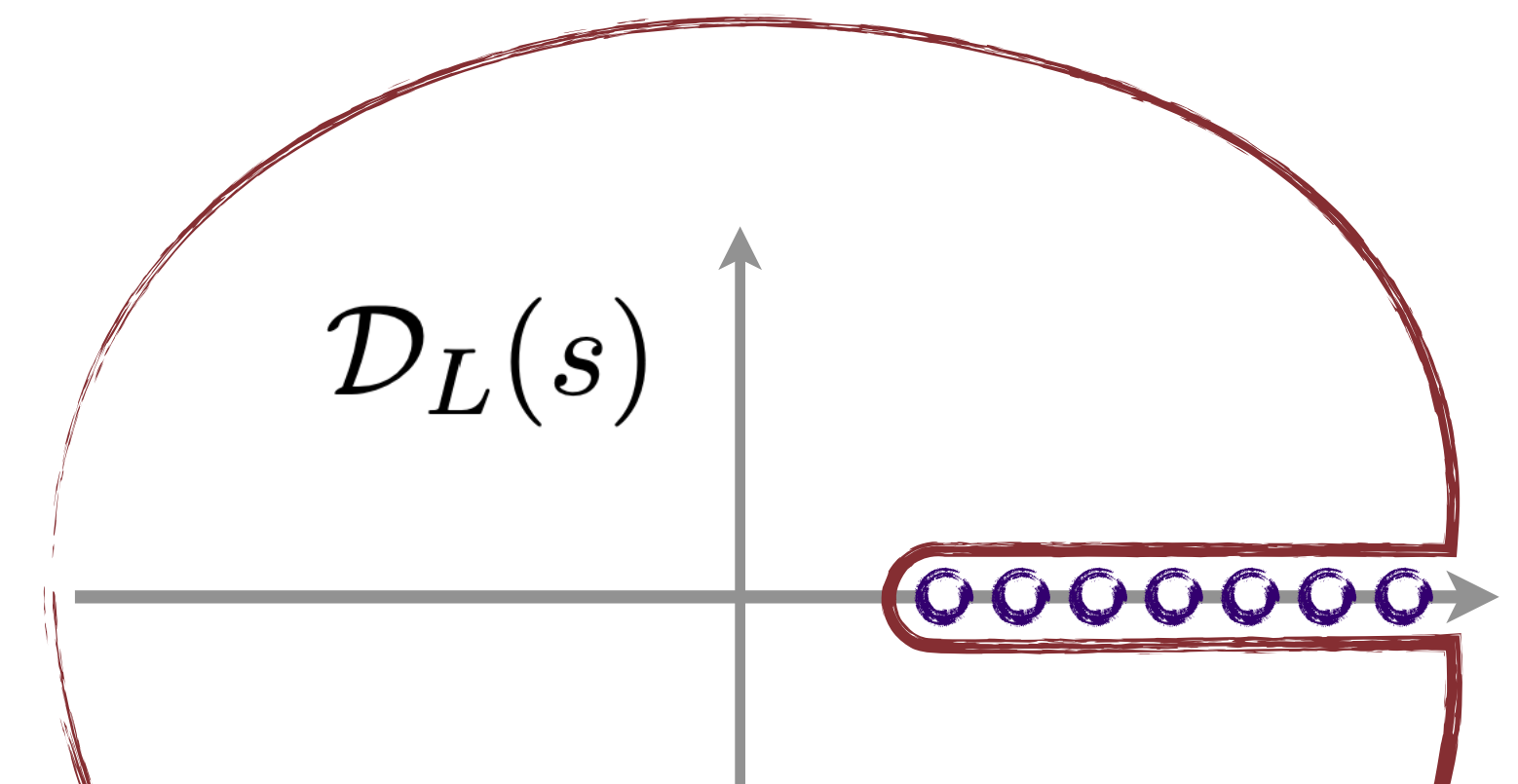


$$\mathbb{T}(E) = \mathbb{V}(E) + \mathbb{V}(E)\mathbb{G}(E)\mathbb{T}(E)$$

$$\mathcal{C}_L \sim \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

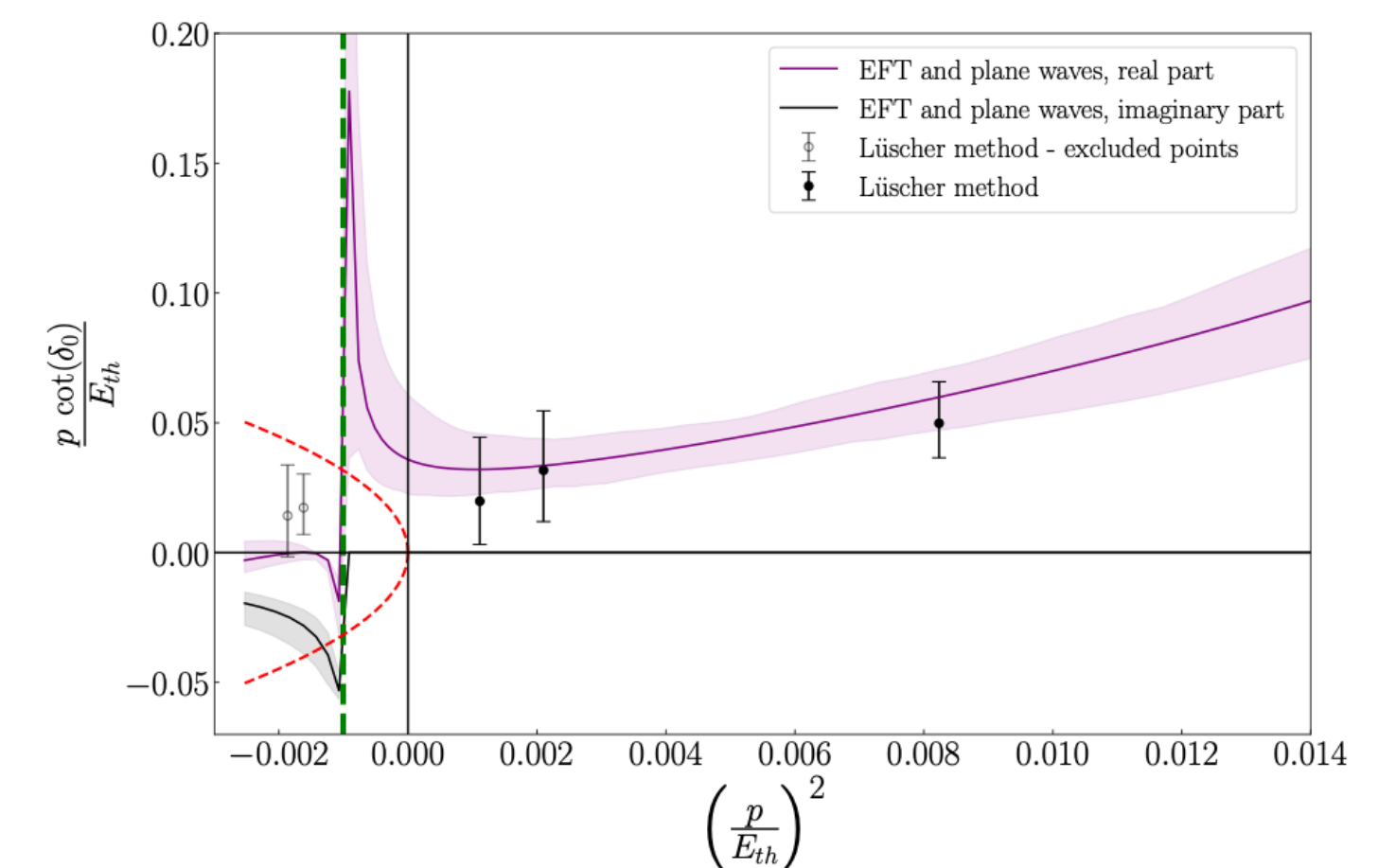
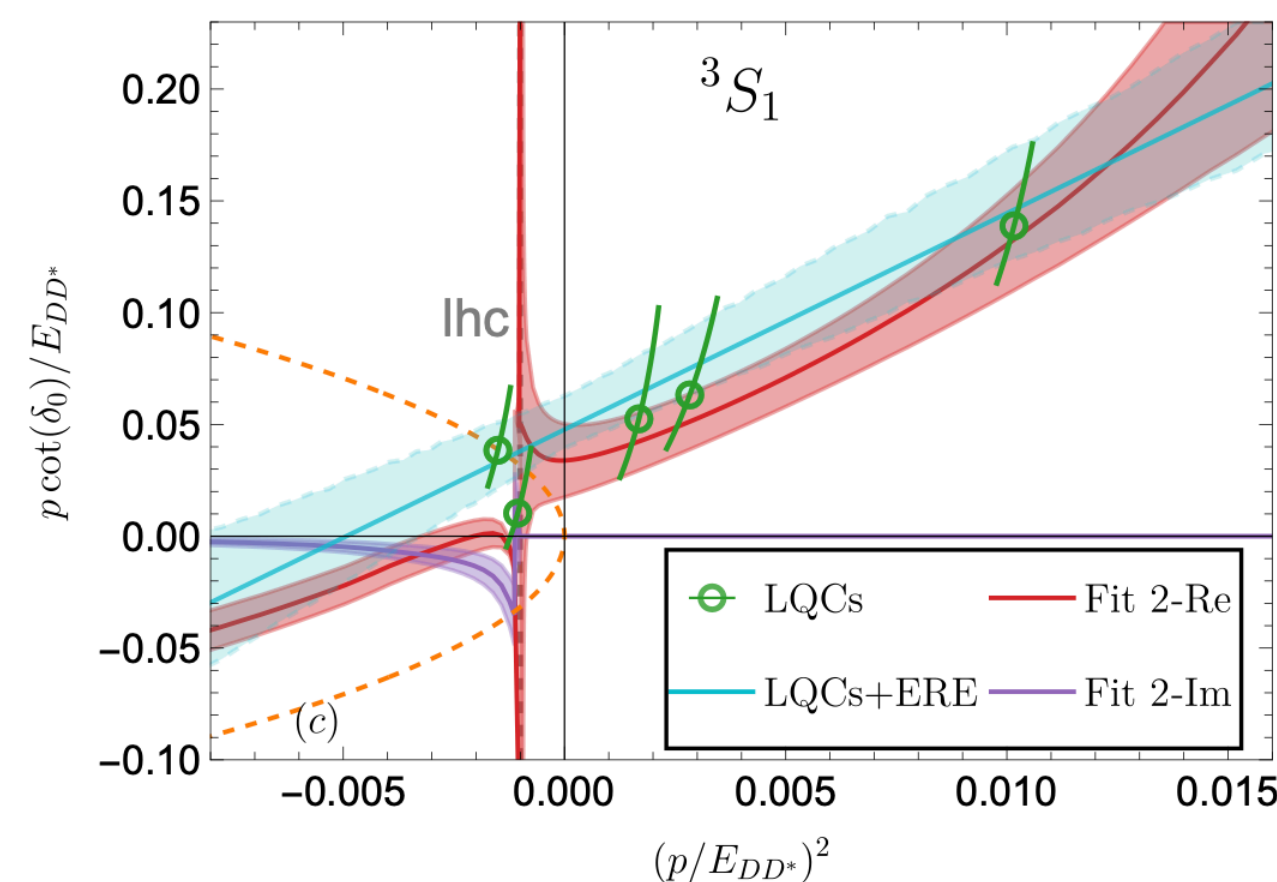
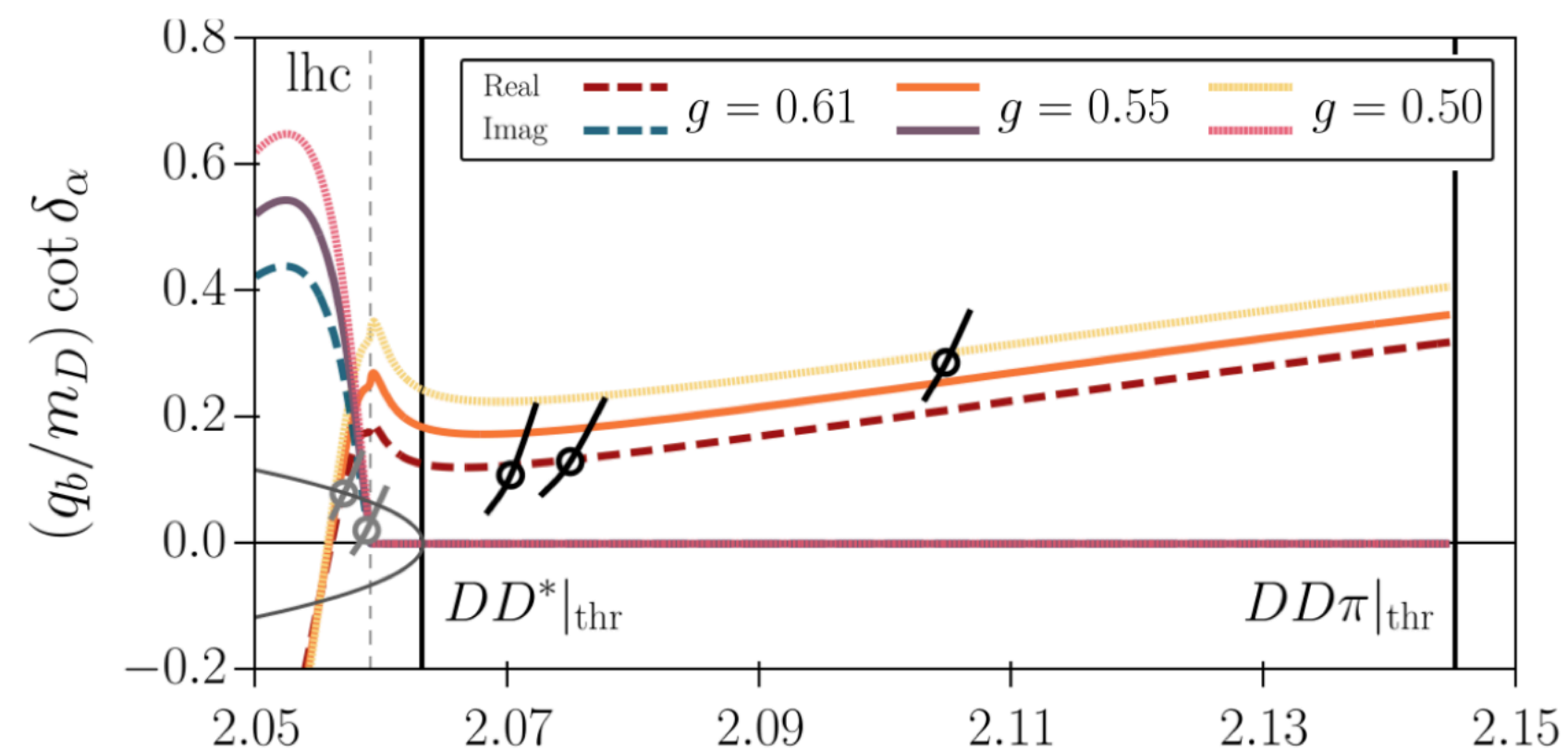
$$\det_{aJm_J\ell S} [\mathcal{M}_0^{-1} + F + \mathcal{C}_L] = 0$$

$$\mathcal{D}_L(s) = 1 + \frac{1}{L^3} \sum_{\mathbf{k}} \frac{1}{s - E^*(\mathbf{k})^2} \mathcal{R}(\mathbf{k}) \mathcal{N}(E^*(\mathbf{k}))$$



Selected LHC applications

- Lachini & Briceño (HadSpec), Resolving the Tcc with pi exchange from Lattice QCD
- Rodas et al. (JPAC), Finite-volume analysis of the H-dibaryon including left-hand cut effects
- Dawid, Romero-López, Sharpe, Finite- and infinite-volume study of DDpi scattering
- Abolnikov et al. Shallow Tbc states from an EFT analysis of B*D* scattering on the lattice
- Abolnikov et al. Internal structure of Tcc from its light-quark mass dependence
- Meng et al. Solving the left-hand cut problem in lattice QCD: Tcc from finite-volume energy levels
- Shirmal et al. Lattice study of ccus tetraquark channel in D*D* scattering
- Prelovsek et al. Doubly heavy tetraquarks from lattice QCD: Incorporating diquark-antidiquark operators...

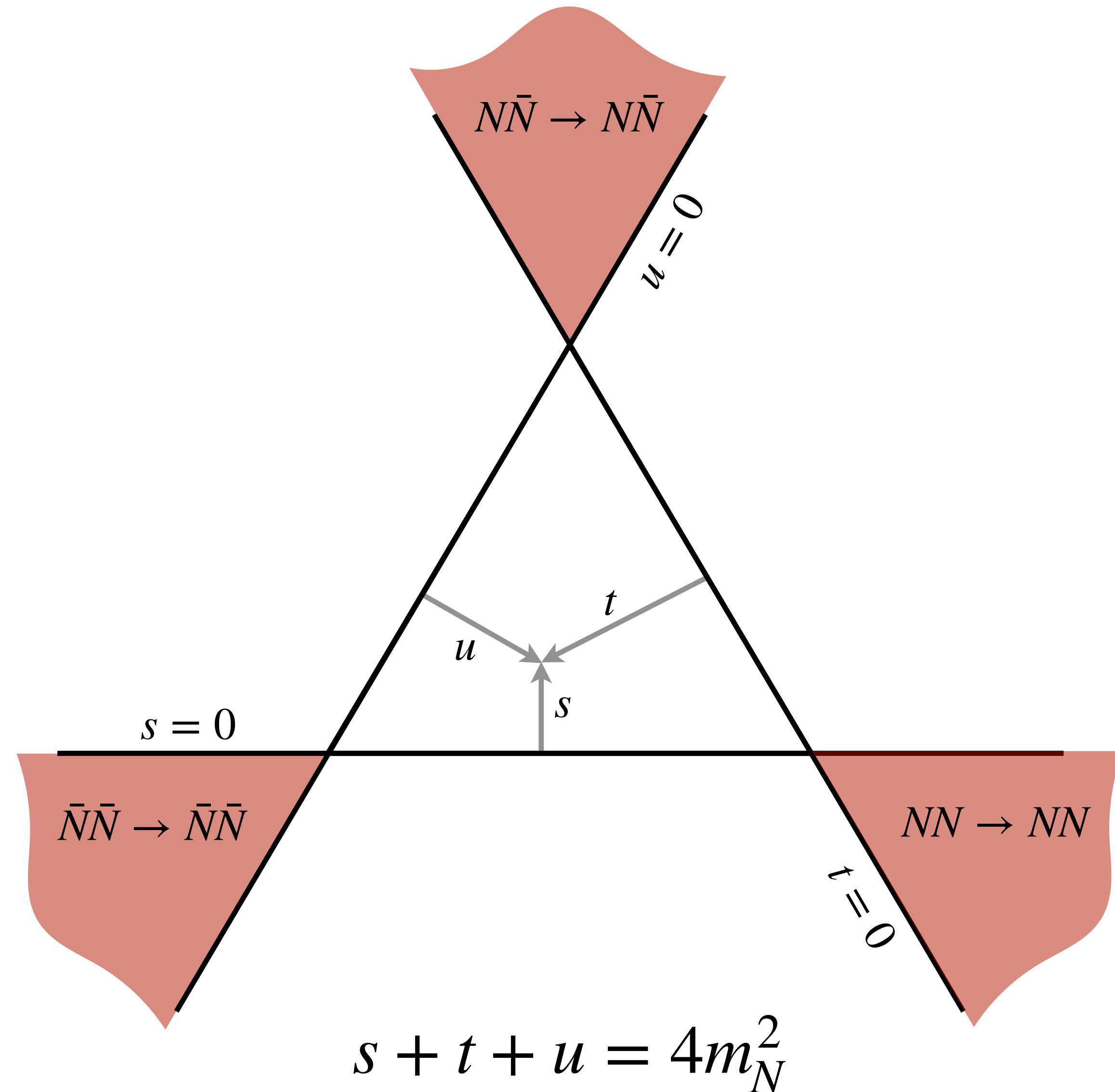


Crossing symmetry

Different amplitudes are boundaries
of the same analytic function

$$\mathcal{M}(s, t, u)$$

in different kinematical regimes.



Crossing symmetry

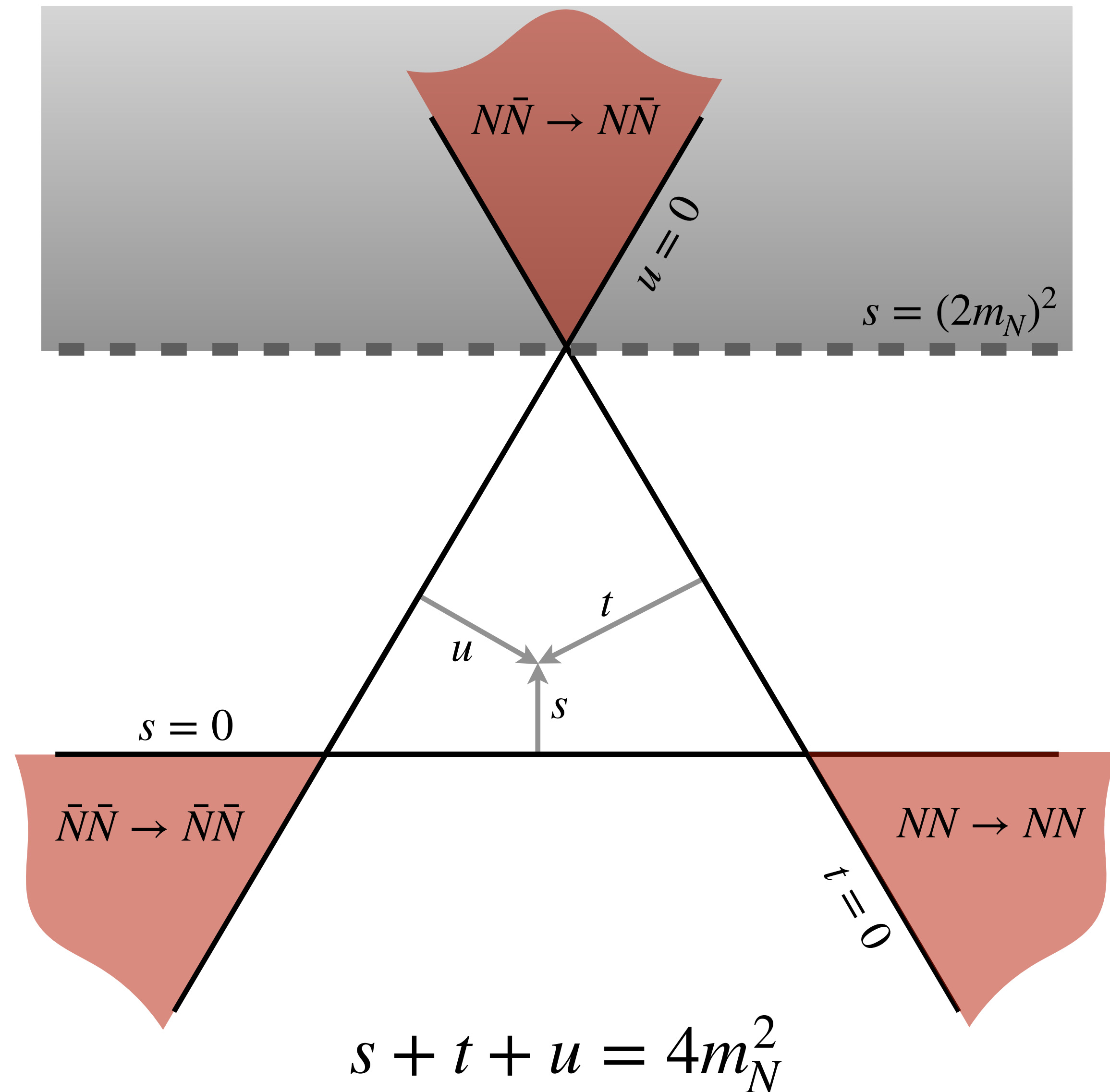
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Cut in the u-channel invariant mass

$$u = s_u < -t = -t_u$$



Crossing symmetry

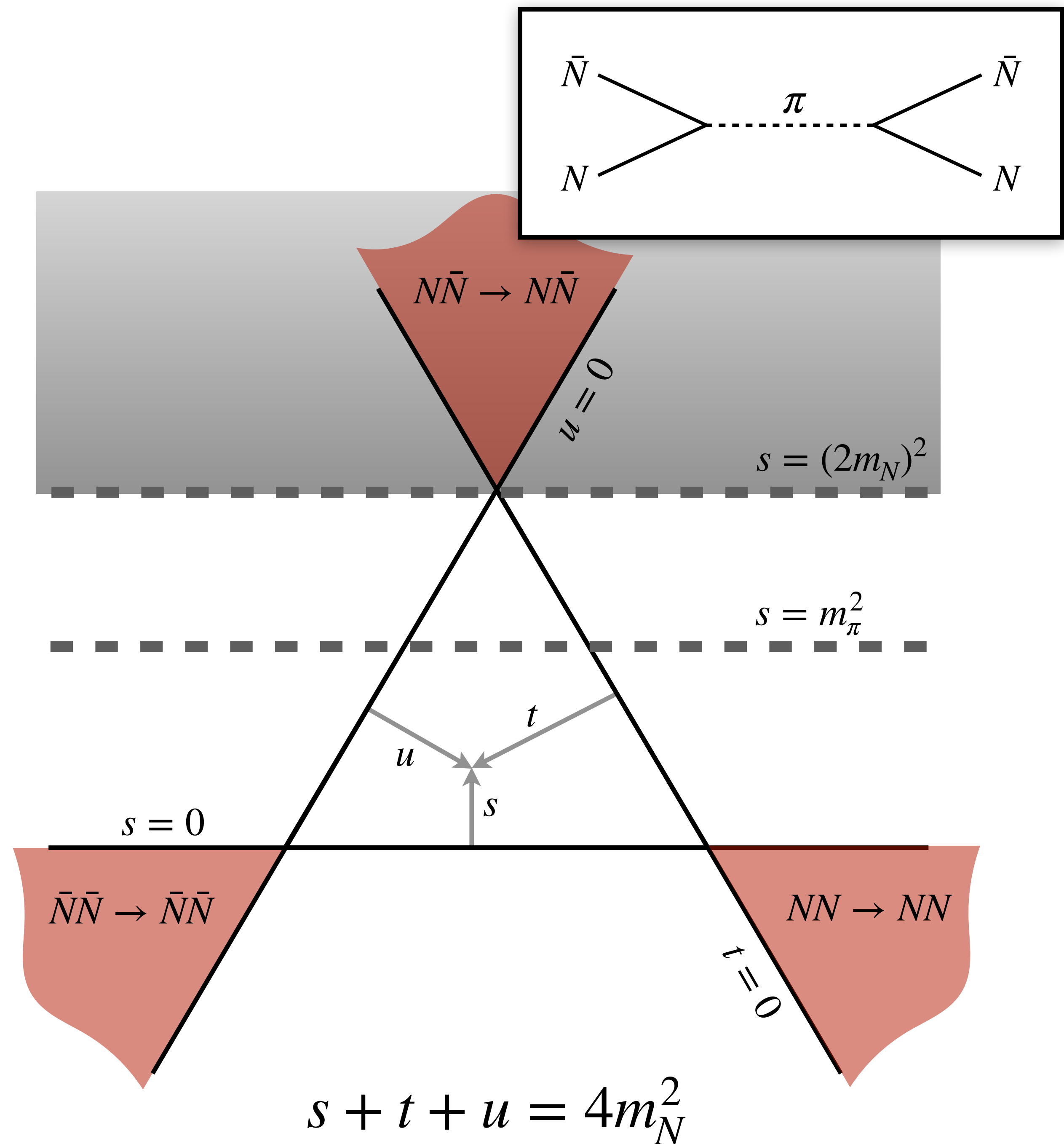
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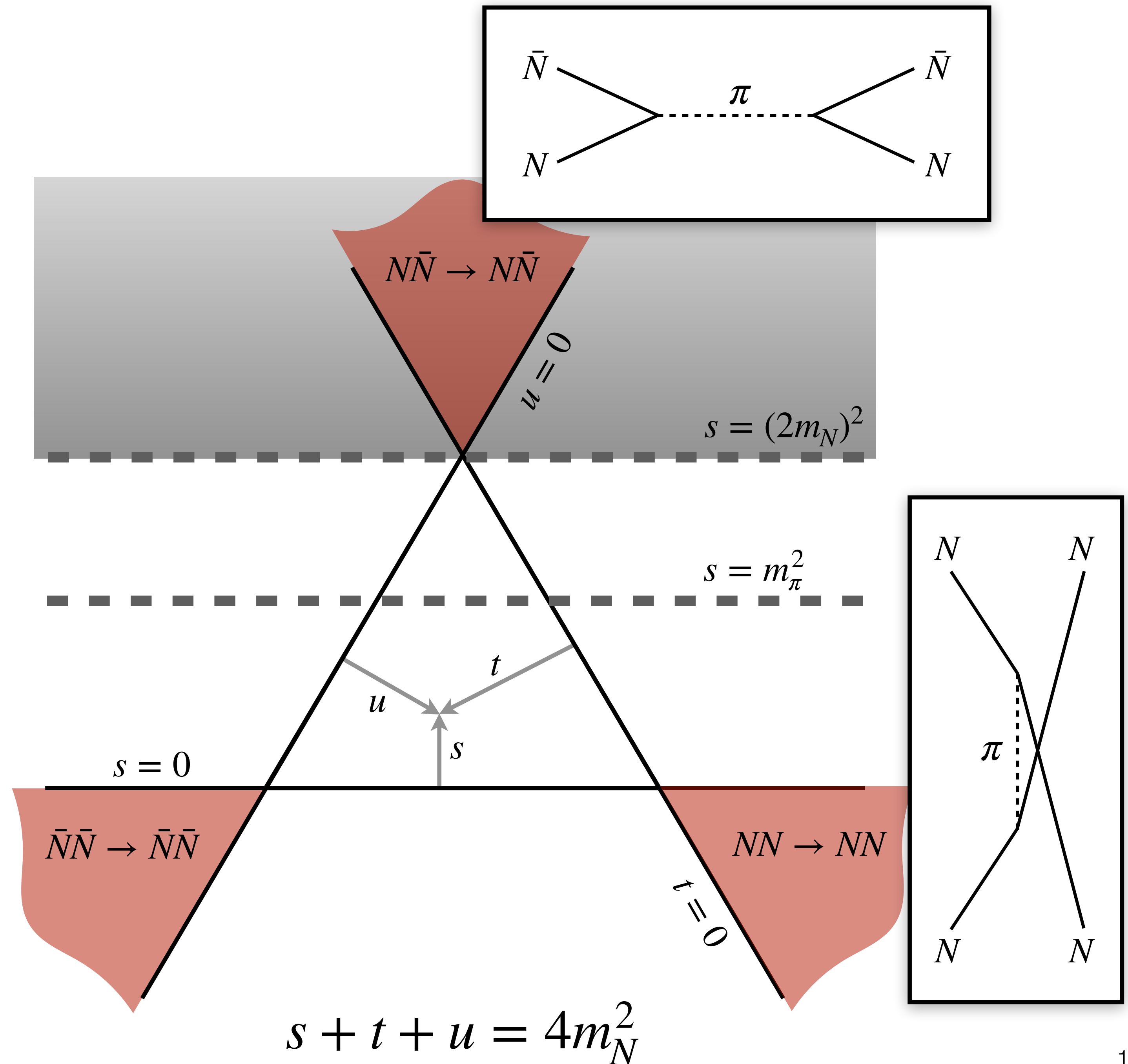
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$$u = s_u < -t = -t_u$$

Pole in the u-channel invariant mass

$$s_u = (2m_N)^2 - m_\pi^2 - t_u$$



Crossing symmetry

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Cut in the u-channel invariant mass

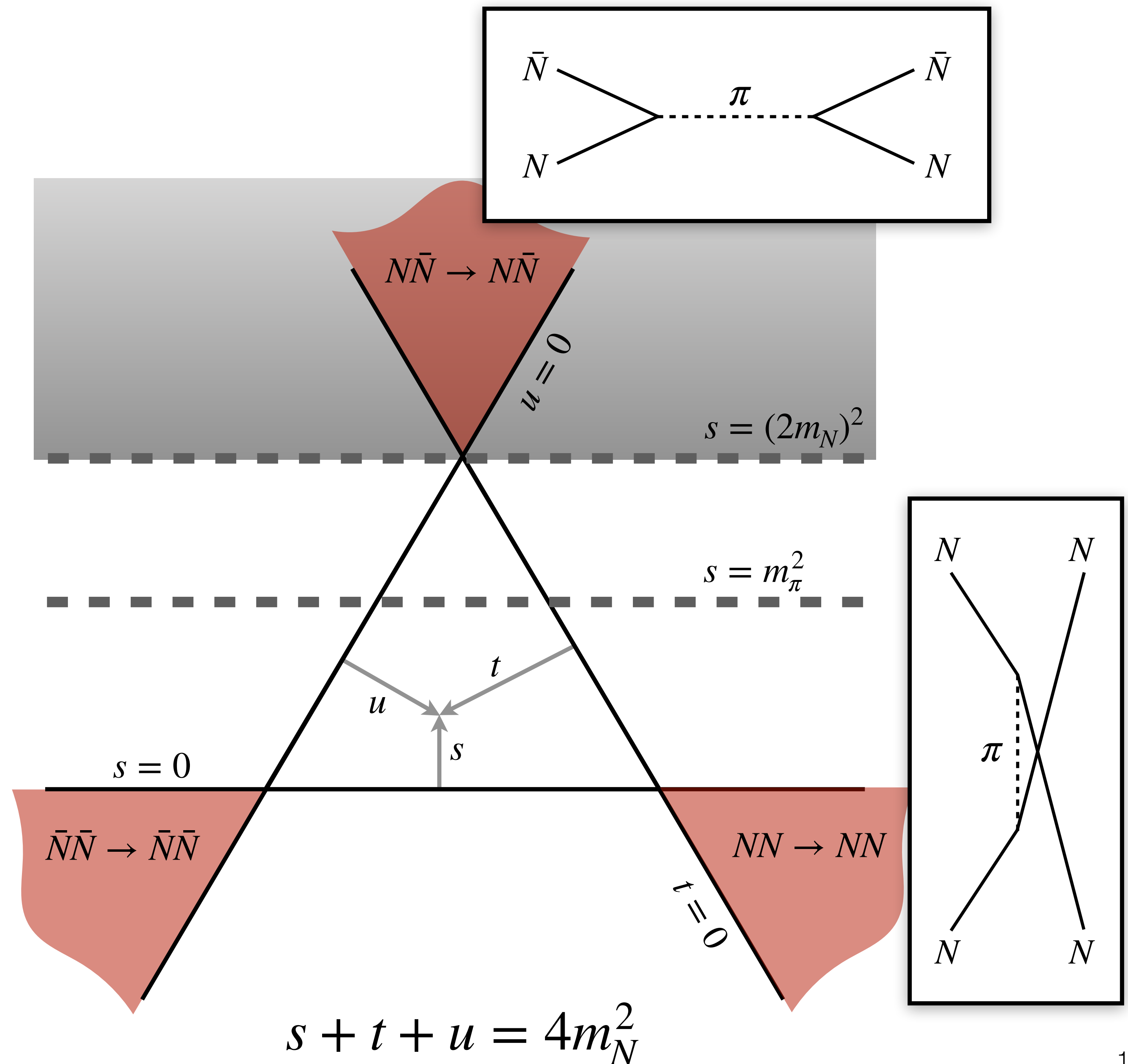
$$u = s_u < -t = -t_u$$

Pole in the u-channel invariant mass

$$s_u = (2m_N)^2 - m_\pi^2 - t_u$$

After the partial-wave projection

$$s_u < (2m_N)^2 - m_\pi^2$$



Technical parts of the proof

Unitarity in the finite volume

$$2 \operatorname{Im} \mathcal{M}_L(\mathbf{p}', \mathbf{p}; \mathbf{P}) = \frac{\xi}{L^3} \sum_{\mathbf{k}} \frac{\pi \delta(E - \omega_1(\mathbf{k}) - \omega_2(\mathbf{P} - \mathbf{k}))}{2\omega_1(\mathbf{k}) 2\omega_2(\mathbf{P} - \mathbf{k})} \mathcal{M}_L^*(\mathbf{k}, \mathbf{p}'; \mathbf{P}) \mathcal{M}_L(\mathbf{k}, \mathbf{p}; \mathbf{P})$$

Partial-wave projection

$$\mathcal{M}(\mathbf{p}', \mathbf{p}; \mathbf{P}) = 4\pi \sum_{\ell' m'_\ell} \sum_{\ell m_\ell} Y_{\ell' m'_\ell}^*(\hat{\mathbf{p}}'^*) \mathcal{M}_{\ell' m'_\ell; \ell m_\ell}(s) Y_{\ell m_\ell}(\hat{\mathbf{p}}^*)$$

Replacement of the finite-volume N function

$$\mathcal{N}_L(p^*, \mathbf{P}) = \mathcal{T}(s) + \frac{1}{L^3} \sum_{\mathbf{k}} \mathcal{L}(\mathbf{k}, \mathbf{P}) \frac{[\mathcal{Y}^*(\mathbf{k}^*) \otimes \mathcal{Y}(\mathbf{k}^*)^T] \operatorname{Im} [\mathcal{T}(E^*(\mathbf{k})^2) - \mathcal{T}(s)]}{E^*(\mathbf{k})^2 - s - i\epsilon} \mathcal{N}_L(k^*, \mathbf{P})$$

Equivalence with Lüscher QC

Lüscher, "Volume dependence of the energy spectrum in massive QFTs"

Lüscher QC $\det_{\ell m_\ell} [\mathcal{K}^{-1}(s) + F(s)] = 0$

Insert relation between the K matrix and N

(obtained from direct comparison of the K matrix parametrization to N/D)

$$\mathcal{K}^{-1}(s) = \left[\mathbb{1} + \text{p.v.} \int \frac{d\mathbf{k}^*}{(2\pi)^3} \mathcal{L}(\mathbf{k}, \mathbf{P}) \frac{\mathcal{N}(E^*(\mathbf{k})^2)}{s - E^*(\mathbf{k})^2} \right] \mathcal{N}^{-1}(s)$$

Replace p.v. integrals in F and K by sums

(up to corrections exponentially suppressed with volume)

$$\text{p.v.} \int \frac{d\mathbf{k}}{(2\pi)^3} \mathcal{L}(\mathbf{k}, \mathbf{P}) \frac{\mathcal{N}(s) - \mathcal{N}(E^*(\mathbf{k})^2)}{s - E^*(\mathbf{k})^2} = \frac{1}{L^3} \sum_{\mathbf{k}} \mathcal{L}(\mathbf{k}, \mathbf{P}) \frac{\mathcal{N}(s) - \mathcal{N}(E^*(\mathbf{k})^2)}{s - E^*(\mathbf{k})^2} + \mathcal{O}(e^{-\Delta L})$$