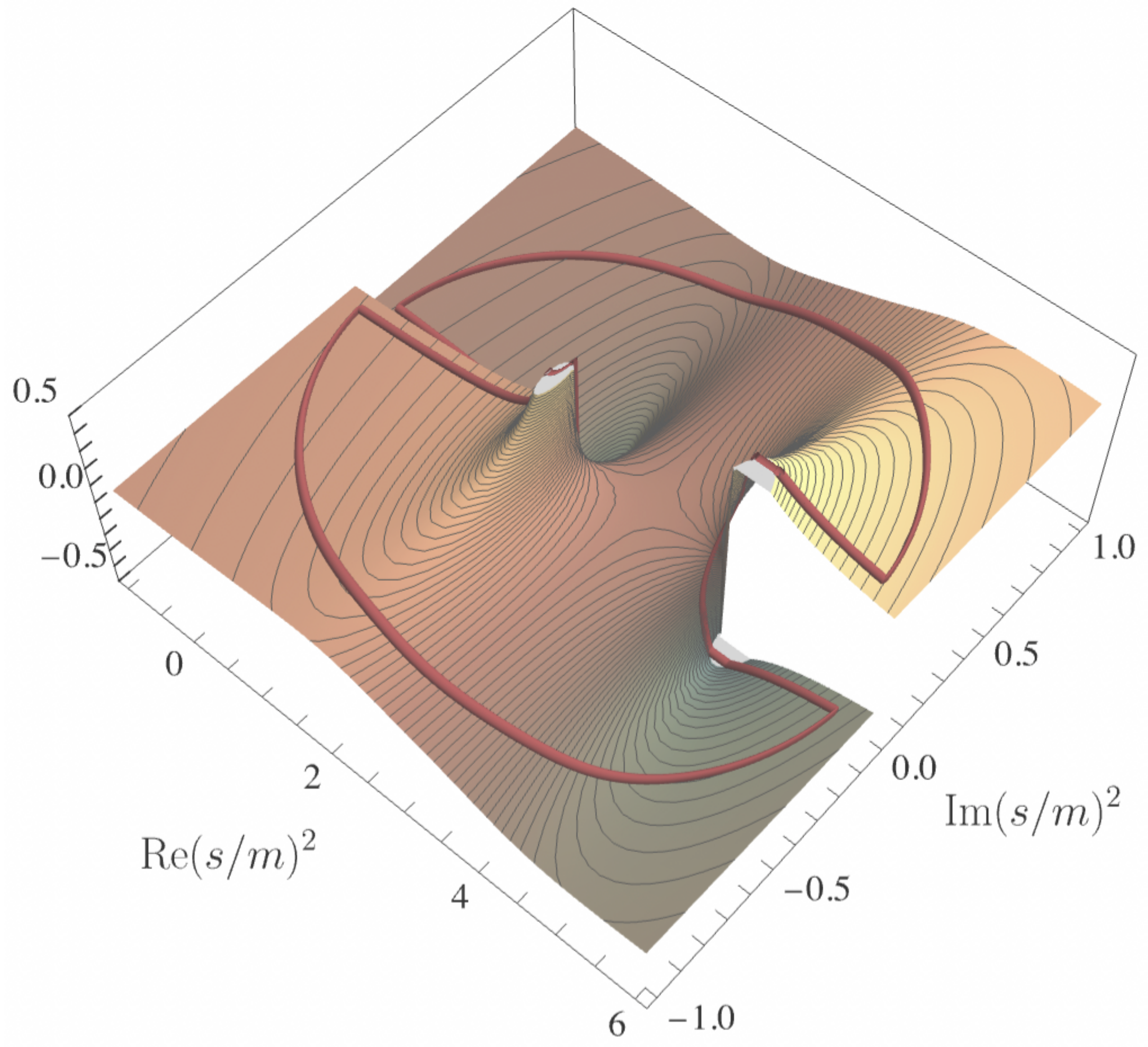
Sebastian M. Dawid

Finite-volume quantization condition from the *N*/*D* representation



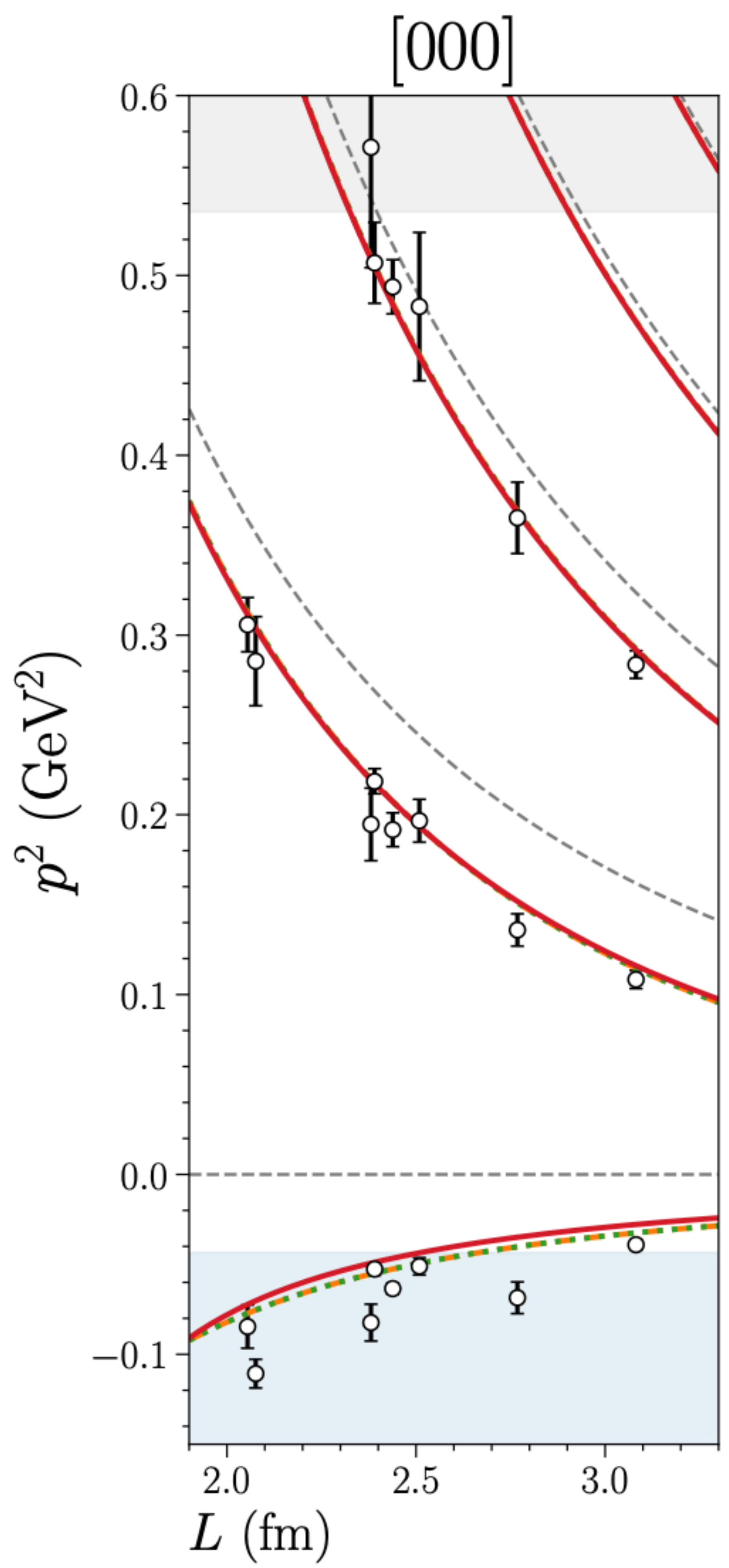
INDIANA UNIVERSITY
BLOOMINGTON

ExoHad
Exotic Hadrons Topical Collaboration

Lattice 2026, University of Maryland, 07.29.2026

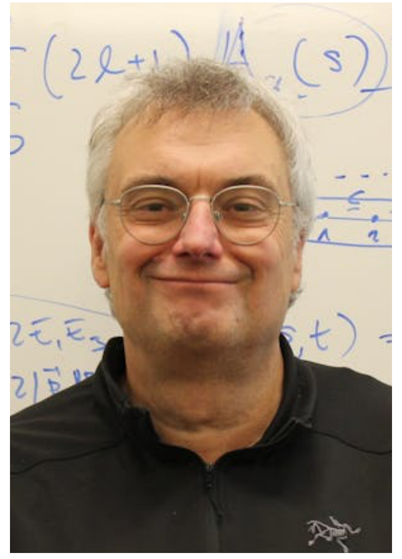
Summary

1) Novel two-body quantization condition derived from the dispersive representation of the amplitude
2) Applicable for energies below the nearest one- and multi-particle-exchange left-hand branch points
3) **Advantage:** conceptually simple separation of singularities, preserves unitarity & analyticity
4) **Current limitations:** modeling of the *N* function & generalization to coupled channels and spins



Relevant literature

Dawid, Jackura, Szczepaniak	**Finite-volume quantization condition from the N/D representation**	(April 2025)
Rodas, Qiu, et al. (JPAC)	**Finite-volume analysis of the H-dibaryon including left-hand cut effects**	(May 2026)



Adam Szczepaniak (IU)

Andrew Jackura (W&M)

Arkaitz Rodas (ODU)

Lin Qiu (ODU)

Related talk

Lin Qiu on Friday, 11:30, Juan Ramon Jimenez room

The left-hand cut problem

Particle exchanges lead to an additional power-law dependence.
Lüscher-like formalisms break down at corresponding energies.

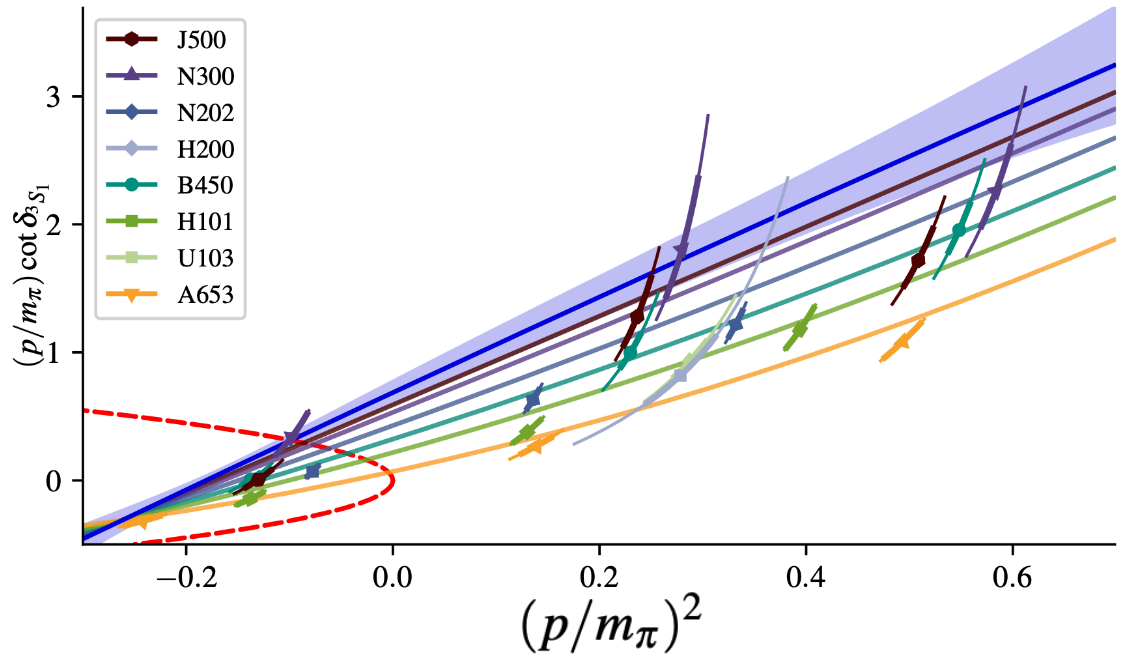
Quantization condition

$$\det_{\ell m_\ell}\left[\mathcal{K}_2^{-1}(E) + F(E, \boldsymbol{P}, L)\right] = 0$$

Real matrix in the angular momentum space

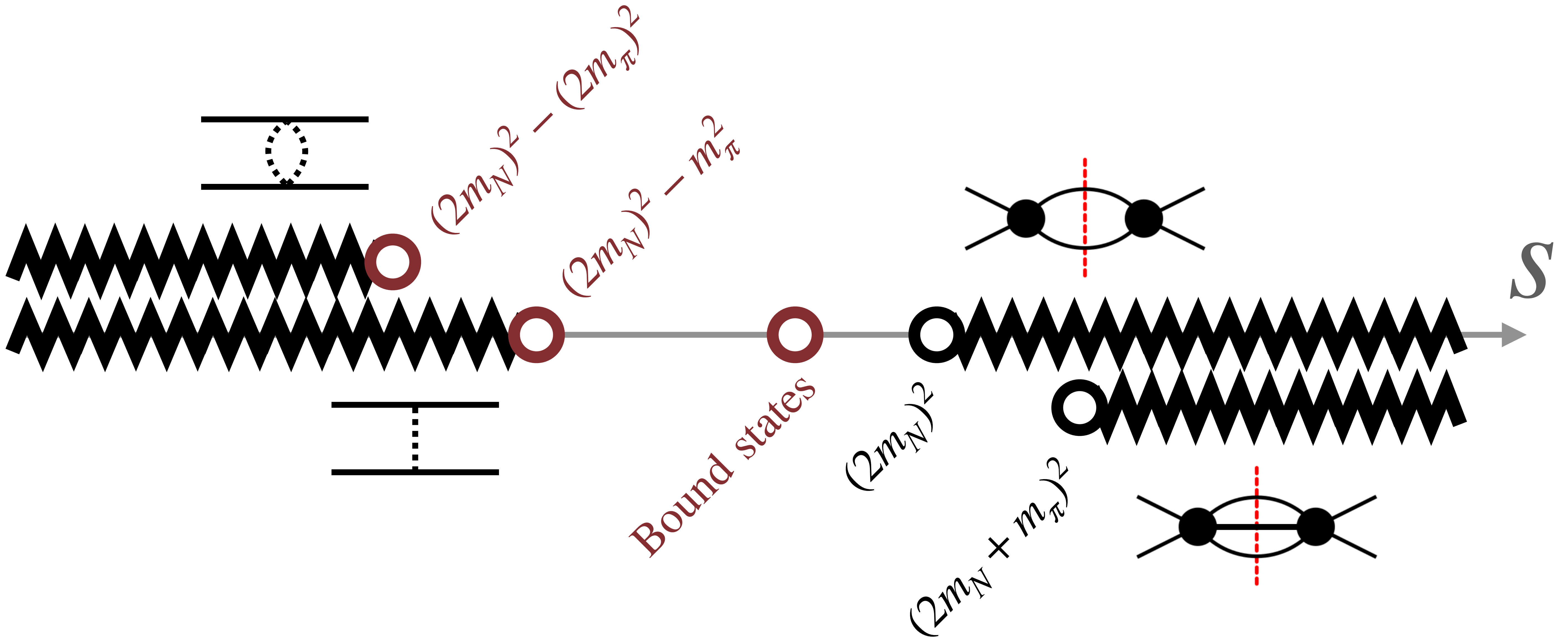
Real, known special function

NN scattering



Green et al. (BaSc)

m_π = 420 MeV



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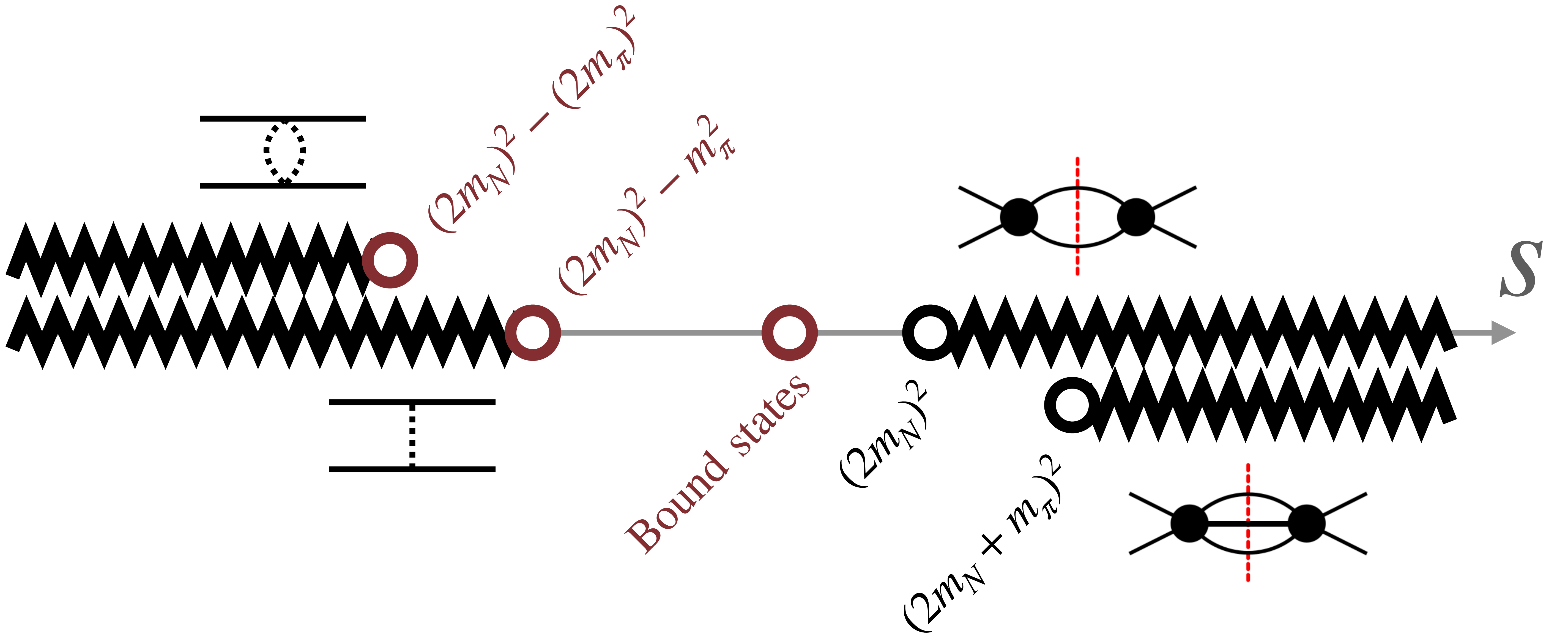
For extra motivation, see an excellent poster by Jorge Baeza Ballesteros



N/D in infinite volume

Chew & Mandelstam, **Theory of low-energy pion-pion interactions**
Bjorken, **Construction of coupled scattering and production amplitudes…**

Elastic unitarity

$$\mathrm{Im}\,\mathcal{M}_\ell(s) = \rho(s)|\mathcal{M}_\ell(s)|^2\,\Theta(s - s_{\mathrm{thr.}})$$

$$\mathrm{Im}\,\mathcal{M}_\ell(s)^{-1} = -\rho(s)\,\Theta(s - s_{\mathrm{thr}})$$

$$\rho(s) \propto \sqrt{s - s_{\mathrm{thr}}}$$

K matrix

$$\mathcal{M}_\ell^{-1}(s) = \mathcal{K}_\ell^{-1}(s) - i\rho(s)$$

finite-volume unitarity +

Sato & Bedaque, **Fitting two nucleons inside a box…**

Lüscher QC

$$\det_{\ell m_\ell}\left[\mathcal{K}^{-1}(s) + F(s)\right] = 0$$

N/D in infinite volume

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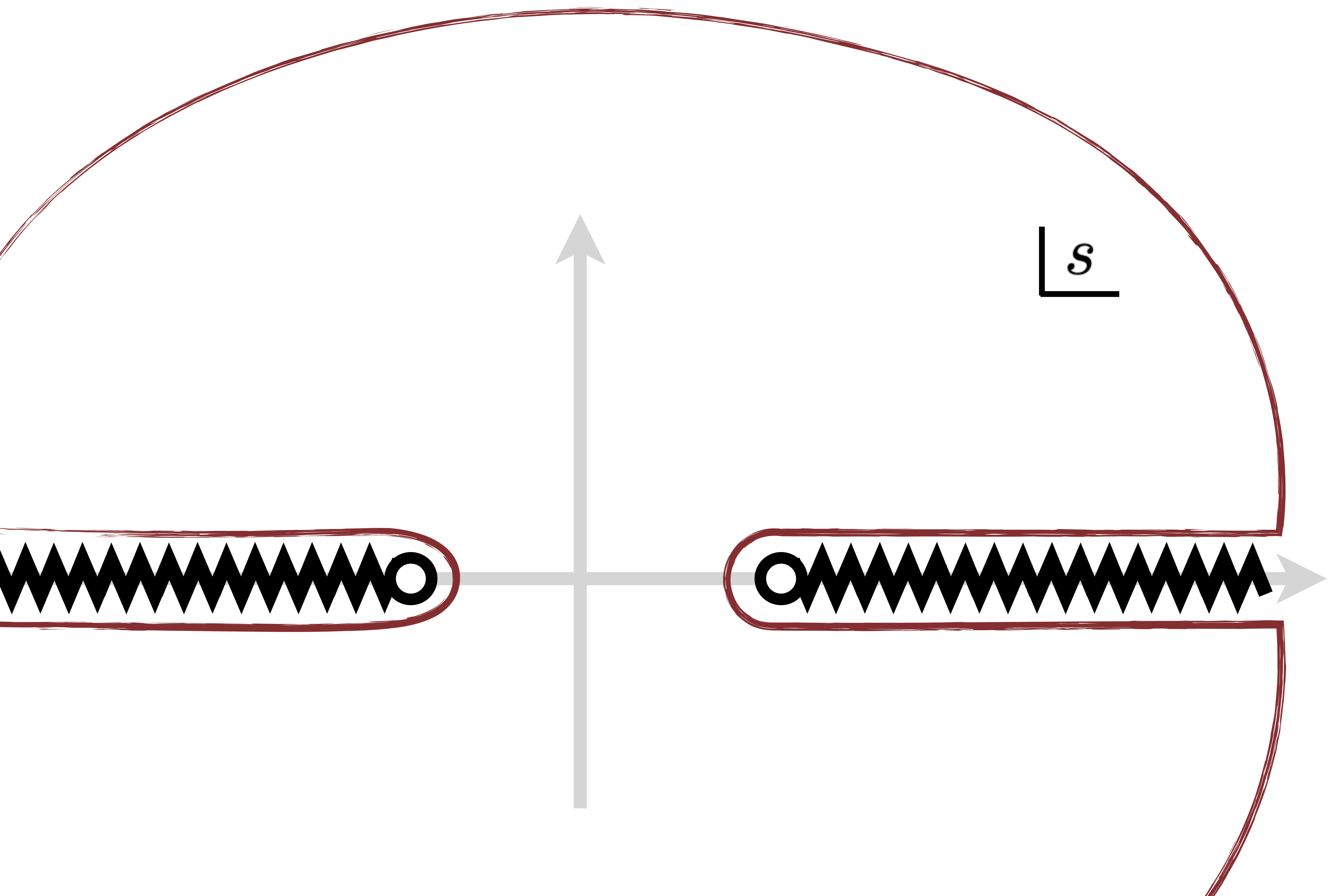
Bjorken, **Construction of coupled scattering and production amplitudes…**

Cauchy theorem
(Riemann-Hilbert problem)

$$\mathcal{M}_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{s_\mathrm{L}} \frac{\mathrm{Im}\,\mathcal{M}_\ell(s')}{s'-s}ds' + \frac{1}{\pi}\int_{s_\mathrm{thr.}}^{\infty} \frac{\mathrm{Im}\,\mathcal{M}_\ell(s')}{s'-s}ds'$$

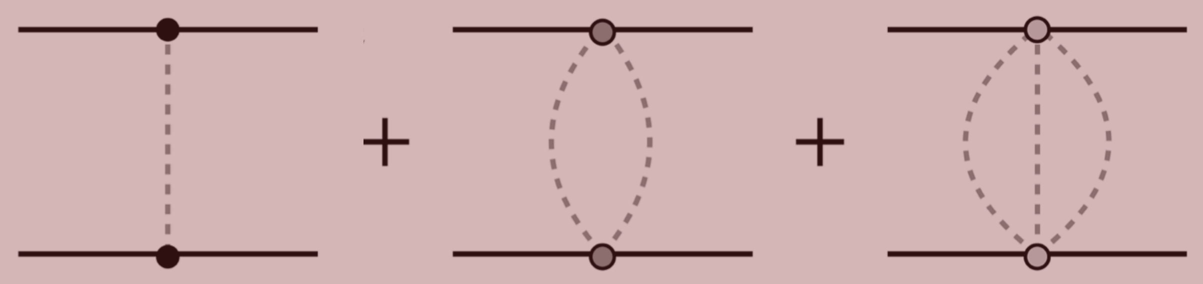


Right- and left-hand cut inputs

– right-hand discontinuity is fixed by experimental data (unitarity)

$$\mathrm{Im}\,\mathcal{M}_\ell(s) = \rho(s)|\mathcal{M}_\ell(s)|^2$$

– left-hand discontinuity is modeled as exchanges of particles

$\mathcal{T}_\ell =$ 

N/D in infinite volume

Chew & Mandelstam, **Theory of low-energy pion-pion interactions**
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Wiener-Hopf decomposition

$$\mathcal{M}_\ell(s) = \mathcal{N}_\ell(s)\, \mathcal{D}_\ell^{-1}(s)$$

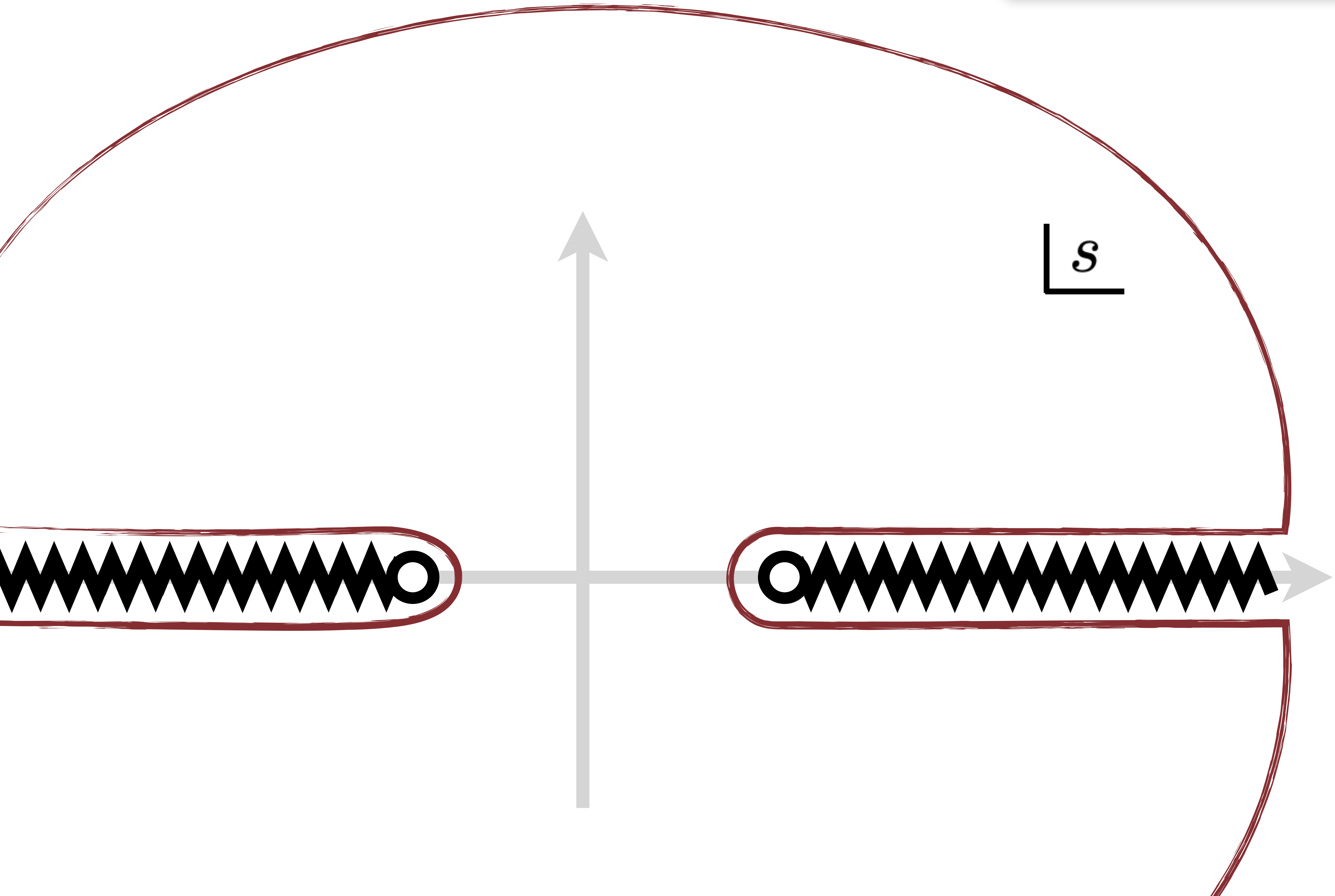
contains (only) the right-hand singularities

contains (only) the left-hand singularities

$$\mathrm{Im}\, \mathcal{D}_\ell(s) = -\rho(s)\, \mathcal{N}(s)\, \Theta(s - s_{\mathrm{thr}})$$

$$\mathcal{M}_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{s_L} \frac{\mathrm{Im}\,\mathcal{M}_\ell(s')}{s'-s}ds' + \frac{1}{\pi}\int_{s_{\mathrm{thr.}}}^{\infty} \frac{\mathrm{Im}\,\mathcal{M}_\ell(s')}{s'-s}ds'$$

reduced to coupled linear integral equations



$$\mathcal{D}_\ell(s) = 1 - \int_{s_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d}s'}{\pi} \frac{\rho(s')}{s' - s - i\epsilon} \mathcal{N}_\ell(s')$$

$$\mathcal{N}_\ell(s) = \int_{-\infty}^{s_{\mathrm{lhc}}} \frac{\mathrm{d}s'}{\pi} \frac{\mathrm{Im}\,\mathcal{M}_\ell(s')}{s' - s - i\epsilon} \mathcal{D}_\ell(s')$$

N/D in infinite volume

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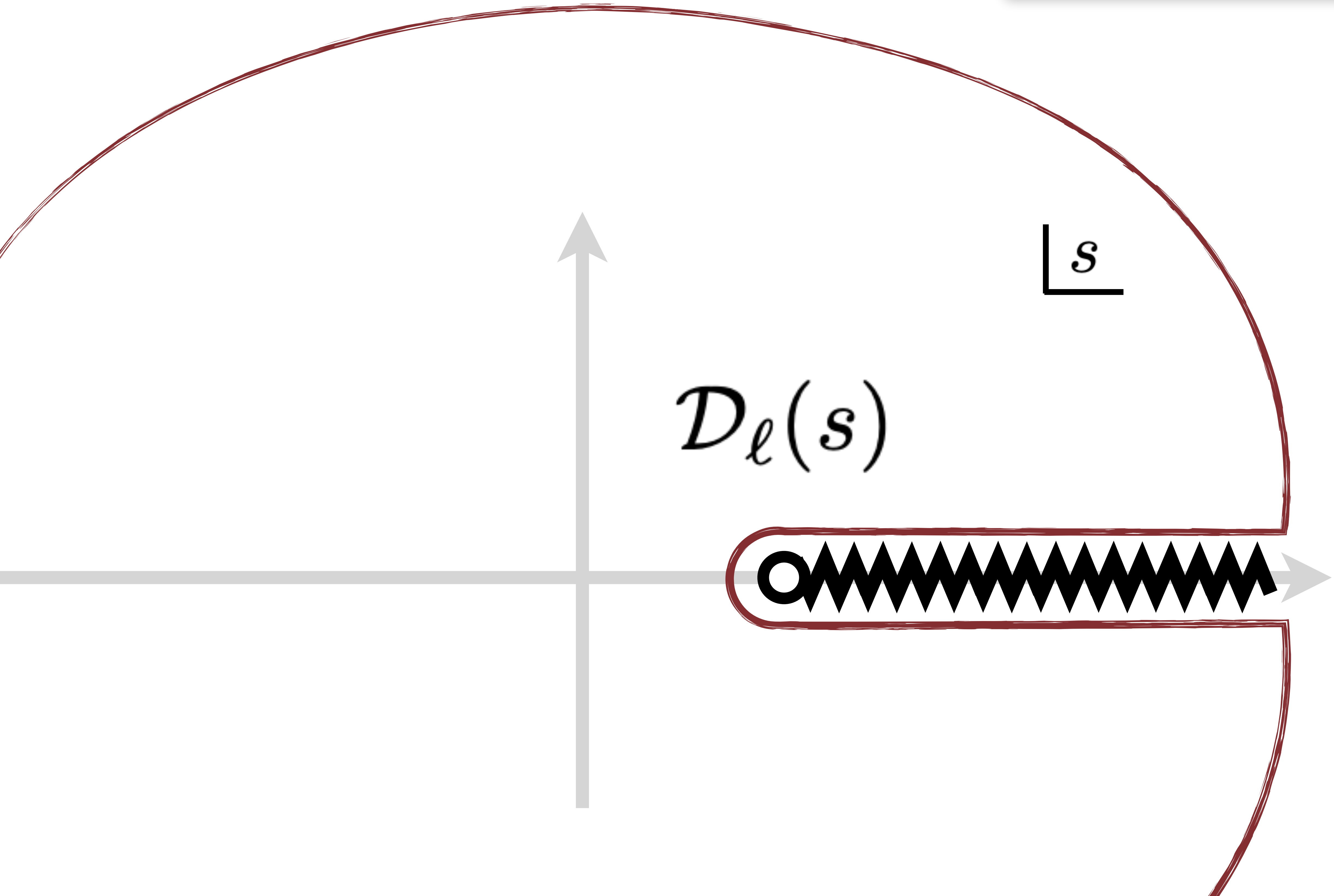

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N/D in finite volume

Dawid, Jackura, Szczepaniak,
Finite-volume quantization condition from the N/D representation

Transition operator in FV

$$T_L(z) = H_{I,L} + H_{I,L}\, G_L(z)\, H_{I,L}\,, \qquad G_L(z) = \frac{1}{z - H_L}$$

$$H_L = H_{0,L} + H_{I,L}$$

first resolvent identity

$$G_L(z) - G_L(z^*) = (z - z^*)\, G_L(z) G_L(z^*)$$

Unitarity in the FV

$$T_L(z) - T_L(z^*) = T_L(z)\, 2\pi i \delta(z - H_{0,L})\, T_L(z^*)$$

$$z = E + i\epsilon$$

unrelated, but note also that

$$T_L(z) - T_L(z^*) = H_{I,L}\, 2\pi i \delta(z - H_L)\, H_{I,L}$$

Right-hand singularities

$$2\,\mathrm{Im}\,\mathcal{M}^{-1}(\boldsymbol{p}', \boldsymbol{p}; \boldsymbol{P}) = \frac{1}{L^3} \sum_{\boldsymbol{k}} \frac{\delta(\,E - \omega_1(\boldsymbol{k}) - \omega_2(\boldsymbol{P} - \boldsymbol{k}))}{2\omega_1(\boldsymbol{k})\omega_2(\boldsymbol{P} - \boldsymbol{k})}$$

N/D in finite volume

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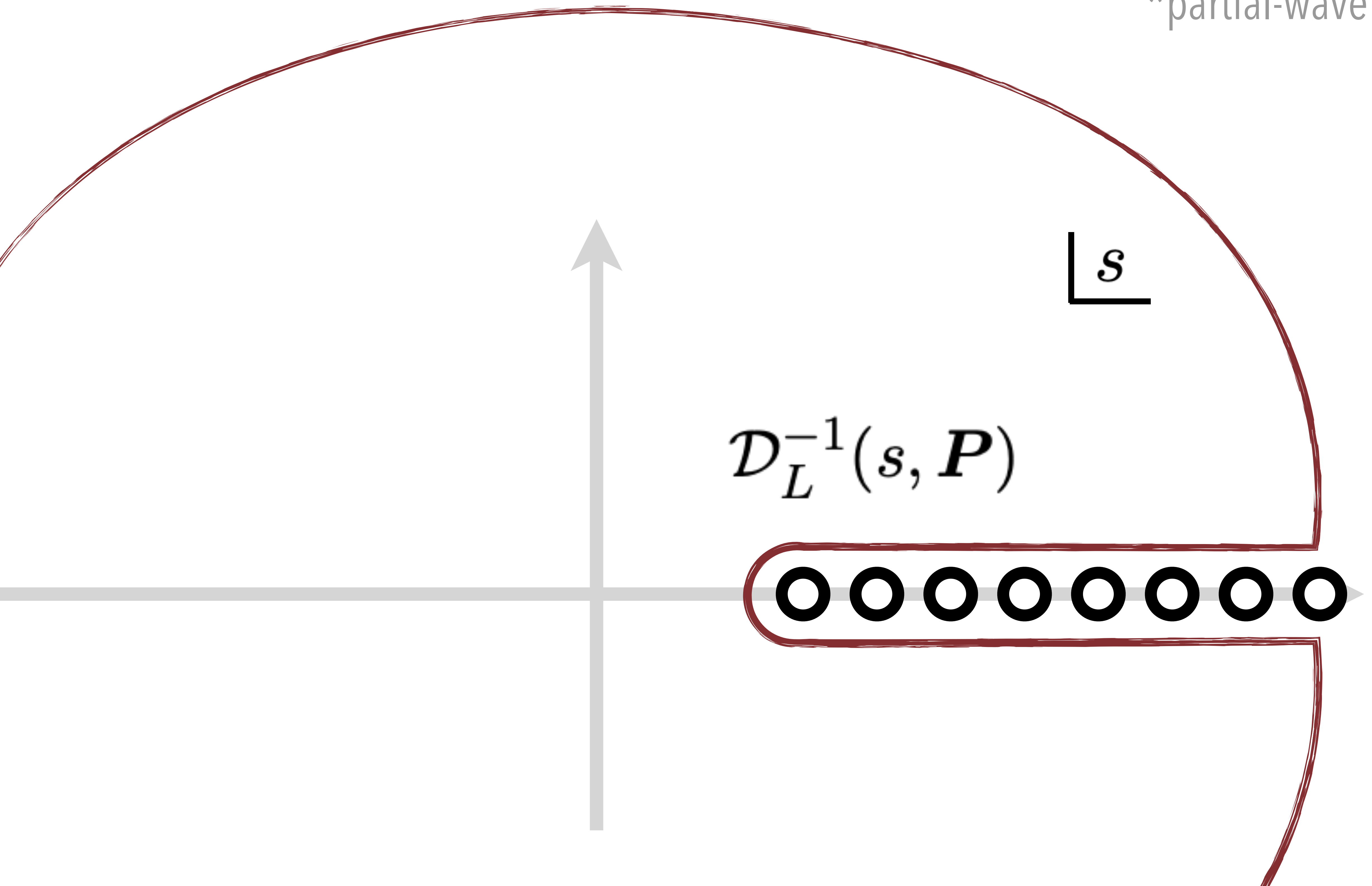
Wiener-Hopf decomposition

$$\mathcal{M}_L(s,\boldsymbol{P}) = \mathcal{N}_L(s,\boldsymbol{P})\mathcal{D}_L^{-1}(s,\boldsymbol{P})$$

explicit ***P*** dependence

matrices in the $(\ell', m'_\ell;\, \ell, m_\ell)$ space of angular momenta* (box breaks rotational symmetry)

*partial-wave projection of the finite-volume on-shell amplitude has to be defined carefully



recall, in the infinite volume

$$\mathrm{Im}\,\mathcal{D}_\ell(s) = -\rho(s)\,\mathcal{N}(s)\,\Theta(s - s_{\mathrm{thr}})$$

In the finite volume analogous relation holds;
simplified S-wave case yields (full version in the article)

$$\mathrm{Im}\,\mathcal{D}_L(s,\boldsymbol{P}) = -\frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{2s}{E}\frac{\delta\left(s - E^\star(\boldsymbol{k})^2\right)}{2\omega_1(\boldsymbol{k})2\omega_2(\boldsymbol{P}-\boldsymbol{k})}\mathcal{N}_L(s,\boldsymbol{P})$$

pick the Pac-Man contour and integrate over complex s!

N/D quantization condition

Dawid, Jackura, Szczepaniak,
Finite-volume quantization condition from the N/D representation

Finite-volume D function

kinematic normalization factor

$$\mathcal{D}_L(s, \boldsymbol{P}) = \mathbb{1} + \frac{1}{L^3} \sum_{\boldsymbol{k}} \frac{\mathcal{L}(\boldsymbol{k}, \boldsymbol{P})}{s - E^\star(\boldsymbol{k})^2} \mathcal{N}_L(E^\star(\boldsymbol{k})^2, \boldsymbol{P})$$

$\omega_1(\boldsymbol{k}^\star) + \omega_2(\boldsymbol{k}^\star)$

– Here I show the simplified S-wave case; full version is found in the article
– FV *N* function above can be replaced by the IV *N* up to exp-suppressed corrections
– This is the unsubtracted version of *D*; subtractions might be necessary to ensure convergence
– The formalism may incorporate "bare" bound-state poles or CDD poles

Quantization condition $\det_{\ell m_\ell} \mathcal{D}_L = 0$

Finite-volume energy levels

$\{E_n\}$

IV numerator function

$\mathcal{N}_\ell(s)$

N/D quantization condition

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$$\mathcal{M}_\ell(s) = \mathcal{N}_\ell(s)\,\mathcal{D}_\ell^{-1}(s)$$

Finite-volume energy levels

$\{E_n\}$

IV numerator function

$\mathcal{N}_\ell(s)$

$$\mathcal{D}_\ell(s) = 1 - \int_{s_{\rm thr}}^{\infty} \frac{ds'}{\pi}\,\frac{\rho(s')}{s'-s-i\epsilon}\,\mathcal{N}_\ell(s')$$

Caveats

Dawid, Jackura, Szczepaniak,
Finite-volume quantization condition from the N/D representation

❖ What does it mean to model the *N* function?

$$\mathcal{N}_\ell(s) = \int_{-\infty}^{s_{\mathrm{lhc}}} \frac{\mathrm{d}s'}{\pi} \frac{\mathrm{Im}\,\mathcal{M}_\ell(s')}{s' - s - i\epsilon} \mathcal{D}_\ell(s')$$

Example

Left-hand singularity given by a left-hand pole at s_0

$$\mathcal{N}_\ell(s) = \frac{g\mathcal{D}_\ell(s_0)}{s - s_0} \quad \Rightarrow \quad \mathcal{M}_\ell^{-1}(s) = -\frac{s - s_0}{g} + \mathcal{I}_{\mathrm{CM}}(s)$$

Example

N obtained from the dispersive integral equation

$$\mathcal{N}_\ell(s) = \mathcal{T}(s) + \int_{s_{\mathrm{thr}}}^{\infty} \frac{ds'}{\pi} \frac{\rho_\ell(s')\,[\mathcal{T}_\ell(s') - \mathcal{T}_\ell(s)]}{s' - s - i\epsilon} \mathcal{N}_\ell(s')$$

❖ Can the FV *N*/*D* incorporate coupled channels and spins?

Currently not.

Bjorken, **Construction of coupled scattering and production amplitudes…**
Bjorken, Nauenberger, **Symmetry of N/D solutions for coupled scattering…**

$$\mathcal{M} = \mathcal{N}\,\mathcal{D}^{-1}$$

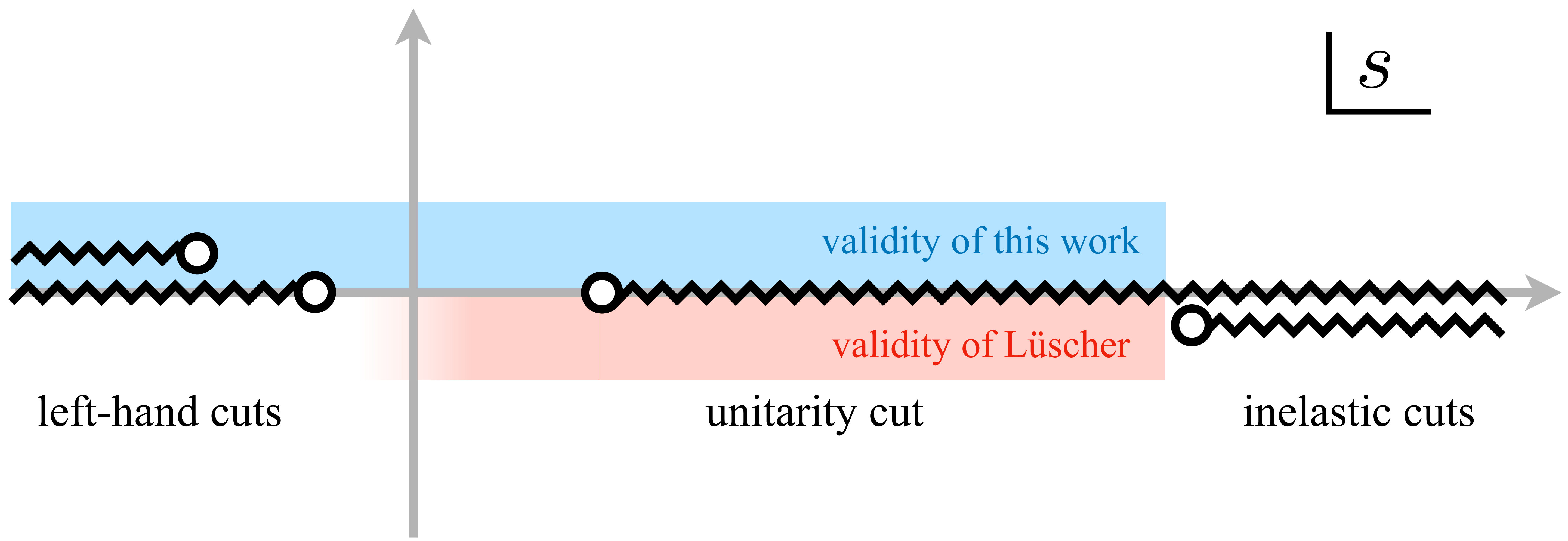
become matrices in the channel and partial-wave space

$$\text{sym. } \mathrm{Im}\,\mathcal{M} \;\Rightarrow\; \text{sym. } \mathcal{M}$$

symmetric discontinuities imply symmetric N/D amplitude

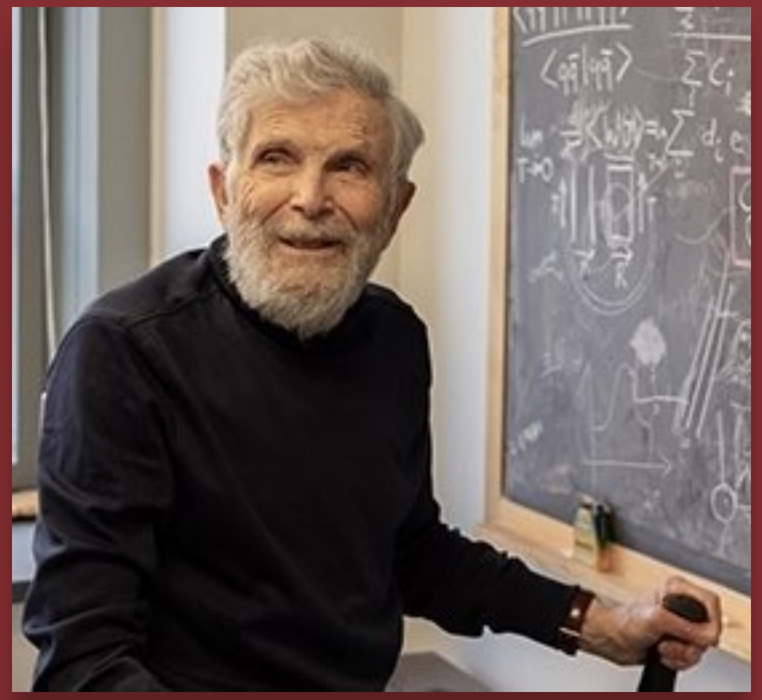
Summary

- N/D separates crossed-channel from direct-channel singularities, also in the finite volume
- Finite-volume energies are determined by the condition $\det[D_L] = 0$ and $N = N_L + O(\exp(-\Delta L))$
- Amplitude is reconstructed by solving the dispersive N/D equations
- Future work: tests & implementation of the formalism, generalization to coupled channels and spins

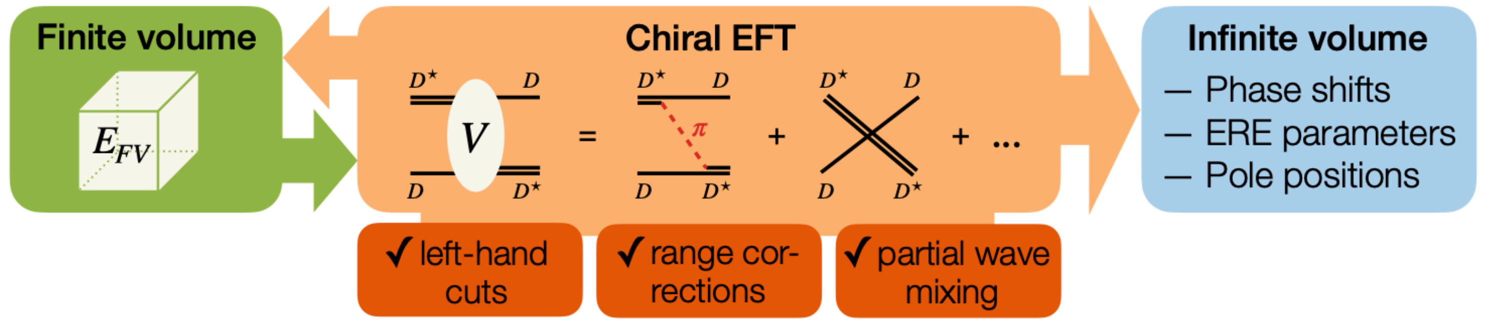


Thank you!

This talk is dedicated to the memory of Marshall Baker (1932-2026), who, together with Schwinger, wrote the first nonperturbative FV QC.



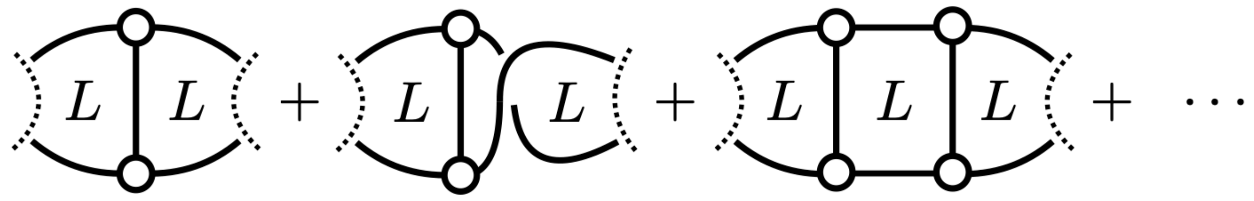
Available approaches



- **Plane-wave basis approach** [Meng & Eppelbaum (2021) and Meng et al. (2024)]
  - ❖ Based on Lippmann-Schwinger equation with chiral EFT potentials and their LECs
  - ❖ Evaluation of the determinant condition in the plane-wave basis

$$\mathbb{T}(E) = \mathbb{V}(E) + \mathbb{V}(E)\mathbb{G}(E)\mathbb{T}(E)$$

- **Two-body RFT approach** [Baião Raposo & Hansen (2024) and Baião Raposo et al. (2025)]
  - ❖ Based on diagrammatic expansion of the Bethe-Salpeter kernel with OPEs singled out
  - ❖ The amplitude is split into a ladder and a "short-range" part with an unknown K matrix

$\mathcal{C}_L \sim$ 

- **Two-body NREFT approach** [Bubna et al. (2024)]
  - ❖ Combines best features of the two above approaches

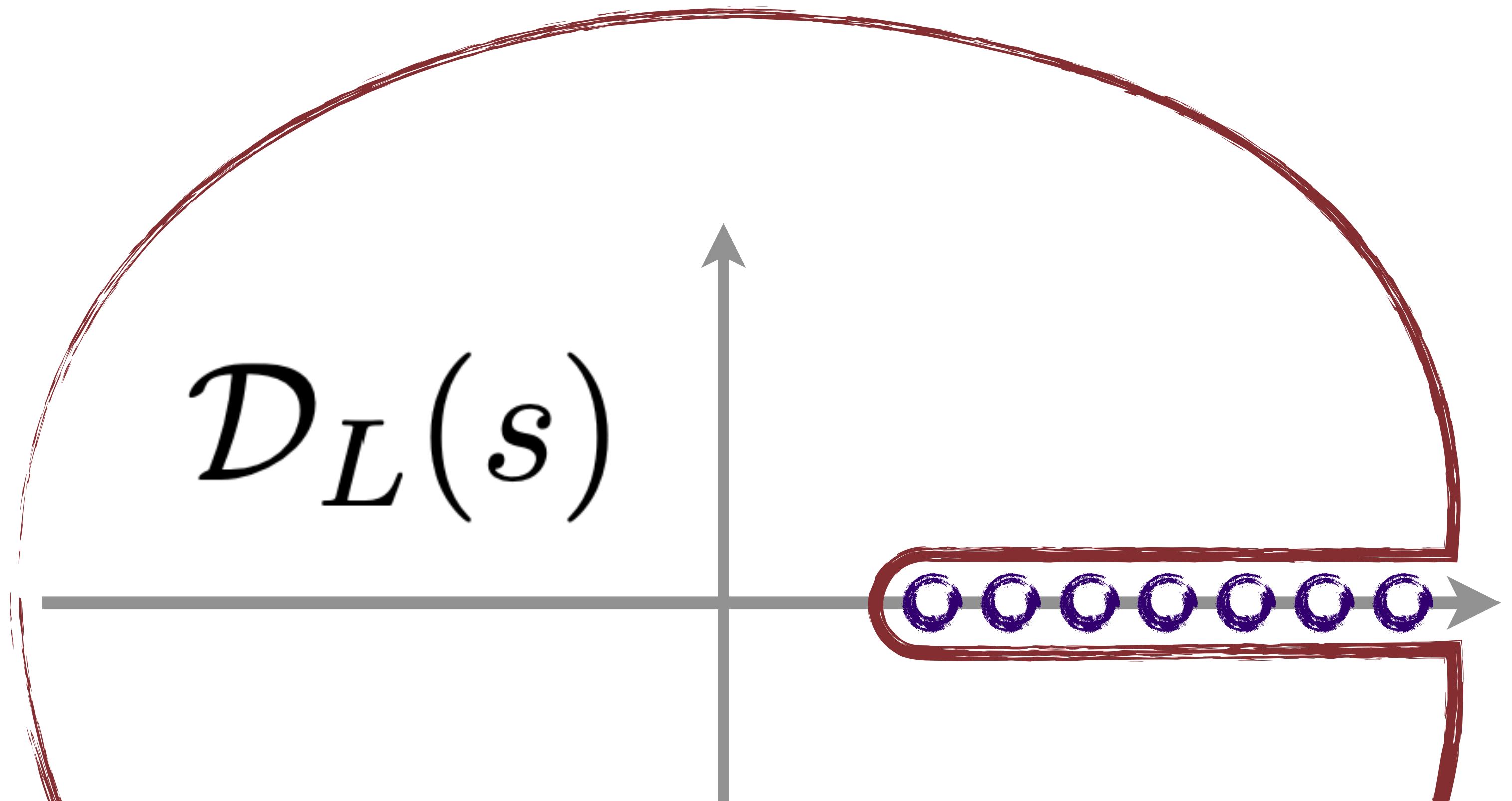
$$\det_{aJm_J\ell S}\left[\mathcal{M}_0^{-1} + F + \mathcal{C}_L\right] = 0$$

- **N/D quantization condition** [Dawid, Jackura, Szczepaniak (2025)]
  - ❖ Application of the Wiener-Hopf decomposition to the FV amplitude
  - ❖ LQCD data constrains models of the numerator function, *N(s)*

$$\mathcal{D}_L(s) = 1 + \frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{s - E^\star(\boldsymbol{k})^2}\mathcal{R}(\boldsymbol{k})\mathcal{N}(E^\star(\boldsymbol{k}))$$

- **Three-body RFT approach** [Hansen, Romero-López, Sharpe (2024)]

$$\det_{k,\ell,m}\left[\mathbb{1} - \boldsymbol{\mathcal{K}}_3(E^\star)\boldsymbol{F}_3(E,\boldsymbol{P},L)\right] = 0$$

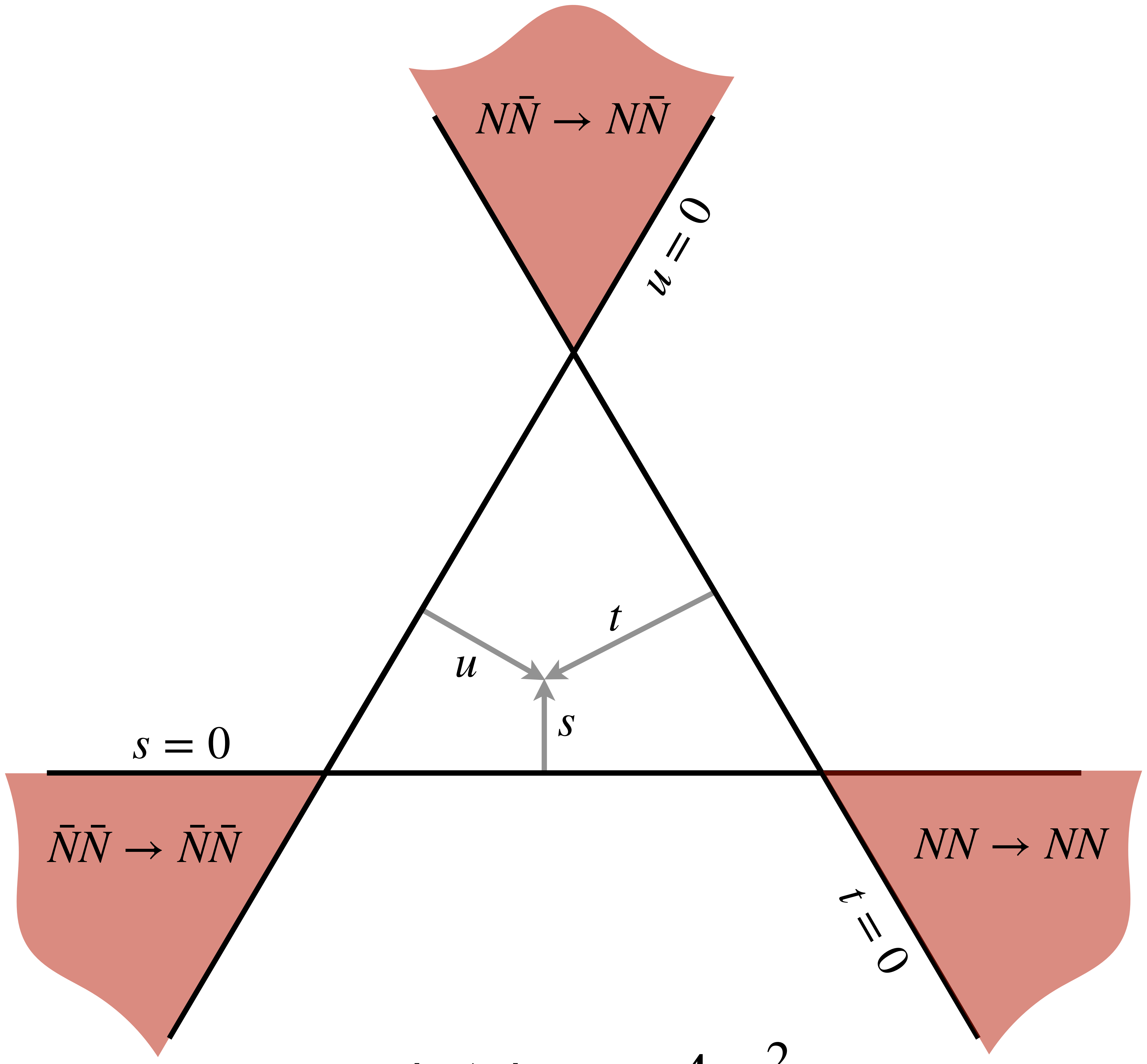


Crossing symmetry

Different amplitudes are boundaries of the same analytic function

$$\mathcal{M}(s,t,u)$$

in different kinematical regimes.



$$s + t + u = 4m_N^2$$

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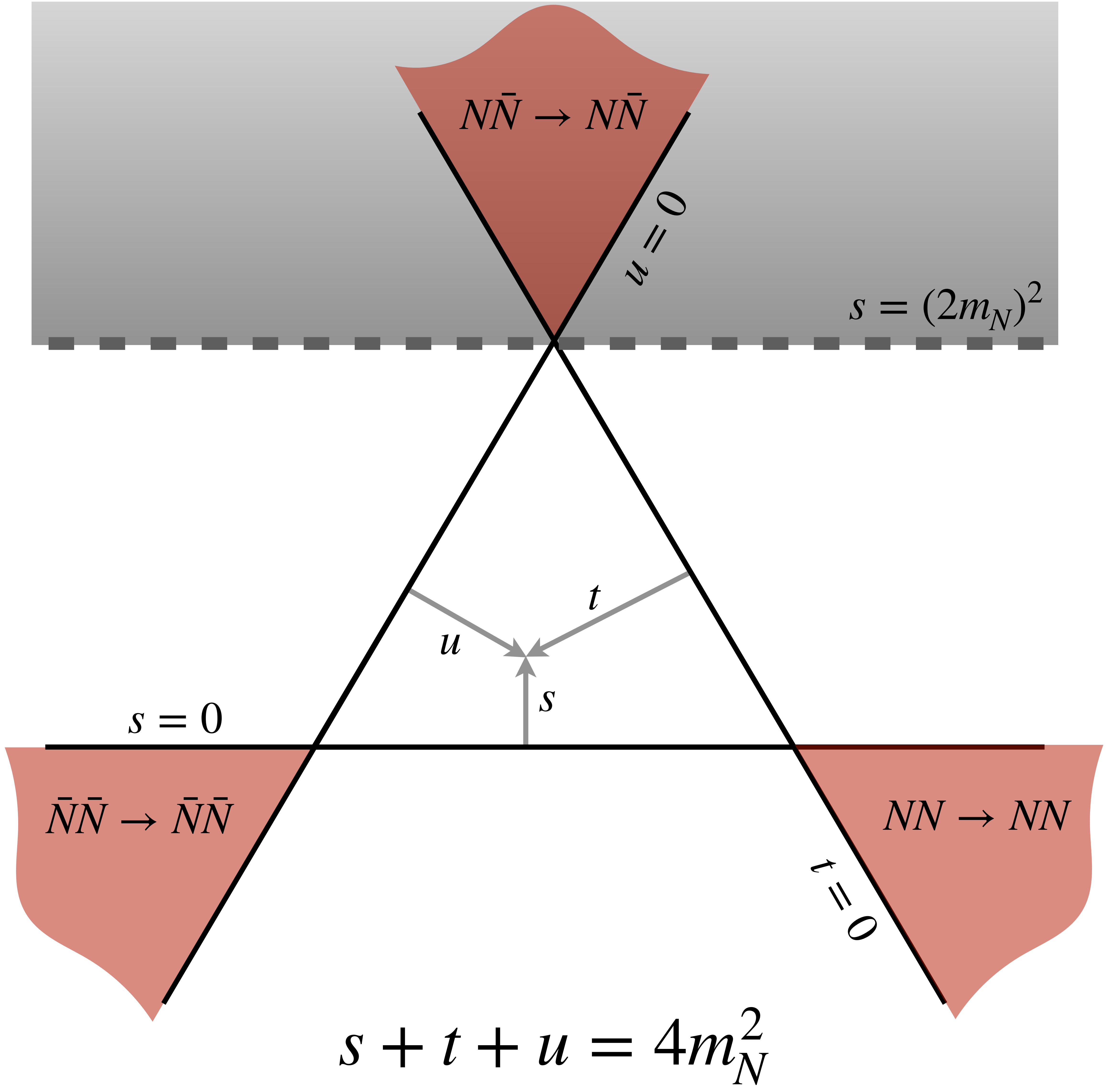
Cut in the u-channel invariant mass

$$u = s_u < -t = -t_u$$



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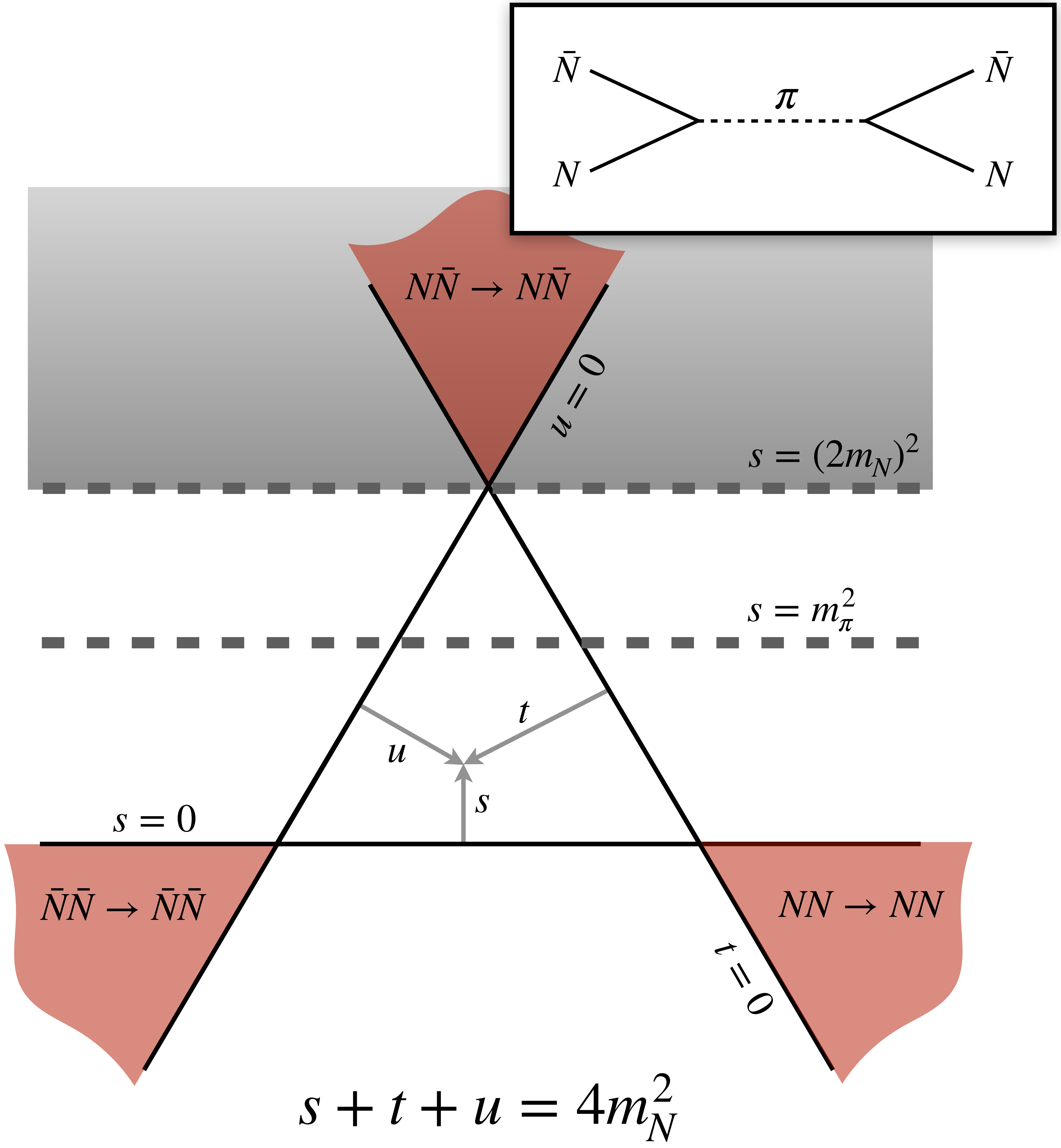

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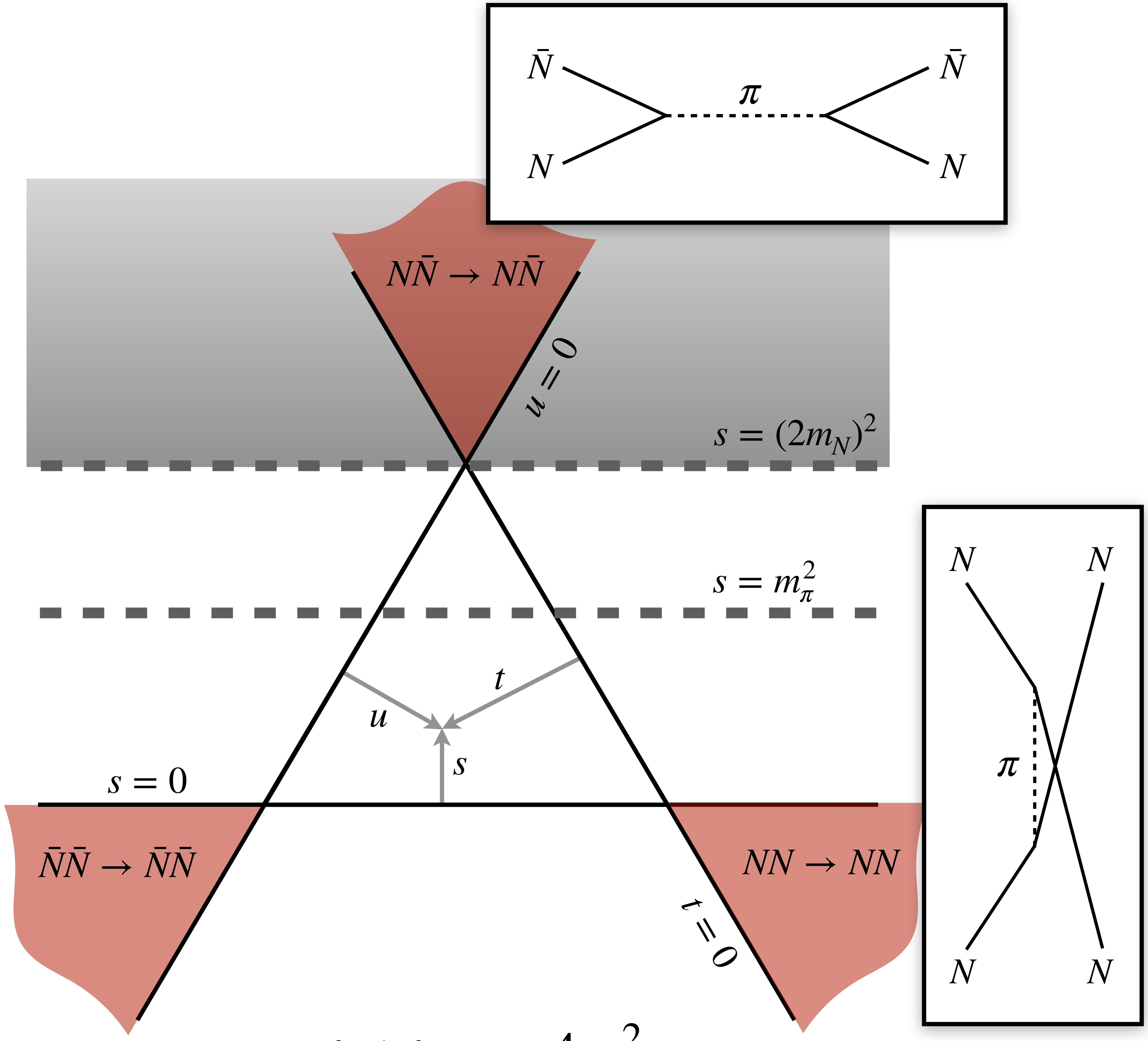
Pole in the u-channel invariant mass

$$s_u = (2m_N)^2 - m_\pi^2 - t_u$$



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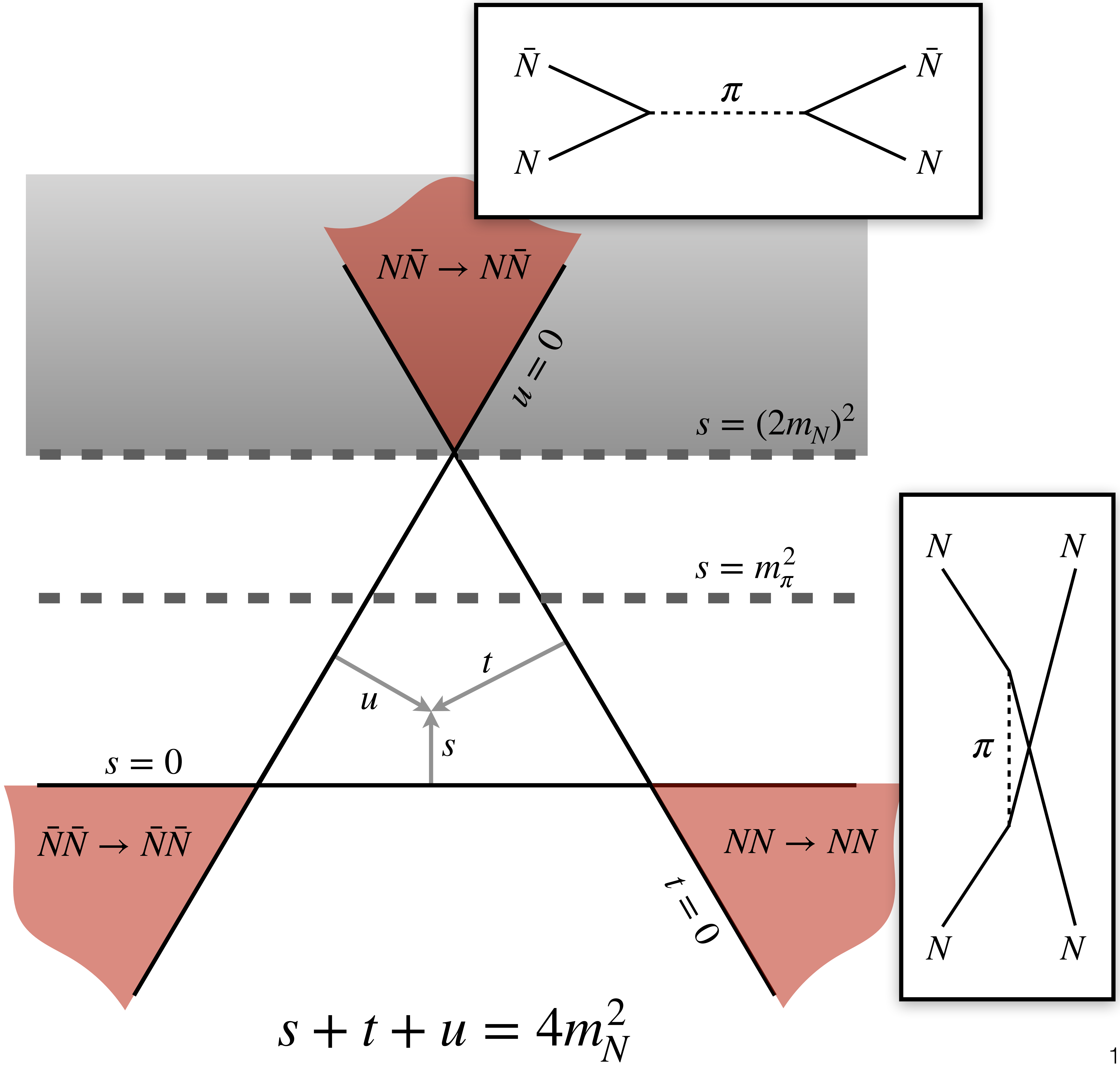
After the partial-wave projection

$$s_u < (2m_N)^2 - m_\pi^2$$



Technical parts of the proof

Unitarity in the finite volume

$$2\,\mathrm{Im}\,\mathcal{M}_L(\boldsymbol{p}',\boldsymbol{p};\boldsymbol{P}) = \frac{\xi}{L^3}\sum_{\boldsymbol{k}} \frac{\pi\delta(E-\omega_1(\boldsymbol{k})-\omega_2(\boldsymbol{P}-\boldsymbol{k}))}{2\omega_1(\boldsymbol{k})\,2\omega_2(\boldsymbol{P}-\boldsymbol{k})}\,\mathcal{M}_L^*(\boldsymbol{k},\boldsymbol{p}';\boldsymbol{P})\,\mathcal{M}_L(\boldsymbol{k},\boldsymbol{p};\boldsymbol{P})$$

Partial-wave projection

$$\mathcal{M}(\boldsymbol{p}',\boldsymbol{p};\boldsymbol{P}) = 4\pi \sum_{\ell' m_\ell'}\sum_{\ell m_\ell} Y^*_{\ell' m_\ell'}(\hat{\boldsymbol{p}}'^{\star})\,\mathcal{M}_{\ell' m_\ell';\ell m_\ell}(s)\,Y_{\ell m_\ell}(\hat{\boldsymbol{p}}^{\star})$$

Replacement of the finite-volume *N* function

$$\mathcal{N}_L(p^\star,\boldsymbol{P}) = \mathcal{T}(s) + \frac{1}{L^3}\sum_{\boldsymbol{k}} \mathcal{L}(\boldsymbol{k},\boldsymbol{P})\,\frac{\left[\mathcal{Y}^*(\boldsymbol{k}^\star)\otimes\mathcal{Y}(\boldsymbol{k}^\star)^T\right]\mathrm{Im}\left[\mathcal{T}(E^\star(\boldsymbol{k})^2)-\mathcal{T}(s)\right]}{E^\star(\boldsymbol{k})^2 - s - i\epsilon}\,\mathcal{N}_L(k^\star,\boldsymbol{P})$$

Equivalence with Lüscher QC

Lüscher, **"Volume dependence of the energy spectrum in massive QFTs"**

Lüscher QC $$\det_{\ell m_\ell}\left[\mathcal{K}^{-1}(s)+F(s)\right]=0$$

Insert relation between the K matrix and *N* (obtained from direct comparison of the K matrix parametrization to N/D)

$$\mathcal{K}^{-1}(s)=\left[\mathbb{1}+\text{p.v.}\int\frac{d\boldsymbol{k}^\star}{(2\pi)^3}\,\mathcal{L}(\boldsymbol{k},\boldsymbol{P})\,\frac{\mathcal{N}(E^\star(\boldsymbol{k})^2)}{s-E^\star(\boldsymbol{k})^2}\right]\mathcal{N}^{-1}(s)$$

Replace p.v. integrals in F and K by sums (up to corrections exponentially suppressed with volume)

$$\text{p.v.}\int\frac{d\boldsymbol{k}}{(2\pi)^3}\,\mathcal{L}(\boldsymbol{k},\boldsymbol{P})\,\frac{\mathcal{N}(s)-\mathcal{N}(E^\star(\boldsymbol{k})^2)}{s-E^\star(\boldsymbol{k})^2}=\frac{1}{L^3}\sum_{\boldsymbol{k}}\mathcal{L}(\boldsymbol{k},\boldsymbol{P})\,\frac{\mathcal{N}(s)-\mathcal{N}(E^\star(\boldsymbol{k})^2)}{s-E^\star(\boldsymbol{k})^2}+\mathcal{O}(e^{-\Delta L})$$