Finite-volume effects in the factorizable contributions to light-meson leptonic decay rates in QCD+QED

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on behalf of the Budapest-Marseille-Wuppertal collaboration

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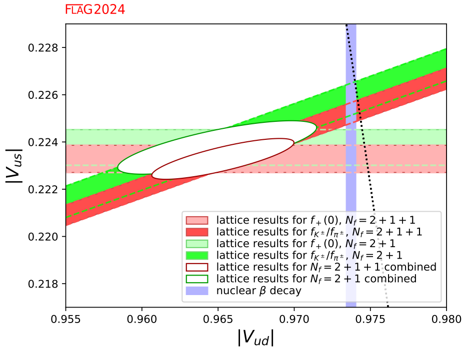
Motivation

Tension in tests of first-row CKM matrix unitarity

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.$$

The ration V_{us}/V_{ud} is extracted from f_K/f_π. The current FLAG average [Aoki et al., Phys. Rev. D 113, 014508]:

$$f_K/f_\pi = 1.1934(19).$$



Because the determination is more precise than 1%, isospin-breaking effects should be included:

$$\underbrace{\left(\frac{m_d - m_u}{\Lambda_{\mathrm{QCD}}}\right)}_{\text{Strong IBE}} \sim 0.01 \qquad \underbrace{\frac{e^2}{4\pi}}_{\text{E.M. IBE}} \sim 0.01.$$

Light-meson decay rate in QCD+QED

The decay rate is computed following the RM123S method [Carrasco et al., Phys. Rev. D 91, 074506] by splitting the decay rate into:

$$\Gamma(P \to l\bar{\nu}_l[\gamma]) = \lim_{V\to\infty} \Big[\underbrace{\Gamma_0(V)}_{\text{lattice}} - \underbrace{\Gamma_0^{\text{pt}}(V)}_{\substack{\text{perturbation}\\ \text{theory}}}\Big] + \underbrace{\lim_{m_\gamma\to 0} \Big[\Gamma_0^{\text{pt}}(m_\gamma) + \Gamma_1(m_\gamma)\Big]}_{\text{perturbation theory}}$$

The IBE correction to the decay rate is defined in the PDG parametrization:

$$\Gamma(P \to l\bar{\nu}_l[\gamma]) = \Gamma_P^{\text{tree}}\left[1 + \delta R_P\right] = \frac{G_F^2}{8\pi}\left|V_{q_1 q_2}\right|^2 M_{P^\pm}\, m_l^2 \left(1 - \frac{m_l^2}{M_{P^\pm}^2}\right)^2 f_P^2\left[1 + \delta R_P\right]$$

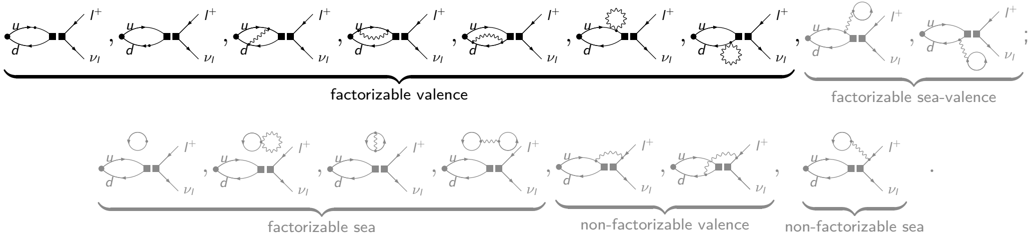
with the correction defined as [Sirlin, Nucl. Phys. B 196, 83–92]:

$$\delta R_P = \underbrace{2\left(\frac{\delta \mathcal{A}_P}{\mathcal{A}_P^0} - \frac{\delta M_P}{M_P^0} + \frac{\delta Z_A}{Z_A^0}\right)}_{\delta R_P^{\text{latt}}} + e^2\left[\frac{1}{2\pi^2}\log\left(\frac{M_Z}{M_W}\right) + \delta\Gamma^{\text{pt}}(\Delta E)\right], \text{ with } \mathcal{A}_\pi^0 = \langle \pi(\vec{0})|\, \bar{u}\gamma_\mu\gamma_5 d\, |0\rangle$$

At this stage, we omit the renormalization constant so $\delta Z_A = 0$. Only one computation is available in the literature [Di Carlo et al., Phys. Rev. D 100, 034514]. More results to the IBE to f_K/f_π [Giusti et al., Phys. Rev. Lett. 120, 072001] [Boyle et al., JHEP 02, 242] [Tuo, Lattice 2026 Mon. 27th 11:30].

Factorizable corrections

The relevant diagrams are:



We focus on the factorizable valence diagrams. The correction to the mass and the amplitude are extracted from the corrections to the correlators:

$$\frac{\delta G^P(x_4)}{G^P(x_4)} \underset{\substack{(T-x_4)\to\infty \\ x_4\to\infty}}{\sim} \left[2\frac{\delta Z_P}{Z_P} - \frac{\delta M_P}{M_P} - \frac{T}{2}\coth\left(M_P T/2\right)\delta M_P + \left(T/2 - x_4\right)\tanh\left[M_P\left(T/2 - x_4\right)\right]\delta M_P\right]$$

$$\frac{\delta G^A(x_4)}{G^A(x_4)} \underset{\substack{(T-x_4)\to\infty \\ x_4\to\infty}}{\sim} \left[\frac{\delta \mathcal{A}_P}{\mathcal{A}_P} + \frac{\delta Z_P}{Z_P} - \frac{\delta M_P}{M_P} - \frac{T}{2}\coth\left(M_P T/2\right)\delta M_P + \left(T/2 - x_4 - 1/2\right)\coth\left[M_P\left(T/2 - x_4 - 1/2\right)\right]\delta M_P\right].$$

Simulation details

- Tree-level-improved Lüscher-Weisz gauge action;
- $N_f = 2+1+1$ staggered fermions;
- Four levels of stout smearing $\rho = 0.125$;
- Physical Pion mass;
- Lattice spacing $a \approx 0.1315$ fm;
- Goldstone pion $\gamma_5 \otimes \xi_5$;
- $\mathrm{QED_L}$ and $\mathrm{QED_r}$;
- Fit with and without the ensemble $M_\pi L = 2.19$;
- Blinded preliminary results.

$M_\pi L$	$L \times T$ [fm]	#confs
2.19	3.16×6.31	716
2.86	4.21×8.24	938
3.55	5.26×8.42	840
4.26	6.31×8.42	904
5.67	8.42×8.42	521

The QCD+QED scheme is fixed by matching $M_{\pi^\pm}$, $M_{K^\pm}$, M_{K^0} to their PDG values [Navas et al., Phys. Rev. D 110, 030001]. The scale is set using w_0 [Boccaletti et al., Nature 653, 373–377]. The tuning is done in the electroquenched approximation.

$$M_{\pi^\pm} = 139.57039 \text{ MeV}, \qquad M_{K^0} = 497.611 \text{ MeV}$$
$$M_{K^\pm} = 493.677 \text{ MeV}, \qquad w_0 = 0.17245 \text{ fm}.$$

Finite volume effects in the QED correction to the mass

Finite volume formula

The QED finite volume power-law corrections are [Di Carlo et al., JHEP 10, 026], [Di Carlo et al., Phys. Rev. D 105, 074509], [Borsanyi et al., Science 347, 1452–1455]:

$$\frac{\delta M_P^i(L) - \delta M_P}{M_P} = \frac{e^2}{2}\left\{ \underbrace{\frac{q_P^2 c_2^i}{4\pi^2 M_P L} + \frac{q_P^2 c_1^i}{2\pi\,(M_P L)^2}}_{\text{universal}} - \underbrace{\frac{c_0^i}{(M_P L)^3}\left(\frac{\langle r_P^2\rangle M_P^2}{3} + \mathcal{C}\right)}_{\text{structure dependent}} + O\left(\frac{1}{(M_P L)^4}\right)\right\}$$

with q_P the meson charge, $\mathcal{C}$ the contribution from the branch cut due to non-locality of $\mathrm{QED}_{\mathrm{L,r}}$ and $\langle r_P^2\rangle$ the meson charge radius with PDG values [Navas et al., Phys. Rev. D 110, 030001]:

$$\sqrt{\langle r^2_{\pi\pm}\rangle} = 0.659(4)\ \mathrm{fm}, \quad \sqrt{\langle r^2_{K\pm}\rangle} = 0.560(32)\ \mathrm{fm}$$

the coefficients c_α^i have the following values:

$$\begin{aligned} c_2^{\mathrm{QED_L}} &= -8.9136, & c_1^{\mathrm{QED_L}} &= -2.8373, & c_0^{\mathrm{QED_L}} &= -1; \\ c_2^{\mathrm{QED_r}} &= -7.9136, & c_1^{\mathrm{QED_r}} &= -1.8373, & c_0^{\mathrm{QED_r}} &= 0 \end{aligned}$$

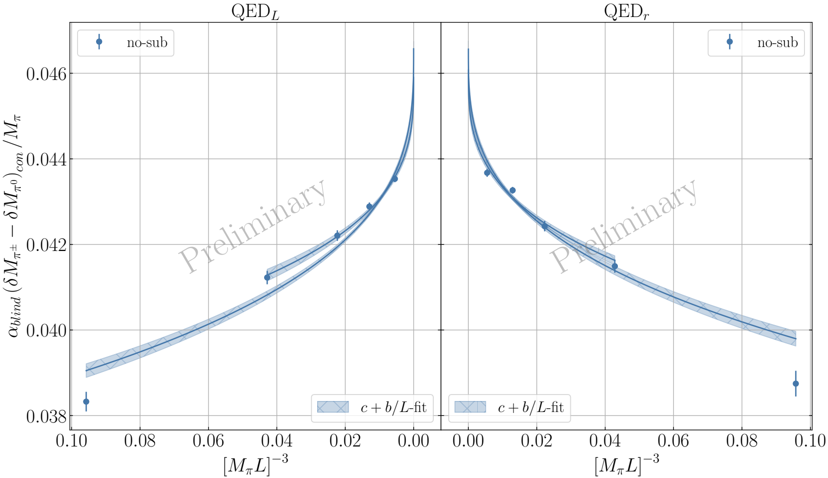
There are some residual effects of the type:

$$\mathrm{res}^{\mathrm{QED_L}}(M_P L) = O\left(e^{-M_P L}\right), \quad \mathrm{res}^{\mathrm{QED_r}}(M_P L) = O\left(e^{-M_P L}, \frac{1}{(M_P L)^5}\right).$$

Finite volume effects in the QED correction to the mass

Effect in the pion mass difference

$$\frac{\delta M^i_{\pi^\pm} - \delta M^i_{\pi^0}}{M_\pi} = \frac{\delta M^i_{\pi^\pm}(L) - \delta M^i_{\pi^0}(L)}{M_\pi} + O\left(\frac{1}{(M_\pi L)}\right) + O\left(e^{-M_\pi L}\right)$$



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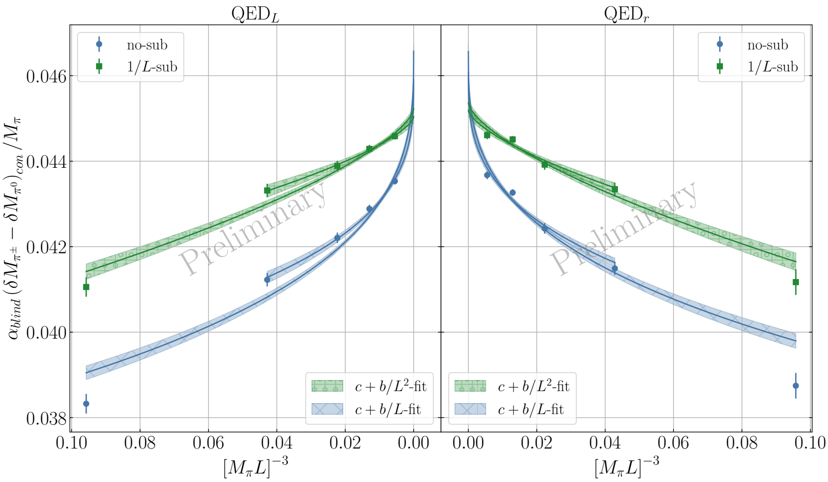
$$\frac{\delta M^i_{\pi^\pm} - \delta M^i_{\pi^0}}{M_\pi} = \frac{\delta M^i_{\pi^\pm}(L) - \delta M^i_{\pi^0}(L)}{M_\pi} - \frac{e^2 c^i_2}{8\pi^2 M_\pi L} \qquad + O\left(\frac{1}{(M_\pi L)^2}\right) + O\left(e^{-M_\pi L}\right)$$



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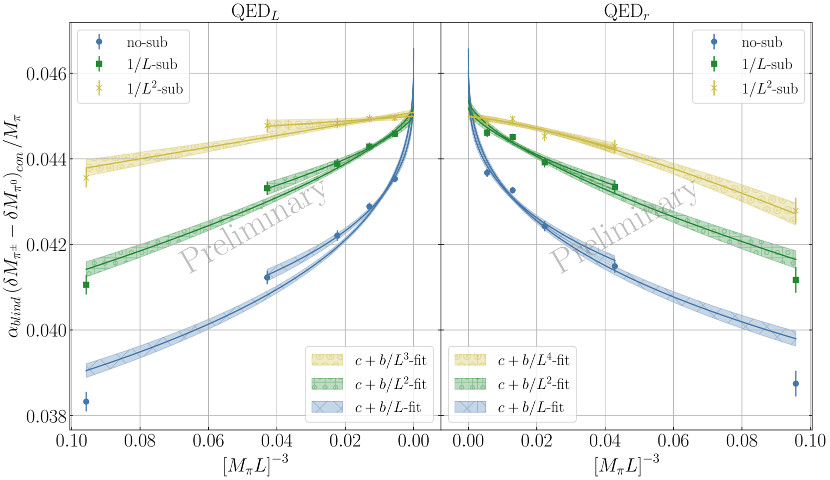

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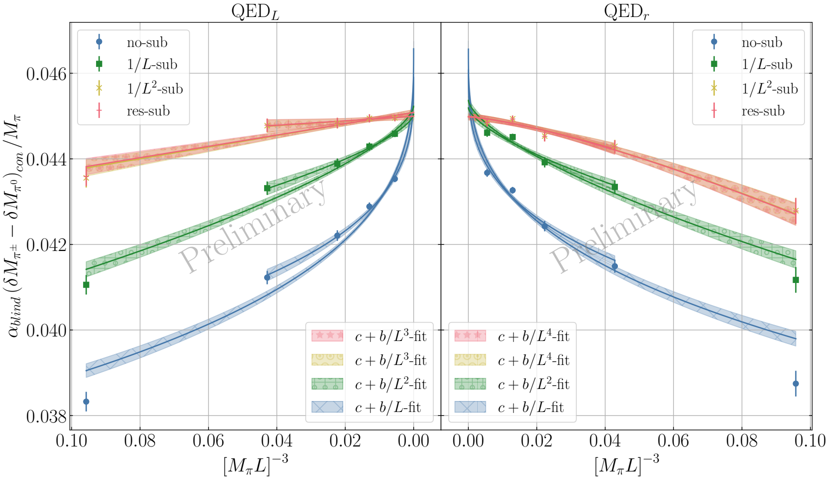


Finite volume effects in the QED correction to the decay rate

Finite volume formula

The pointlike finite volume decay rate is:

$$\Gamma_0^{\mathrm{pt}}(L) = \left(1 + \frac{e^2}{8\pi^2} Y_P^{\mathrm{pt}}(L)\right) \Gamma_P^{\mathrm{tree}}$$

for the factorizable diagrams, the finite volume expression is [Lubicz et al., Phys. Rev. D 95, 034504], [Di Carlo et al., Phys. Rev. D 105, 074509]:

$$Y_P^{\mathrm{fc,pt}}(L) = \underbrace{-\log\left(\frac{M_W}{M_P}\right) - 3 + 2\left(1 + \frac{c_3^i}{4\pi} - \log(4\pi)\right) + \log\left[(M_P L)^2\right]}_{\mathrm{IR}^i(M_P L)} + \frac{2c_2^i}{M_P L} + \frac{2\pi c_1^i}{(M_P L)^2} + O\left(e^{-M_P L}\right)$$

The factorizable structure-dependent contribution is [Di Carlo et al., Phys. Rev. D 105, 074509]:

$$\Delta Y_P^{\mathrm{fc,SD}}(L) = -\frac{2\pi\langle r_P^2\rangle M_P^2 c_1^i}{3\,(M_P L)^2}.$$

The coefficients are:

$$c_3^{\mathrm{QED_L}} = 3.8219,\, c_3^{\mathrm{QED_r}} = 4.8219.$$

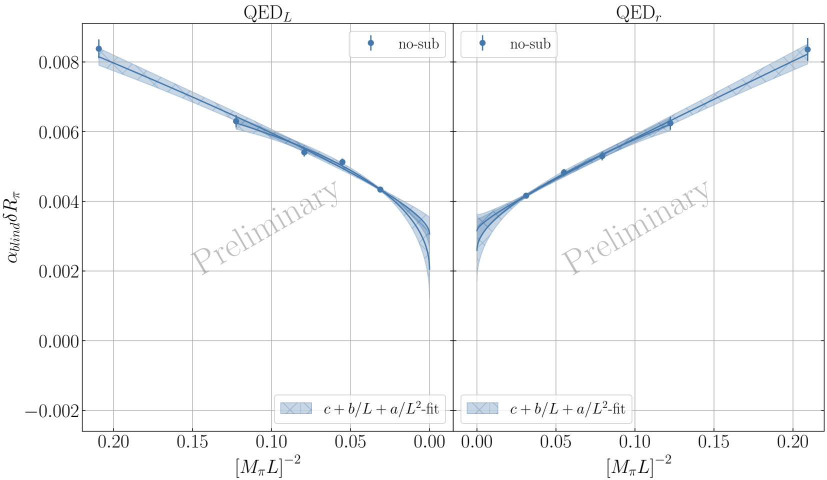
For the decay rate, the residue is:

$$\mathrm{res}^{\mathrm{QED_L}}(M_P L) = O\left(e^{-M_P L}\right),\ \mathrm{res}^{\mathrm{QED_r}}(M_P L) = O\left(e^{-M_P L}, \frac{1}{(M_P L)^5}\right).$$

Finite volume effects in the QED correction to the decay rate

Pion channel

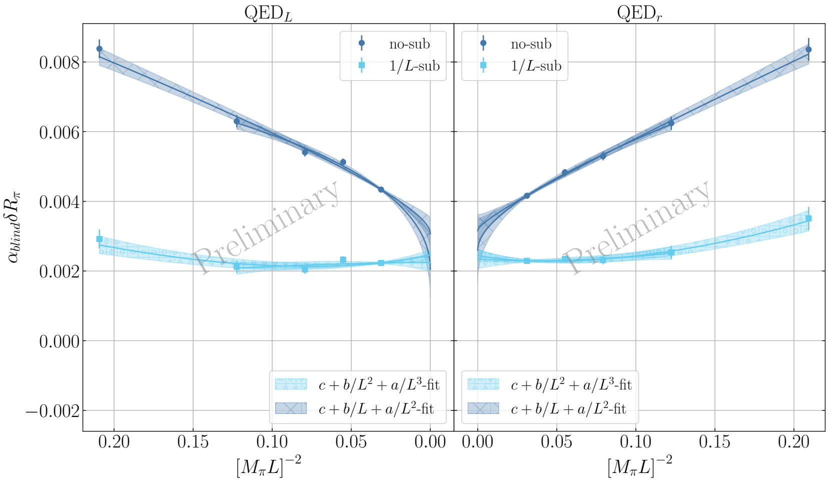
$$\delta R^i_\pi = \delta R^i_\pi(L) - \mathrm{IR}^i\,(M_\pi L) \qquad + O\left(\frac{1}{(M_\pi L)^2}\right) + O\left(e^{-M_\pi L}\right)$$



Finite volume effects in the QED correction to the decay rate

Pion channel

$$\delta R_\pi^i = \delta R_\pi^i(L) - \mathrm{IR}^i(M_\pi L) - \frac{e^2 c_2^i}{4\pi^2 M_\pi L} + O\left(\frac{1}{(M_\pi L)^2}\right) + O\left(e^{-M_\pi L}\right)$$



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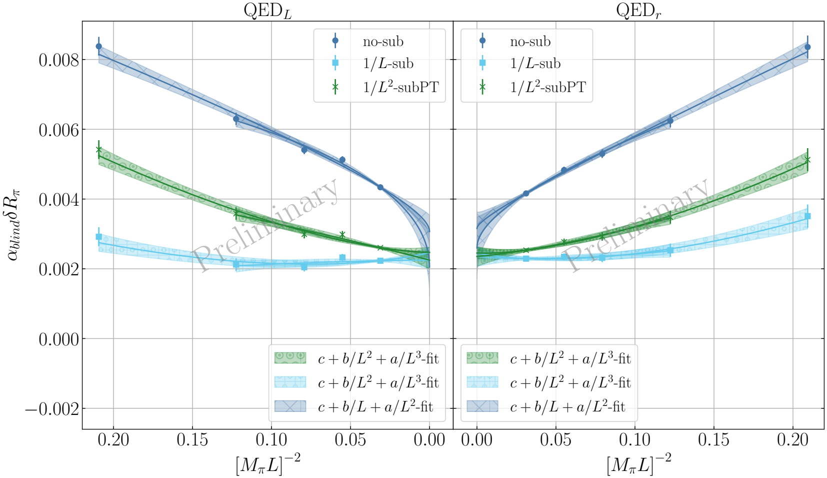

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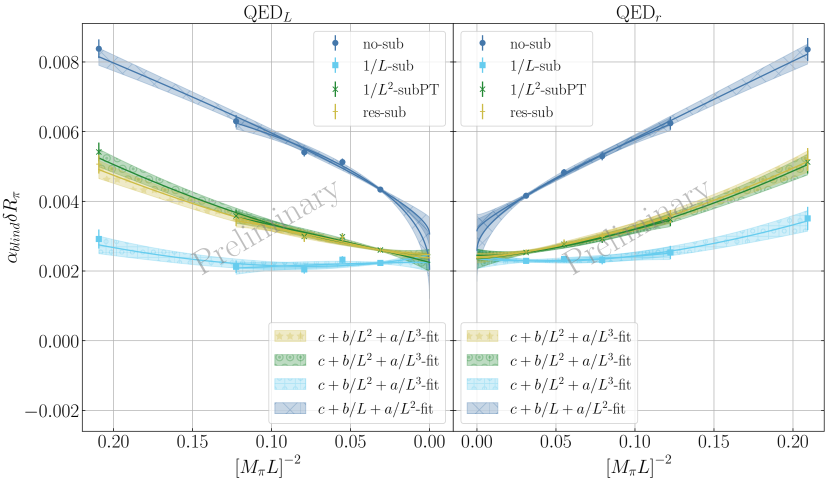

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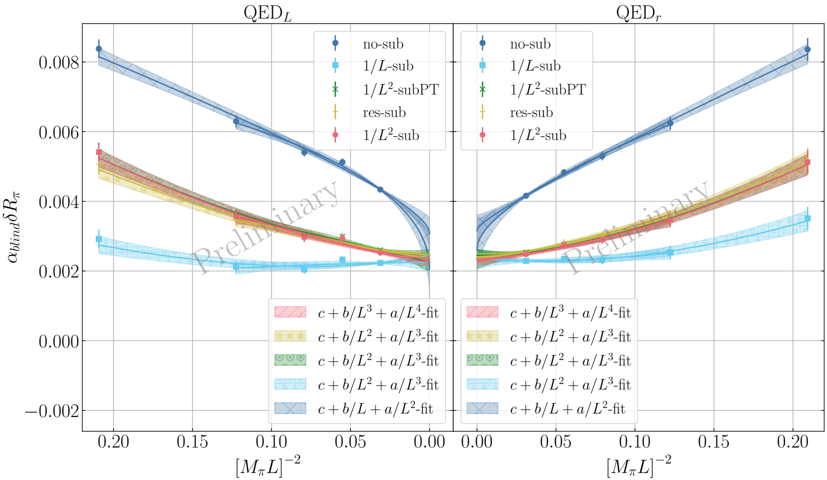

Finite volume effects in the QED correction to the decay rate

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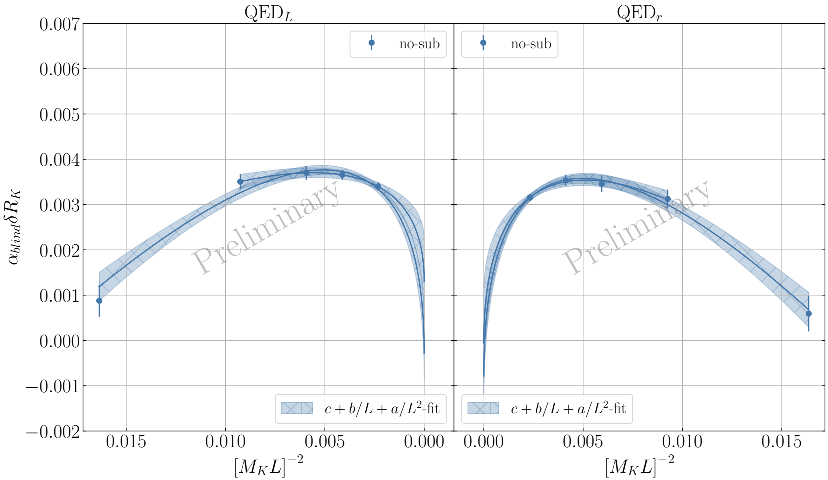
$$\delta R_K^i = \delta R_K^i(L) - \mathrm{IR}^i\,(M_K L) \qquad + O\left(\frac{1}{(M_K L)^2}\right) + O\left(e^{-M_\pi L}\right)$$



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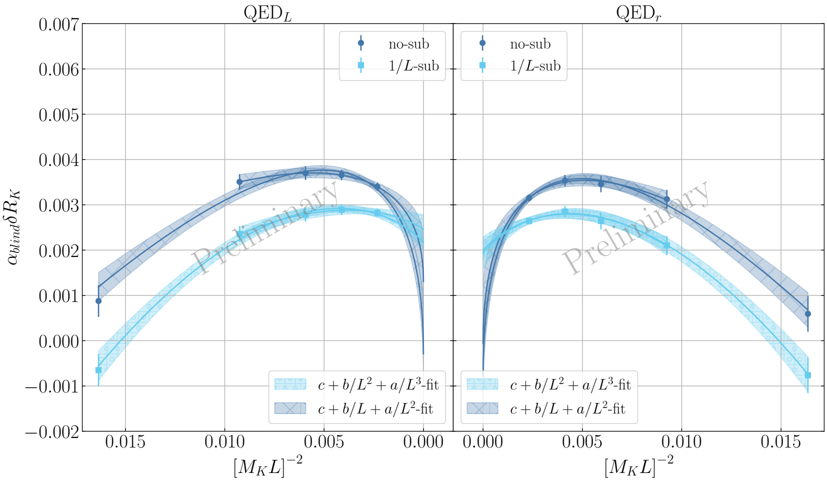

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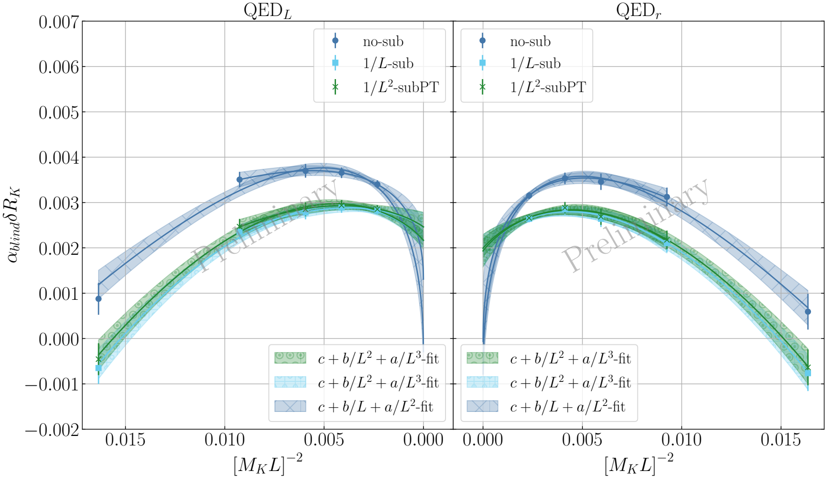

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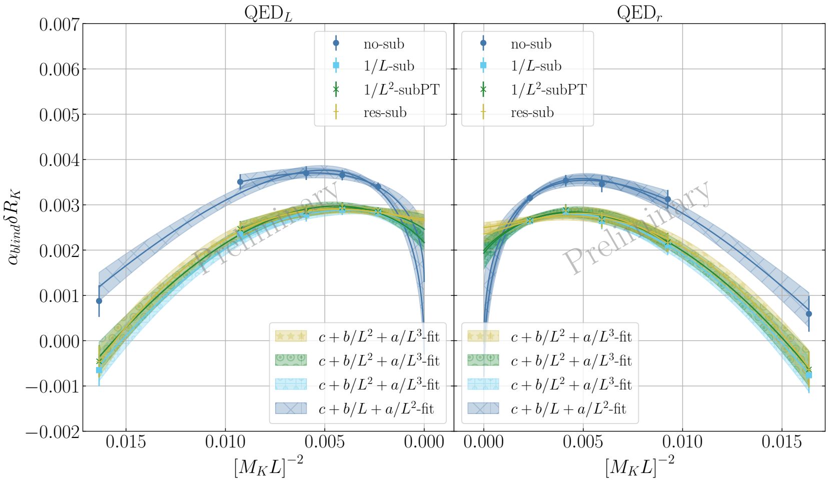

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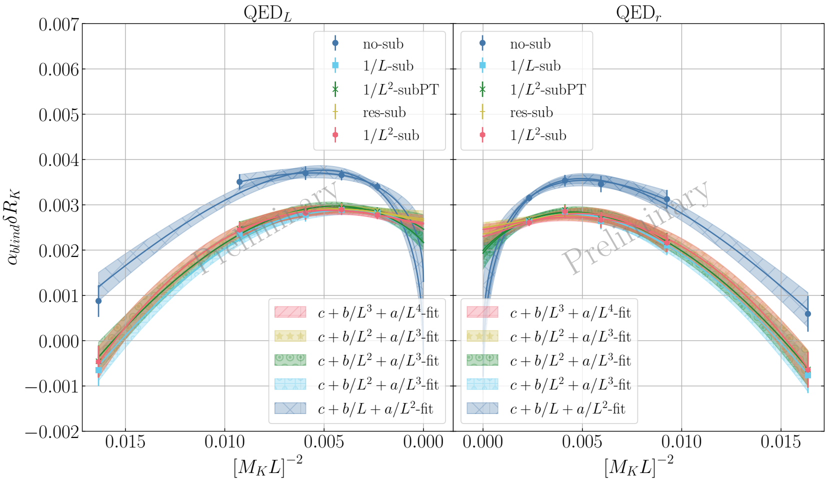

Conclusions and outlook

- We observe sizable higher-order finite volume effects in the factorizable contribution;
- $M_\pi L \approx 2.19$ might present some big QCD exponential effects;
- QED_r and QED_L seem to give similar results for the mass and the factorizable contribution to the decay rate;
- The residual QED exponential effects seem to be irrelevant in the observed range of $M_\pi L$;
- We are working on studying the effect for finer lattice spacings;
- We are working on implementing the non-factorizable diagrams and the renormalization of the electroweak hamiltonian;

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Thank you for your attention!

Backup: Extracting the correction from the data

We determine the mass correction from a plateau fit to:

$$\delta M_P^{\text{eff,loc}}(x_4) = \frac{1}{\Delta} \frac{\frac{\delta G(x_4+\Delta)+\delta G(x_4-\Delta)}{G(x_4+\Delta)+G(x_4-\Delta)} - \frac{\delta G(x_4)}{G(x_4)}}{\sqrt{\left[\frac{2G(x_4)}{G(x_4+\Delta)+G(x_4-\Delta)}\right]^2 - 1}}.$$

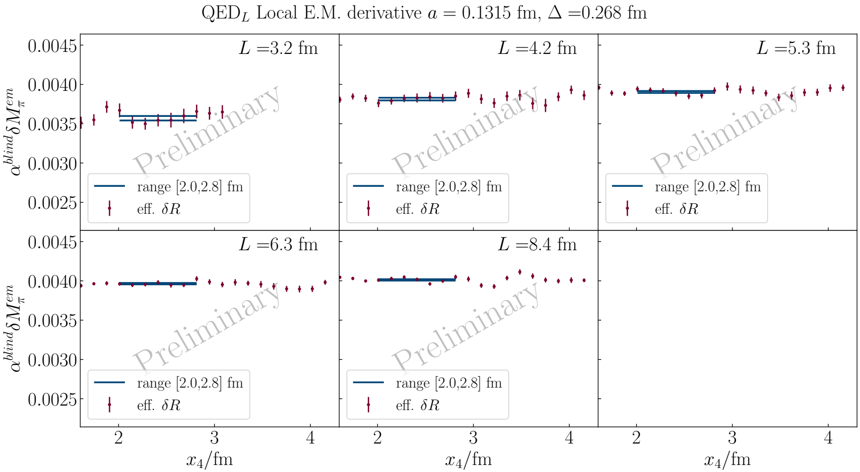
The correction δR_P is then obtained as:

$$\begin{aligned}\delta R_P =& 2\frac{\delta G^A(x_4)}{G^A(x_4)} - \frac{\delta G^P(x_4)}{G^P(x_4)} - \frac{\delta M_P}{M_P} + \delta M_P \left\{ \frac{T}{2} \tanh\left(M_P T/2\right) \right. \\ & - 2(T/2 - x_4 - 1/2) \coth\left[M_P\left(T/2 - x_4 - 1/2\right)\right] \\ & \left. + \left(T/2 - x_4\right) \tanh\left[M_P\left(T/2 - x_4\right)\right] \right\}\end{aligned}$$

We use even values of Δ to reduce the effect of the oscillating parity partner.

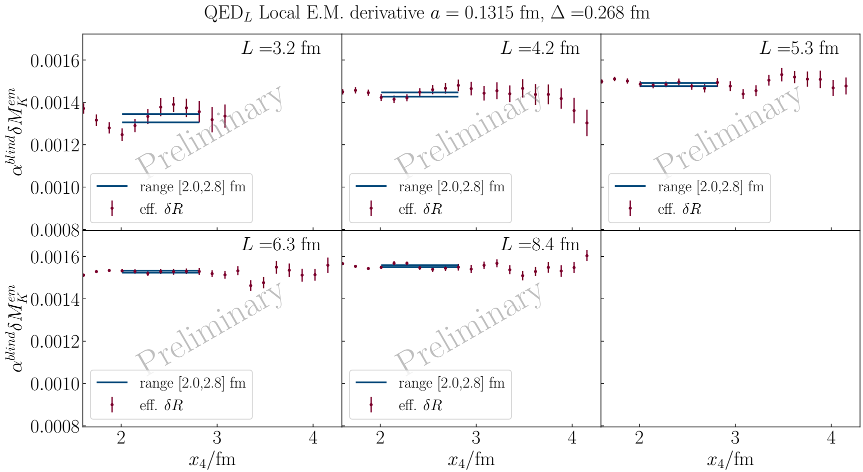
Backup: Example of effective mass correction

Pion

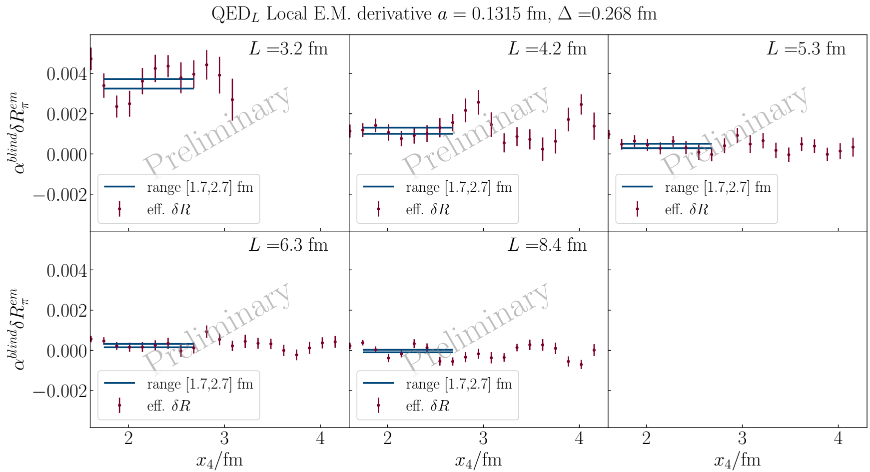


Backup: Example of effective mass

Kaon



Backup: Example of effective δR_π



Backup: Example of effective δR_K