Encoding Entropy & Spatial Entanglement

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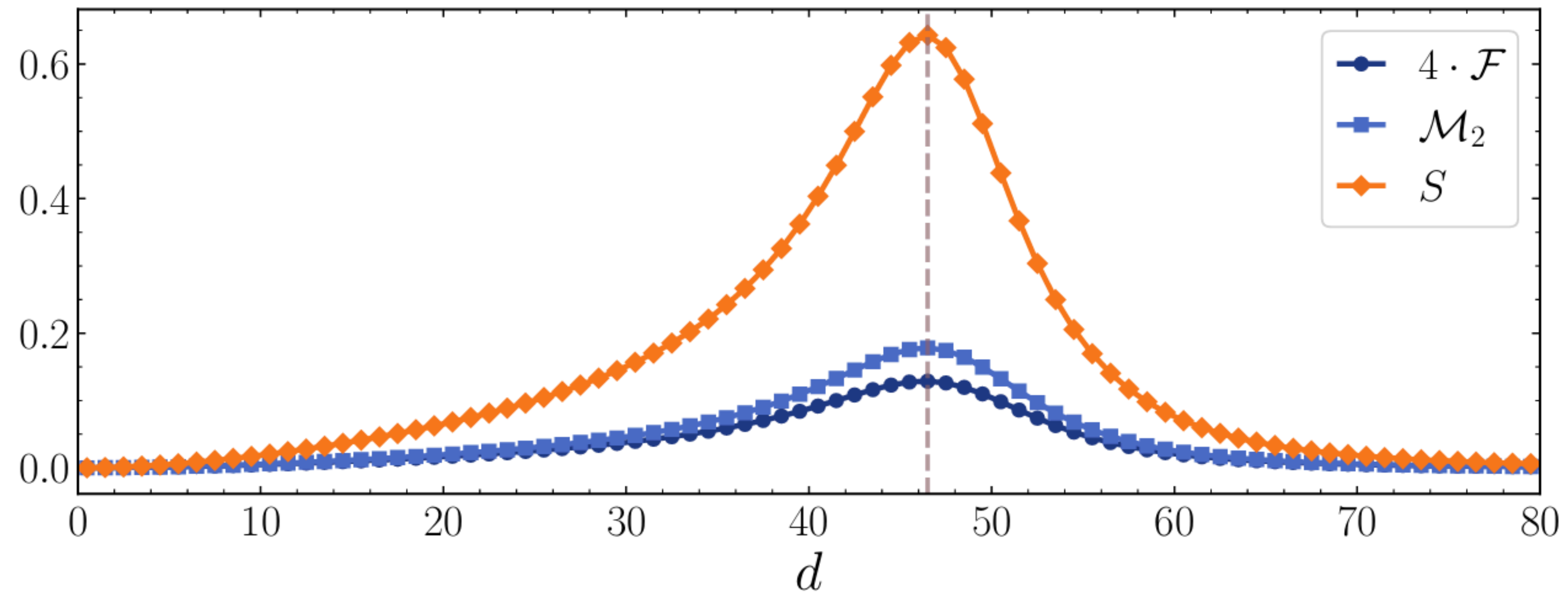
(In collaboration with Emil Mathew, Zohreh Davoudi, and Indrakshi Raychowdhury.)

Entanglement entropy and quantum simulations

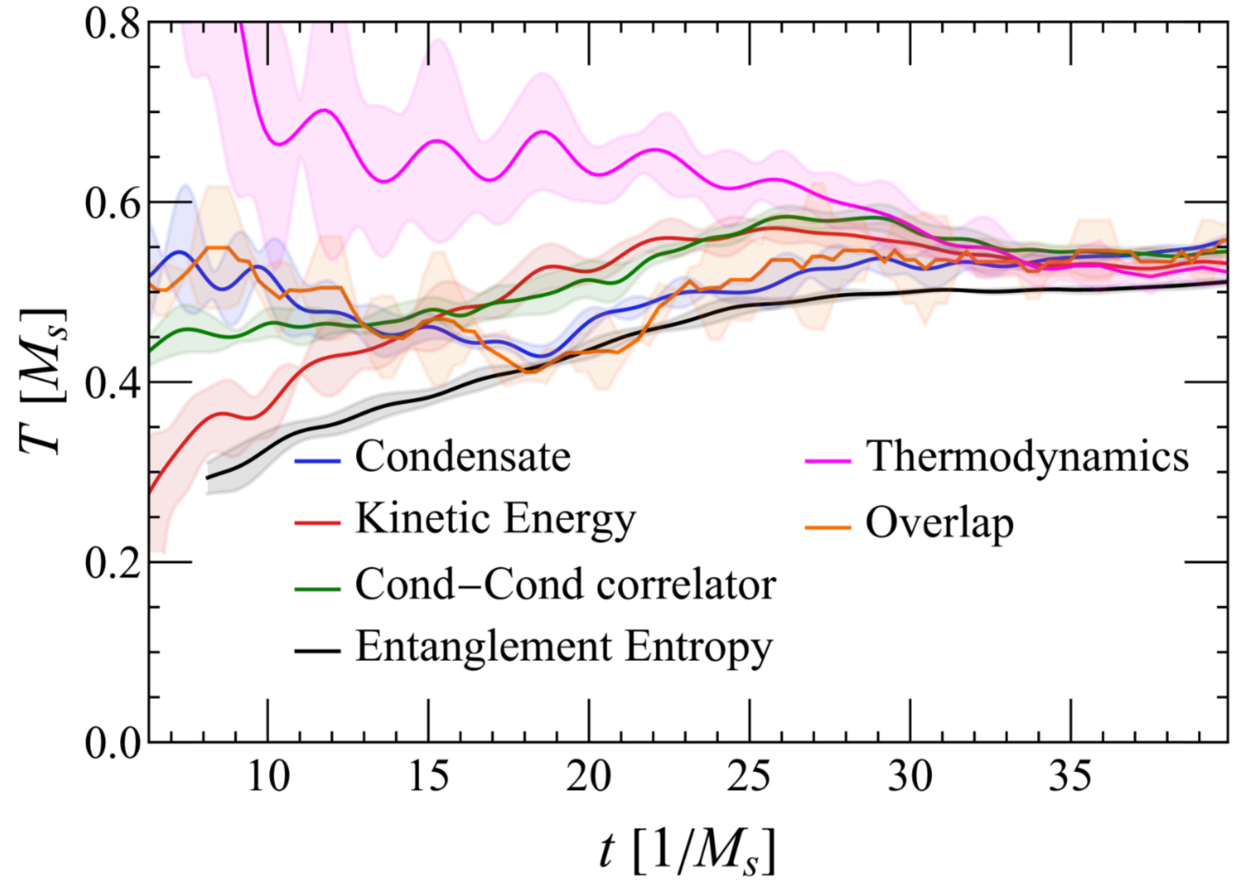
PHYSICS DIAGNOSTIC

String breaking transition



Grieninger, Savage, and Zemlevskiy (2026)

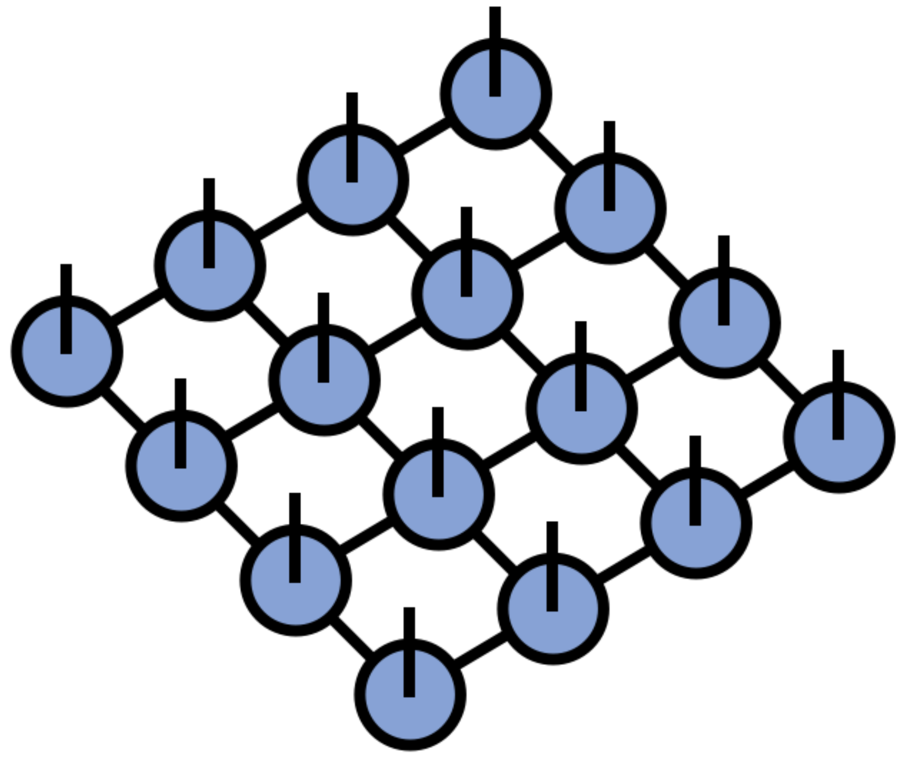
Jet temperature



Florio, Frenklakh, Grieninger et al. (2025)

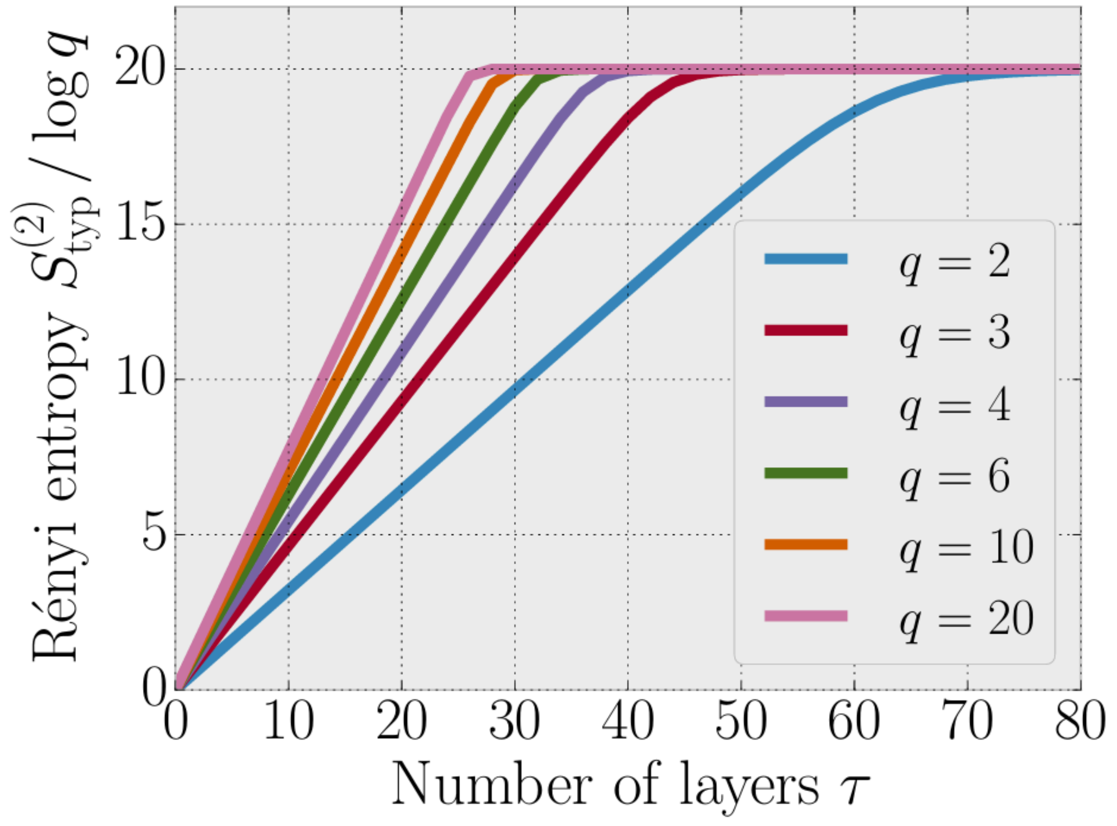
COST DIAGNOSTIC

Bond dimension



www.tensornetwork.org

Circuit depth



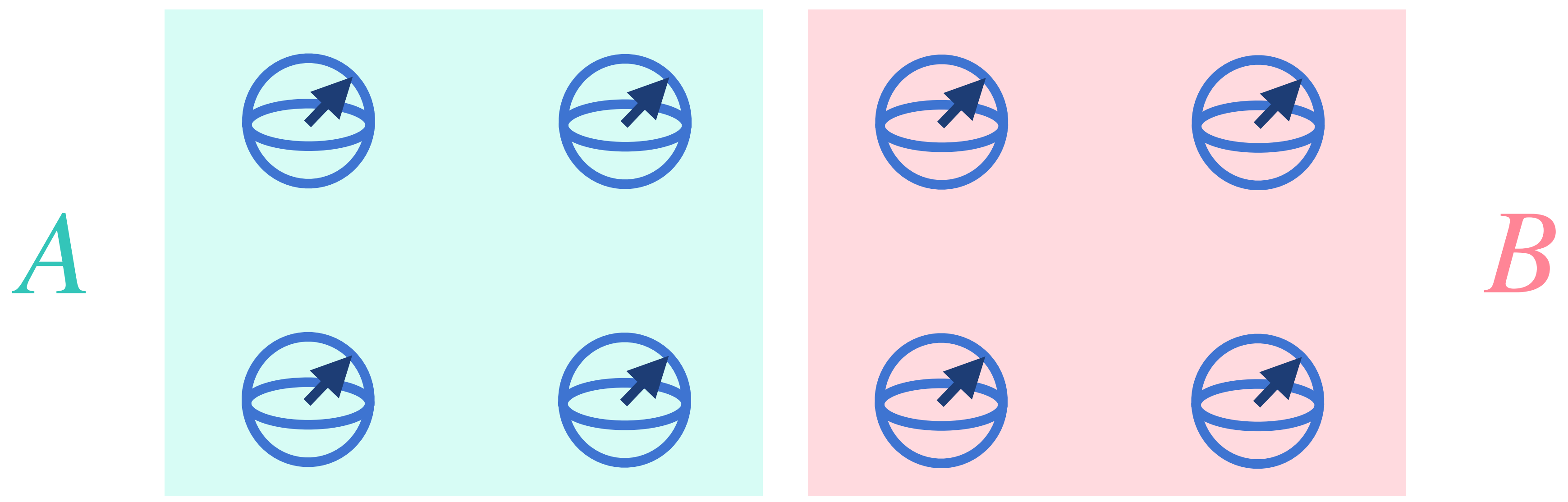
von Keyserlingk, Rakovszky, Pollmann, and Sondhi (2017)

Entanglement entropy

STATE

$$\rho = |\phi\rangle\langle\phi|$$

SUBSYSTEM



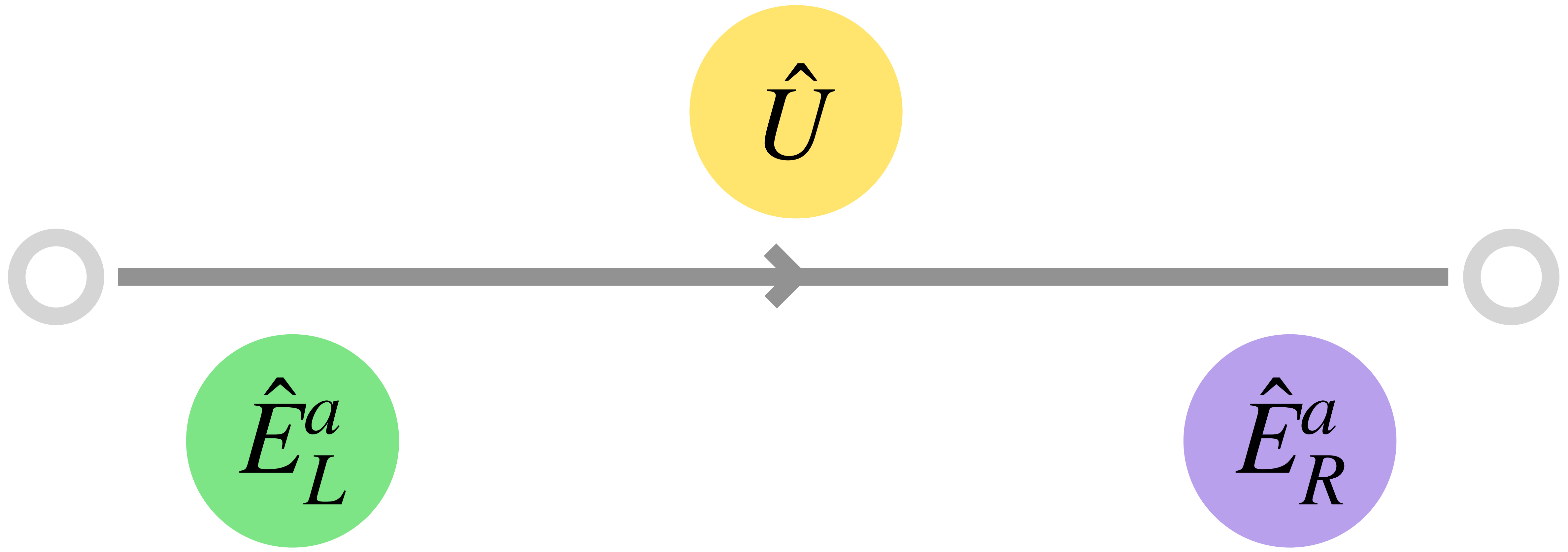
$$\mathbb{H} = \mathbb{H}_A \otimes \mathbb{H}_B$$

$$\rho_A = \mathrm{Tr}_B\, \rho$$

$$S_A(\rho) = -\,\mathrm{Tr}\, \rho_A \log \rho_A$$

SU(2) gauge link

GAUGE LINK l

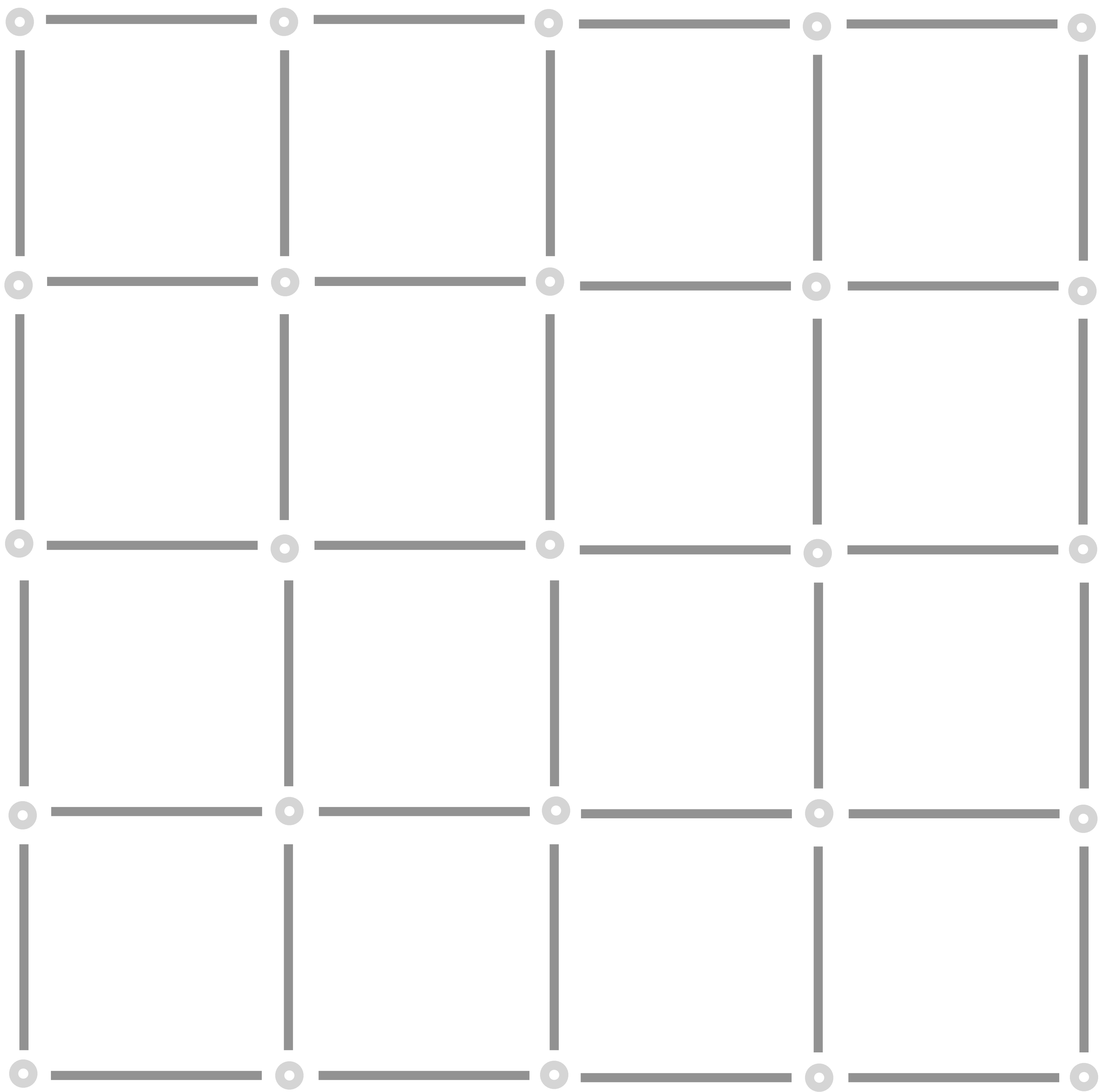


$(a = 1,2,3)$

$$\mathbb{H}_l \cong L^2\left(SU(2)\right) \qquad |j, m_L, m_R\rangle$$

Zohar and Burrello (2015)

SU(2) lattice gauge theory

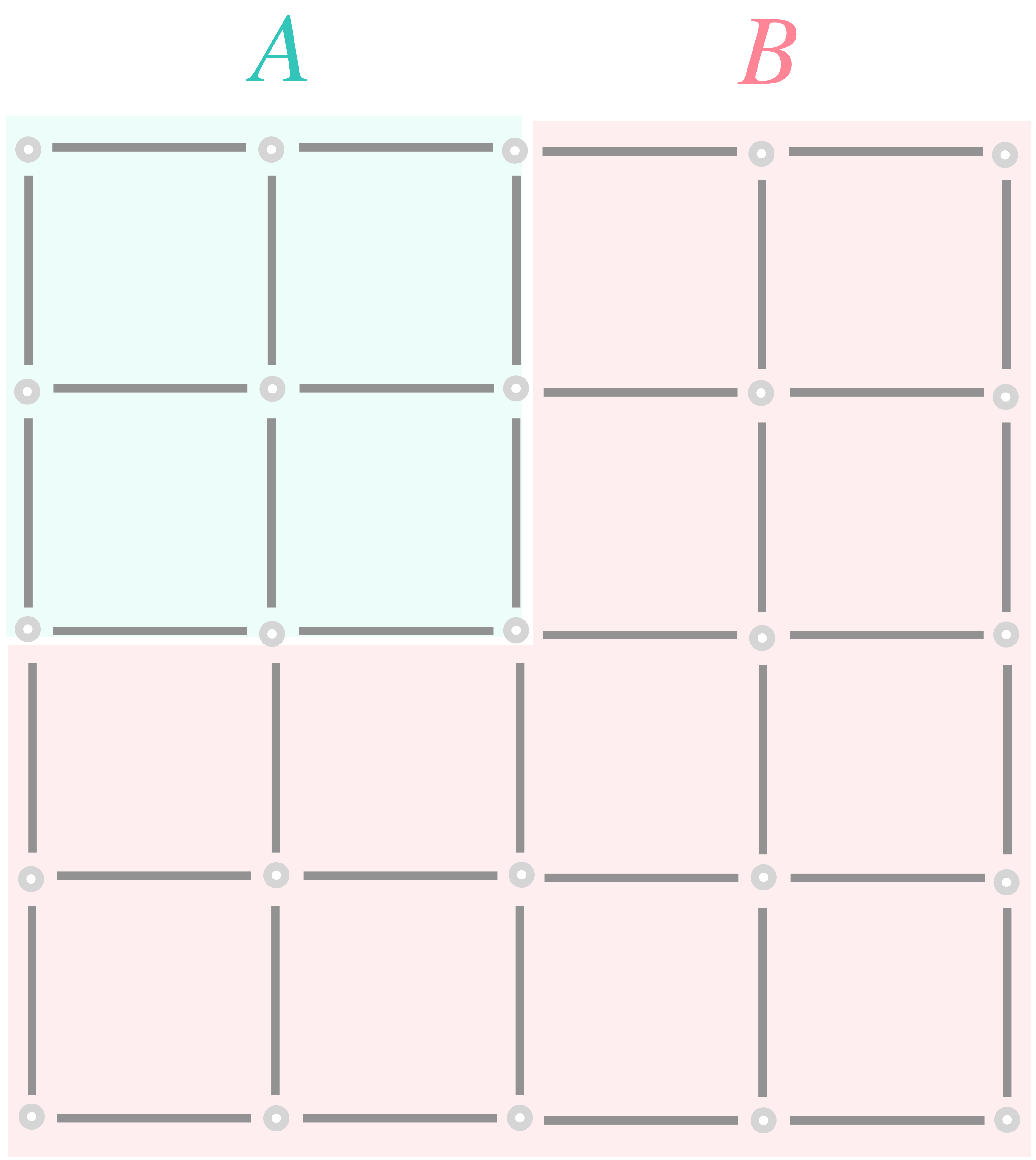


EXTENDED HILBERT SPACE

$$\mathbb{H}^{\text{ext}} = \bigotimes_l \mathbb{H}_l$$

Buividovich and Polikarpov (2008), Donnelly (2012, 2014), Aoki et al. (2015), Ghosh et al. (2015), Van Acoleyen et al. (2015)

SU(2) lattice gauge theory



EXTENDED HILBERT SPACE

$$\mathbb{H}^{\text{ext}} = \bigotimes_l \mathbb{H}_l$$

SUBSYSTEMS

$$\mathbb{H}^{\text{ext}} = \mathbb{H}_A^{\text{ext}} \otimes \mathbb{H}_B^{\text{ext}}$$

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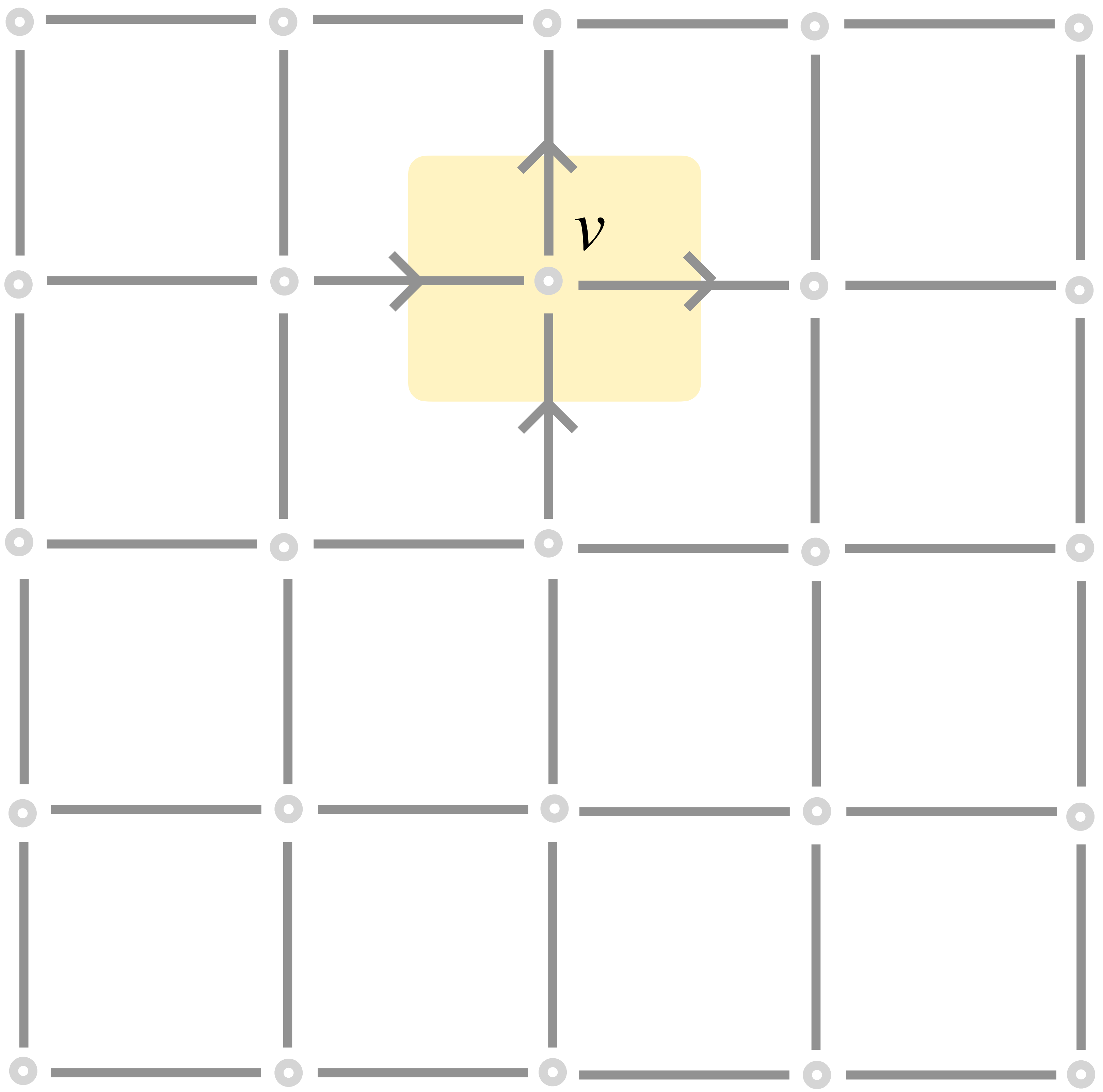


GAUSS'S LAW OPERATORS

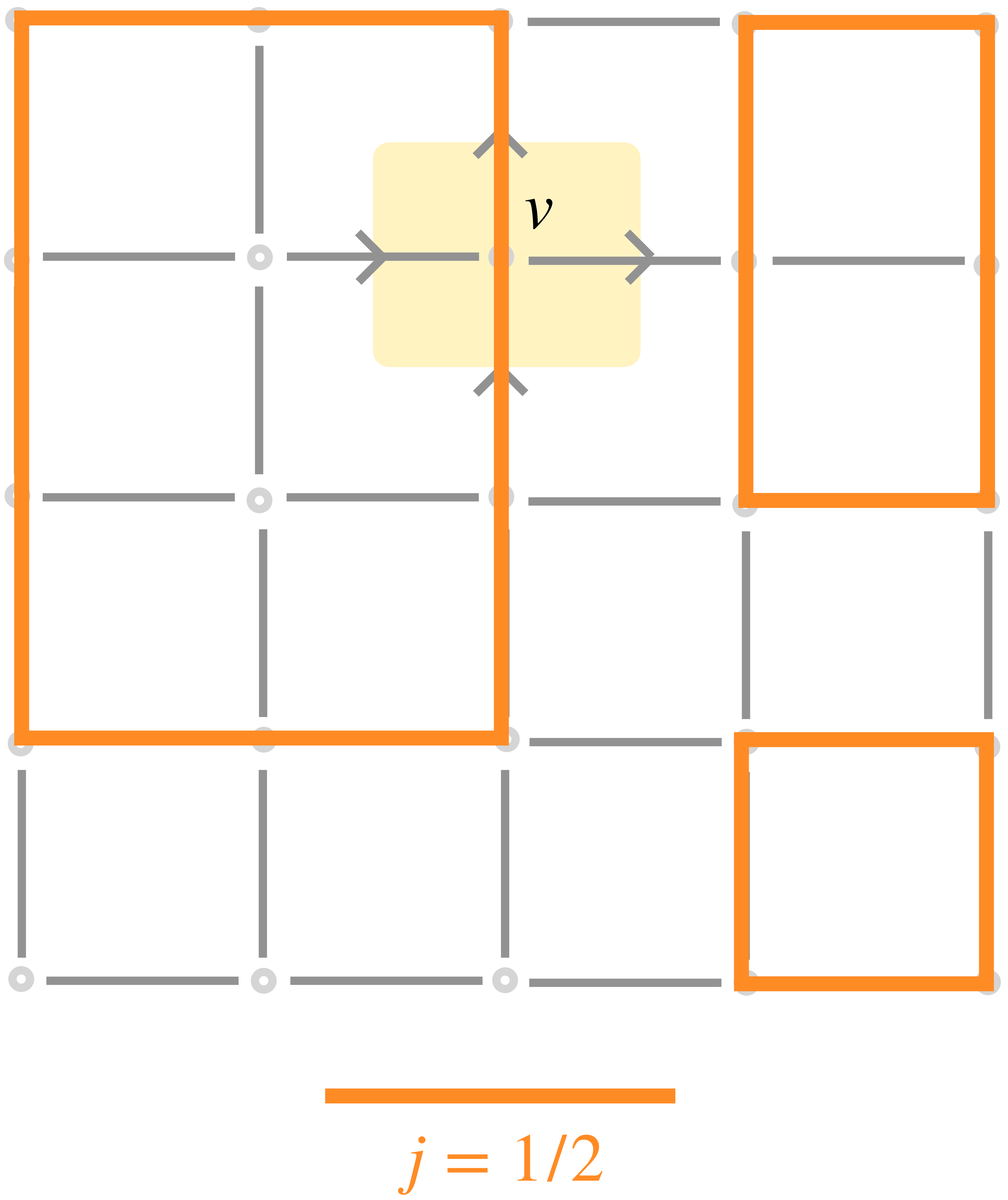
$$\hat{G}^a(v) = \sum_{\substack{l \text{ incoming} \\ \text{at } v}} \hat{E}^a_R(l) - \sum_{\substack{l \text{ outgoing} \\ \text{from } v}} \hat{E}^a_L(l)$$

STATES

$$\mathbb{H}^{\text{phys}} = \{ |\phi\rangle, \hat{G}^a(v)|\phi\rangle = 0 \}$$

Buividovich and Polikarpov (2008), Donnelly (2012, 2014), Aoki et al. (2015), Ghosh et al. (2015), Van Acoleyen et al. (2015)

SU(2) lattice gauge theory



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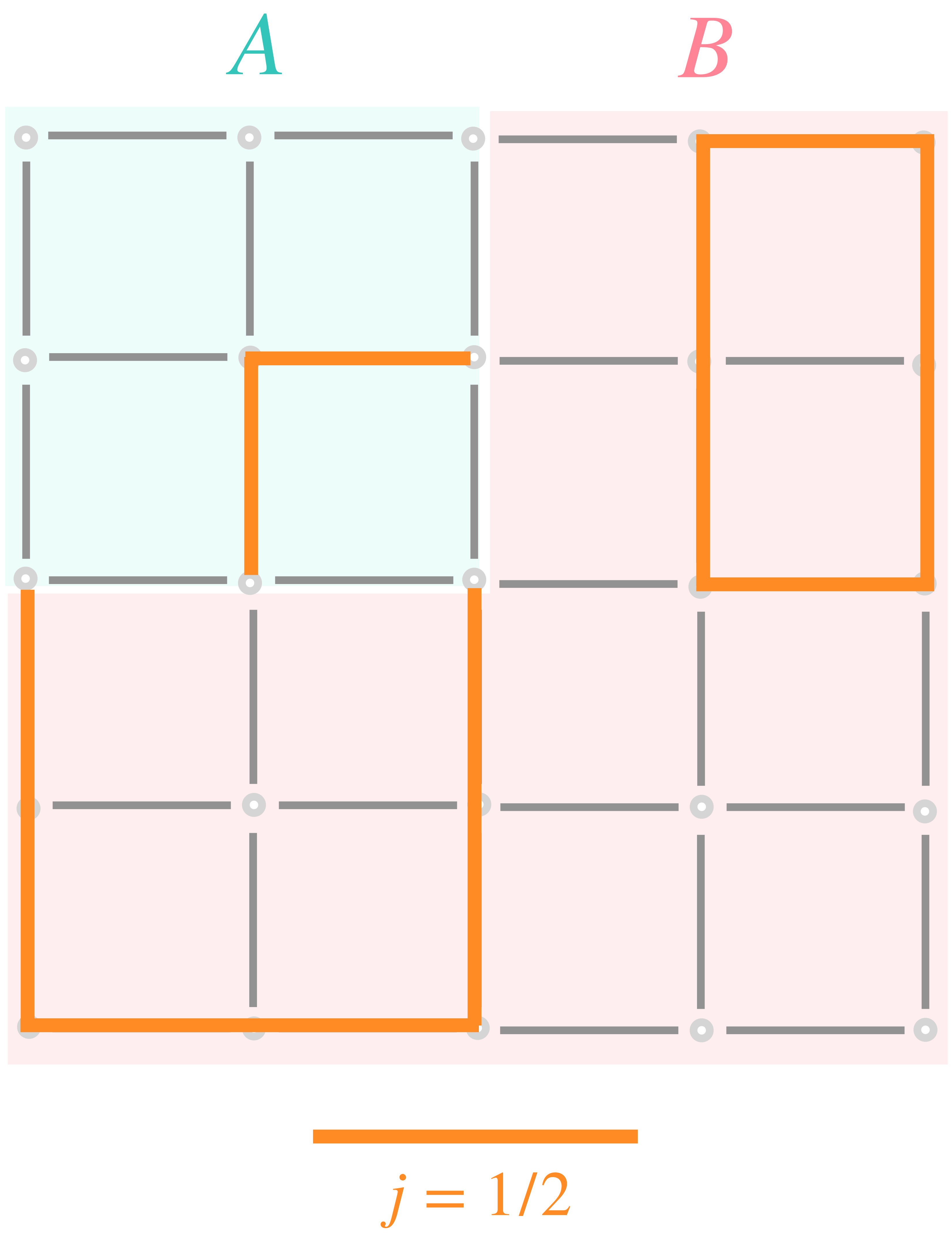

GAUSS'S LAW OPERATORS

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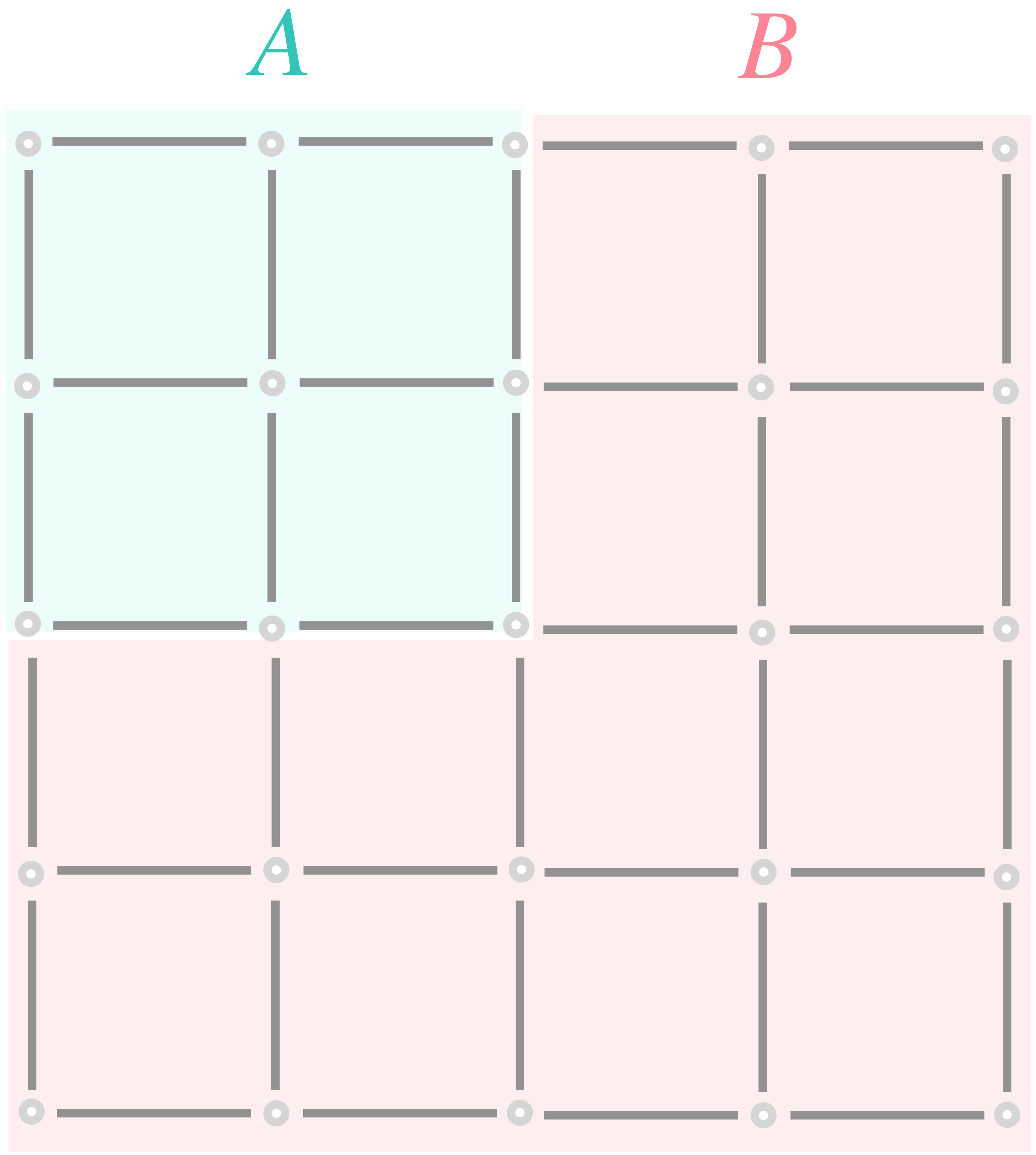
STATES

$$\mathbb{H}^{\text{phys}} = \{ |\phi\rangle, \hat{G}^a(v)|\phi\rangle = 0 \}$$

$$\mathbb{H}^{\text{phys}} \neq \mathbb{H}^{\text{phys}}_A \otimes \mathbb{H}^{\text{phys}}_B$$

Buividovich and Polikarpov (2008), Donnelly (2012, 2014), Aoki et al. (2015), Ghosh et al. (2015), Van Acoleyen et al. (2015)

SU(2) lattice gauge theory



STATE | SUBSYSTEM

$$|\phi\rangle \in \mathbb{H}^{\text{phys}} \quad \subset \quad \mathbb{H}^{\text{ext}} = \mathbb{H}_A^{\text{ext}} \otimes \mathbb{H}_B^{\text{ext}}$$

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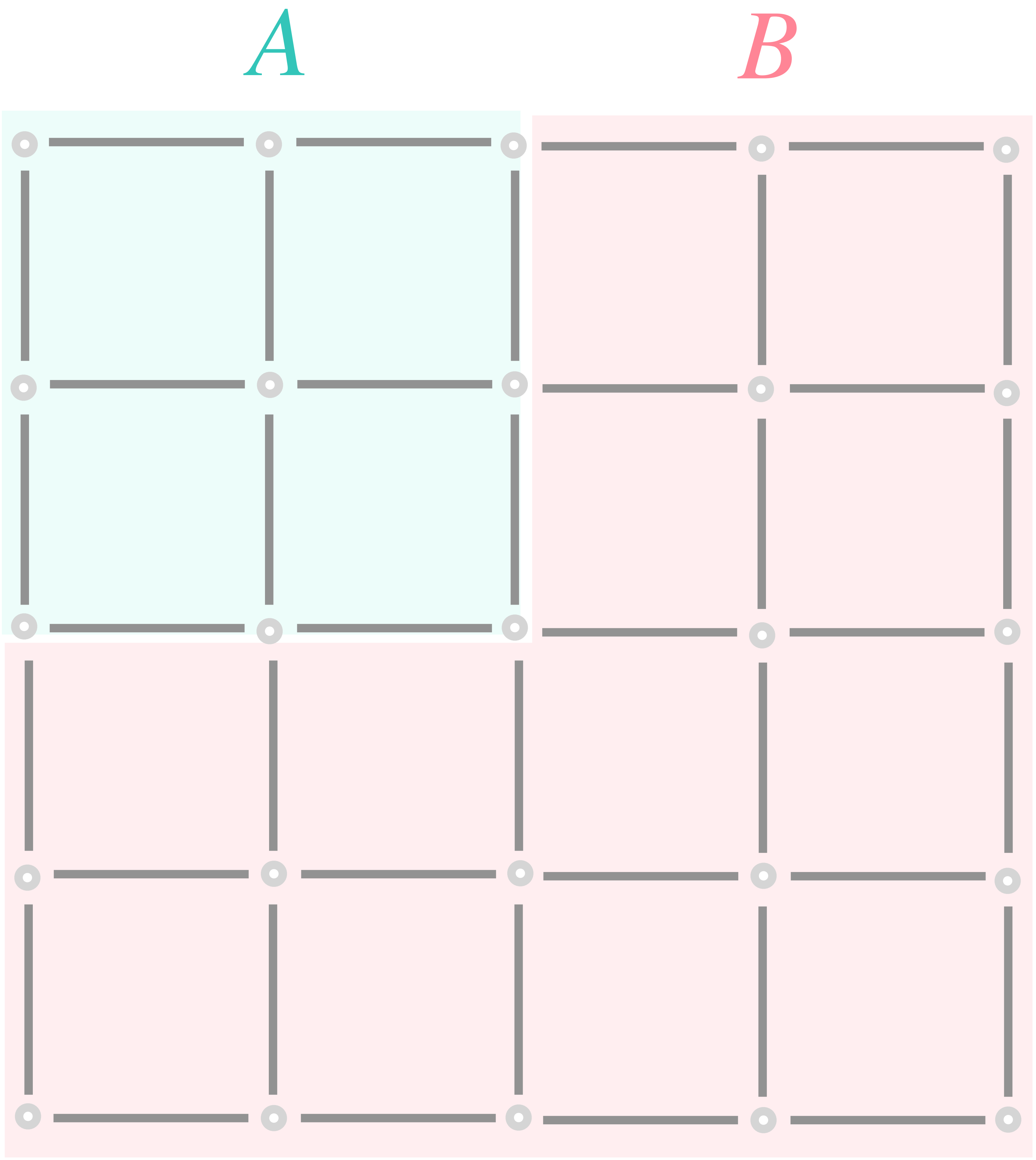

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$$S_A(\rho) = -\operatorname{Tr} \rho_A \log \rho_A$$

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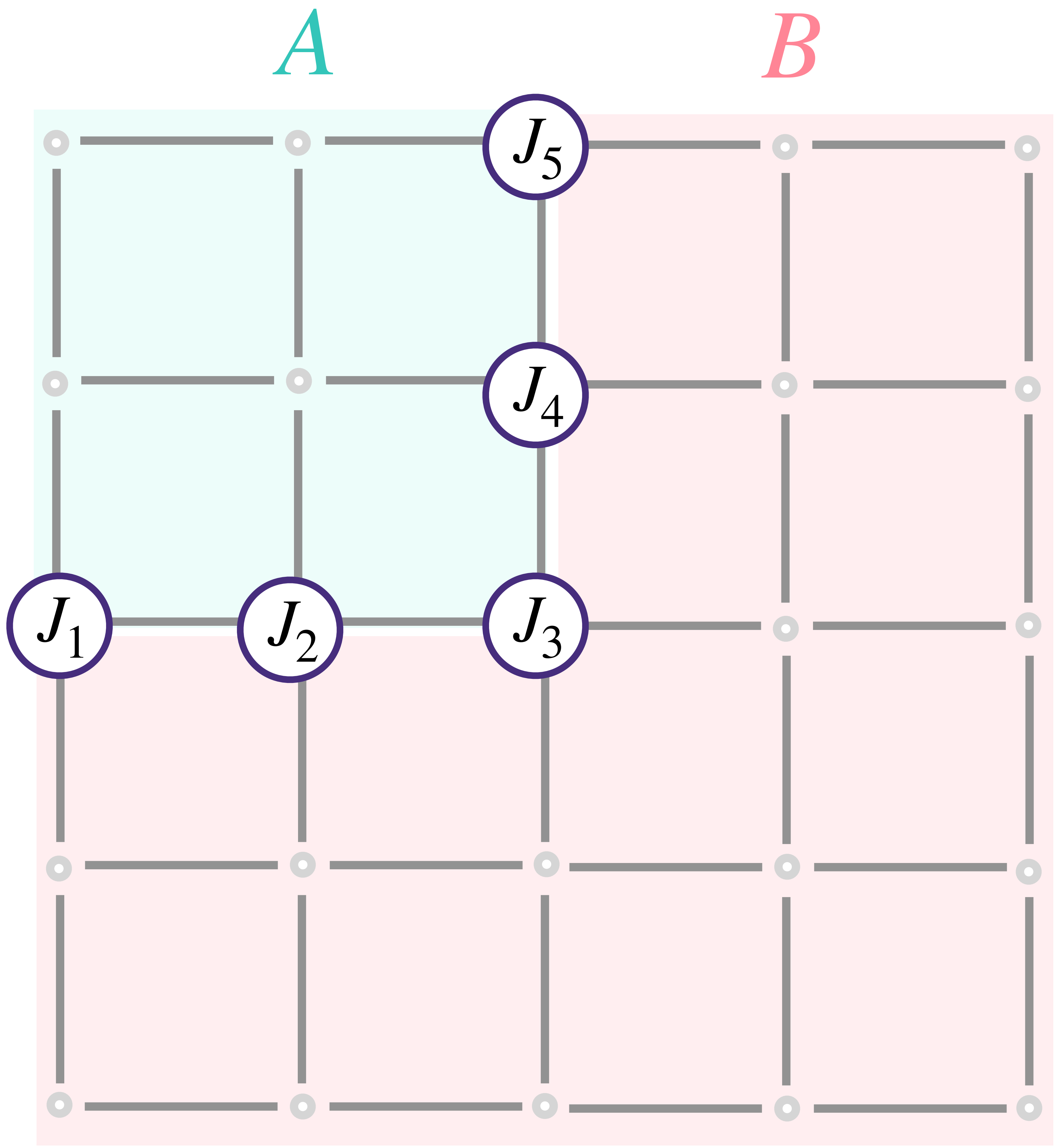
SU(2) lattice gauge theory



$$\boldsymbol{J} = (J_1, J_2, J_3, J_4, J_5)$$

STATE | SUBSYSTEM

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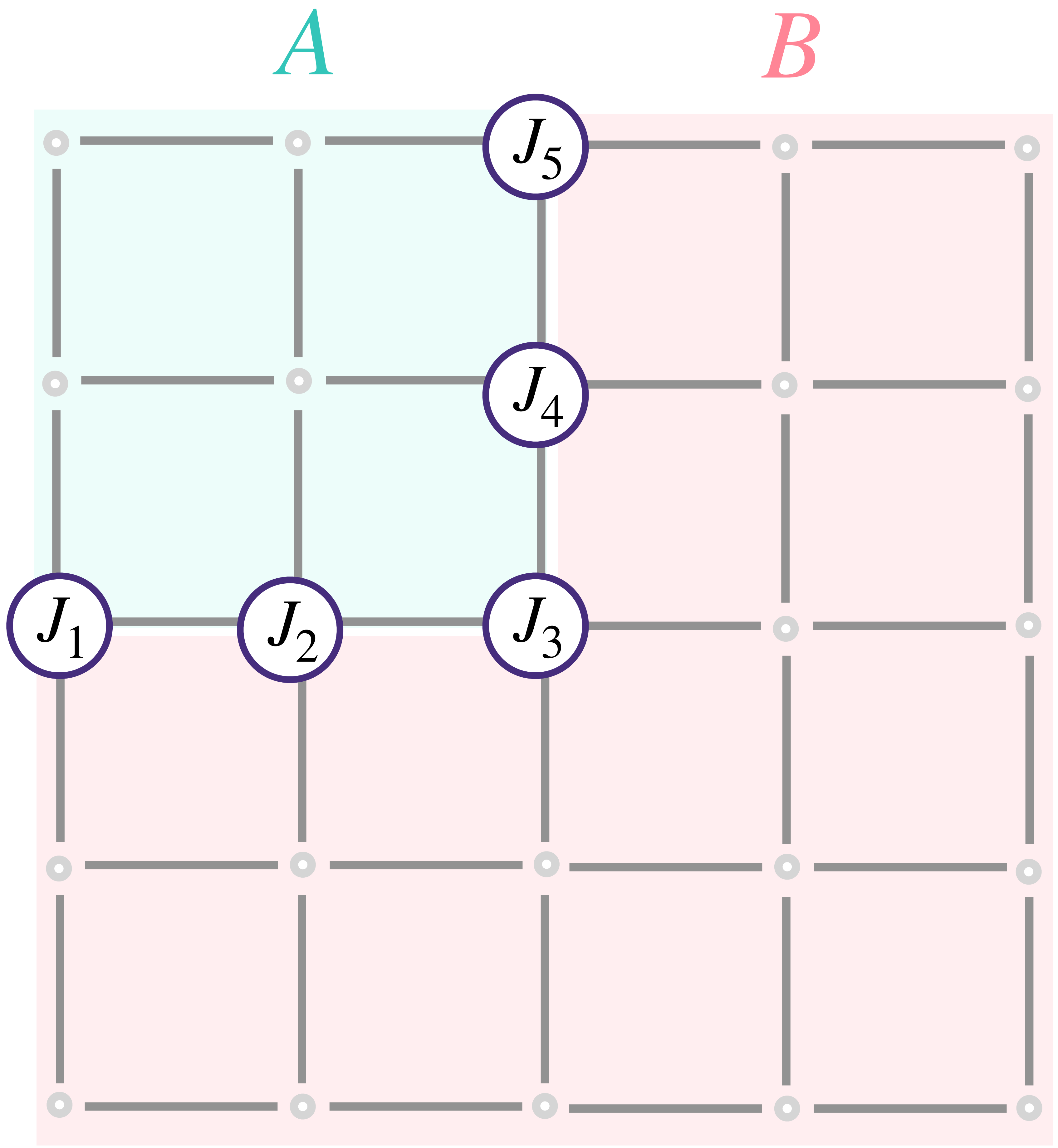

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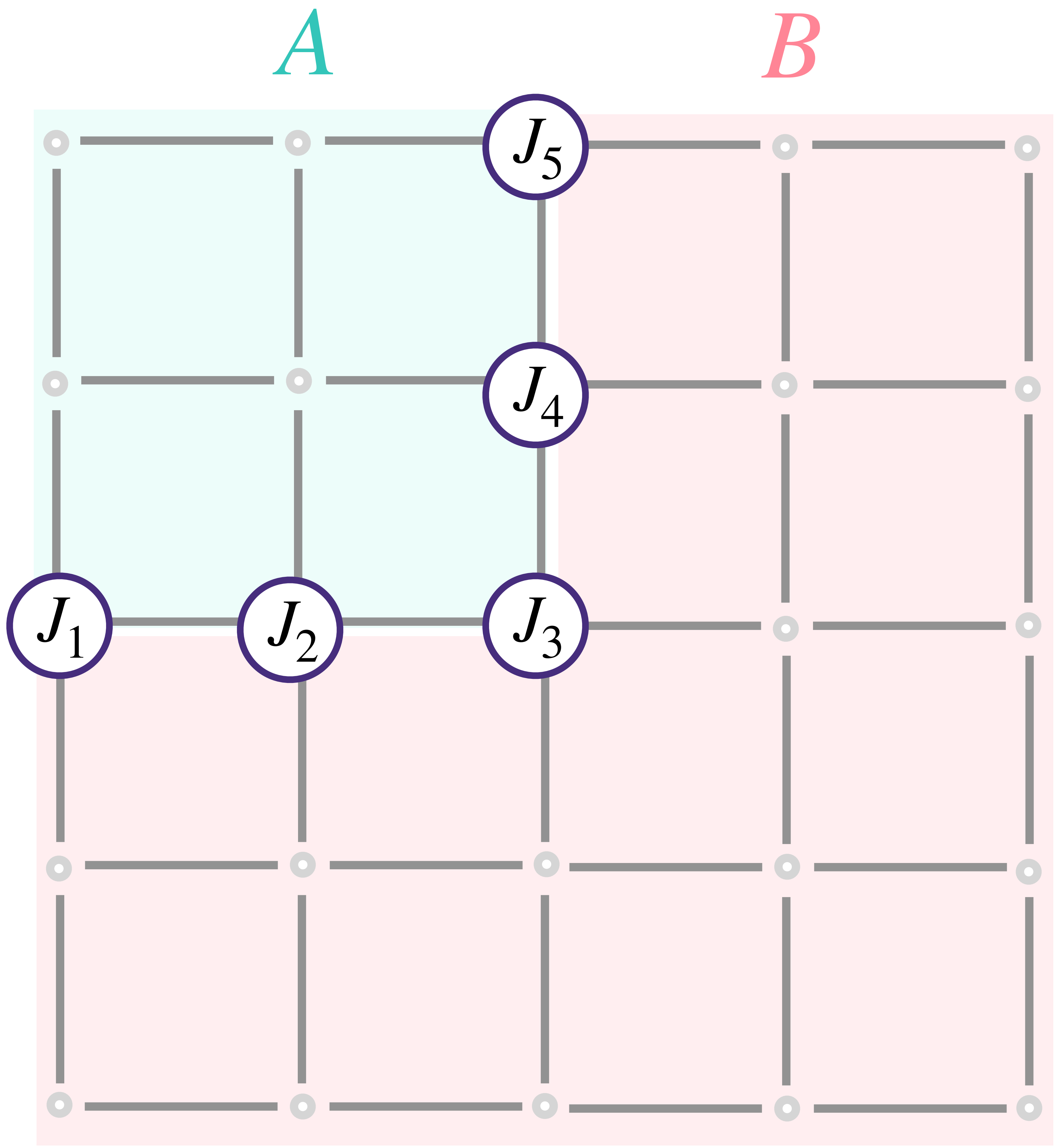
$$\rho_A = \operatorname{Tr}_B \rho$$

$$S_A(\rho) = -\operatorname{Tr} \rho_A \log \rho_A$$

$$\underbrace{-\sum_{\boldsymbol{J}} p_{\boldsymbol{J}} \log p_{\boldsymbol{J}}}_{\text{Classical}} + \underbrace{\sum_{\boldsymbol{J}} p_{\boldsymbol{J}} \log d_{\boldsymbol{J}}}_{\text{Representation}} + \underbrace{\sum_{\boldsymbol{J}} p_{\boldsymbol{J}} S_{\text{vN}}(\hat{\rho}_{A,\boldsymbol{J}})}_{\text{Quantum}}$$

Buividovich and Polikarpov (2008), Donnelly (2012, 2014), Aoki et al. (2015), Ghosh et al. (2015), Van Acoleyen et al. (2015)

SU(2) lattice gauge theory



$$\boldsymbol{J} = (J_1, J_2, J_3, J_4, J_5)$$

STATE | SUBSYSTEM

$$|\phi\rangle \in \mathbb{H}^{\text{phys}} \quad \subset \quad \mathbb{H}^{\text{ext}} = \mathbb{H}_A^{\text{ext}} \otimes \mathbb{H}_B^{\text{ext}}$$

$$\rho_A = \mathrm{Tr}_B\, \rho$$

$$S_A(\rho) = -\,\mathrm{Tr}\, \rho_A \log \rho_A$$

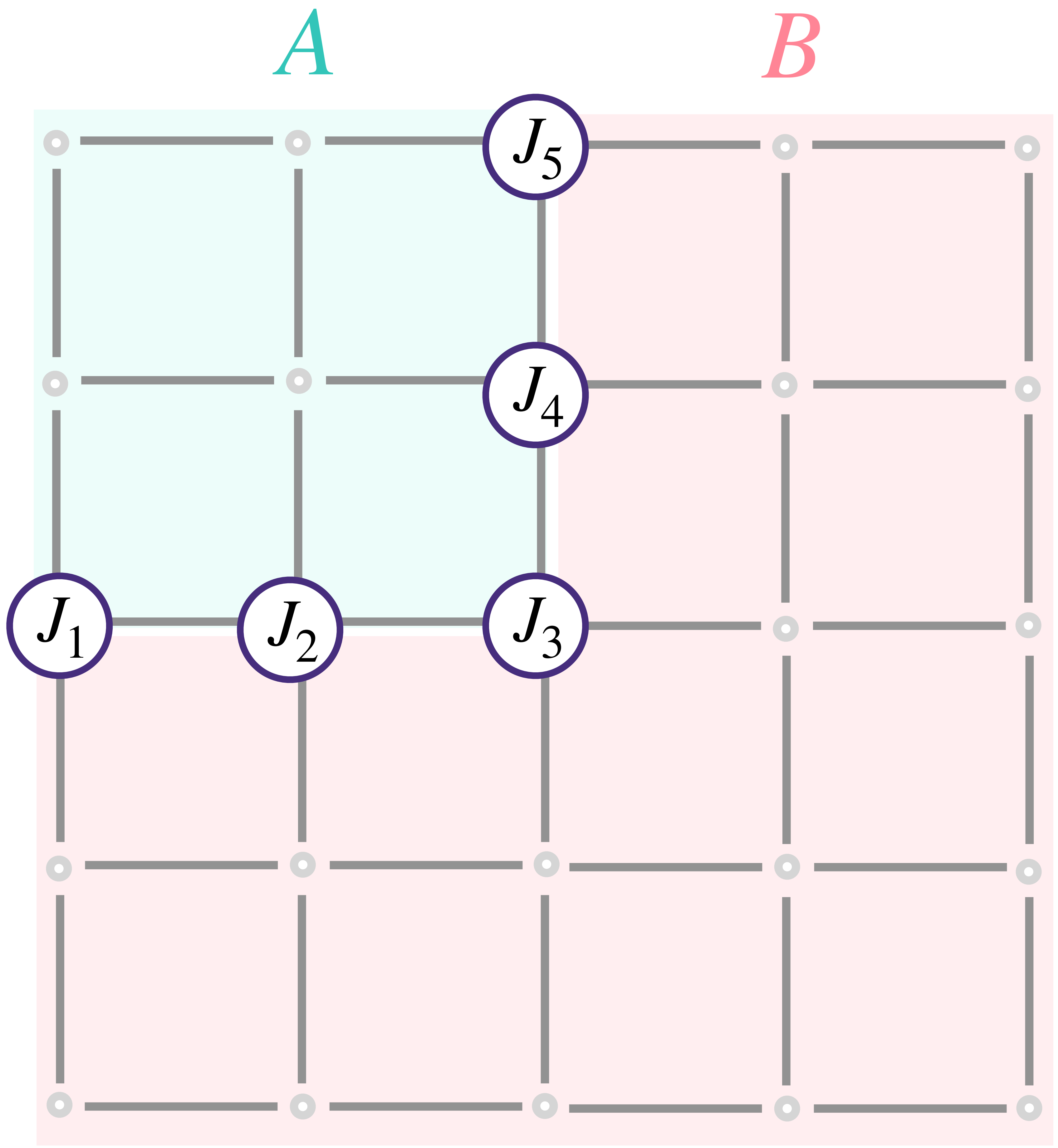
$$-\sum_{\boldsymbol{J}} p_{\boldsymbol{J}} \log p_{\boldsymbol{J}} + \sum_{\boldsymbol{J}} p_{\boldsymbol{J}} \log d_{\boldsymbol{J}} + \sum_{\boldsymbol{J}} p_{\boldsymbol{J}} S_{\text{vN}}(\hat{\rho}_{A,\boldsymbol{J}})$$

Classical | *Representation* | *Quantum*

EXTENDED-HILBERT-SPACE DEFINITION

Buividovich and Polikarpov (2008), Donnelly (2012, 2014), Aoki et al. (2015), Ghosh et al. (2015), Van Acoleyen et al. (2015)

SU(2) lattice gauge theory



$$\boldsymbol{J} = (J_1, J_2, J_3, J_4, J_5)$$

STATE	SUBSYSTEM
$\vert\phi\rangle \in \mathbb{H}^{\text{phys}}$ $\subset$	$\mathbb{H}^{\text{ext}} = \mathbb{H}^{\text{ext}}_A \otimes \mathbb{H}^{\text{ext}}_B$

$$\rho_A = \mathrm{Tr}_B\, \rho$$

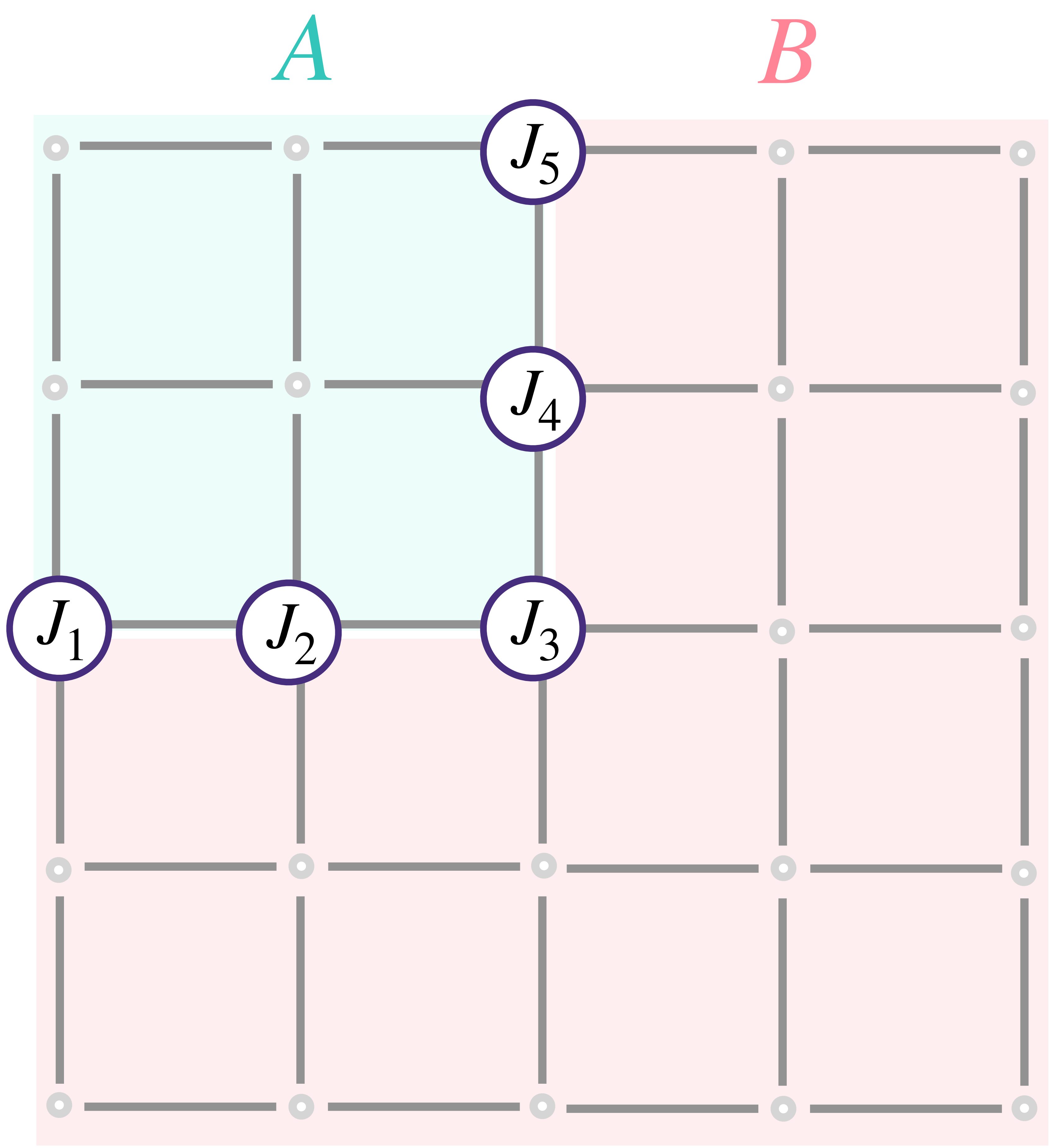
$$S_A(\rho) = -\mathrm{Tr}\, \rho_A \log \rho_A$$

$$-\sum_{\boldsymbol{J}} p_{\boldsymbol{J}} \log p_{\boldsymbol{J}} + \sum_{\boldsymbol{J}} p_{\boldsymbol{J}} \log d_{\boldsymbol{J}} + \sum_{\boldsymbol{J}} p_{\boldsymbol{J}} S_{\mathrm{vN}}(\hat{\rho}_{A,\boldsymbol{J}})$$

Classical | *Representation* | *Quantum*

S_A depends upon the extended Hilbert space **of the Kogut-Susskind encoding.**

SU(2) lattice gauge theory



$$\boldsymbol{J} = (J_1, J_2, J_3, J_4, J_5)$$

STATE | SUBSYSTEM

$$|\phi\rangle \in \mathbb{H}^{\text{phys}} \quad \subset \quad \mathbb{H}^{\text{enc,KS}} = \mathbb{H}_A^{\text{enc,KS}} \otimes \mathbb{H}_B^{\text{enc,KS}}$$

$$\rho_A^{\text{KS}} = \text{Tr}_B\, \rho$$

Encoding Entropy

$$S_A^{\text{KS}}(\rho) = -\,\text{Tr}\,\rho_A^{\text{KS}} \log \rho_A^{\text{KS}} =$$

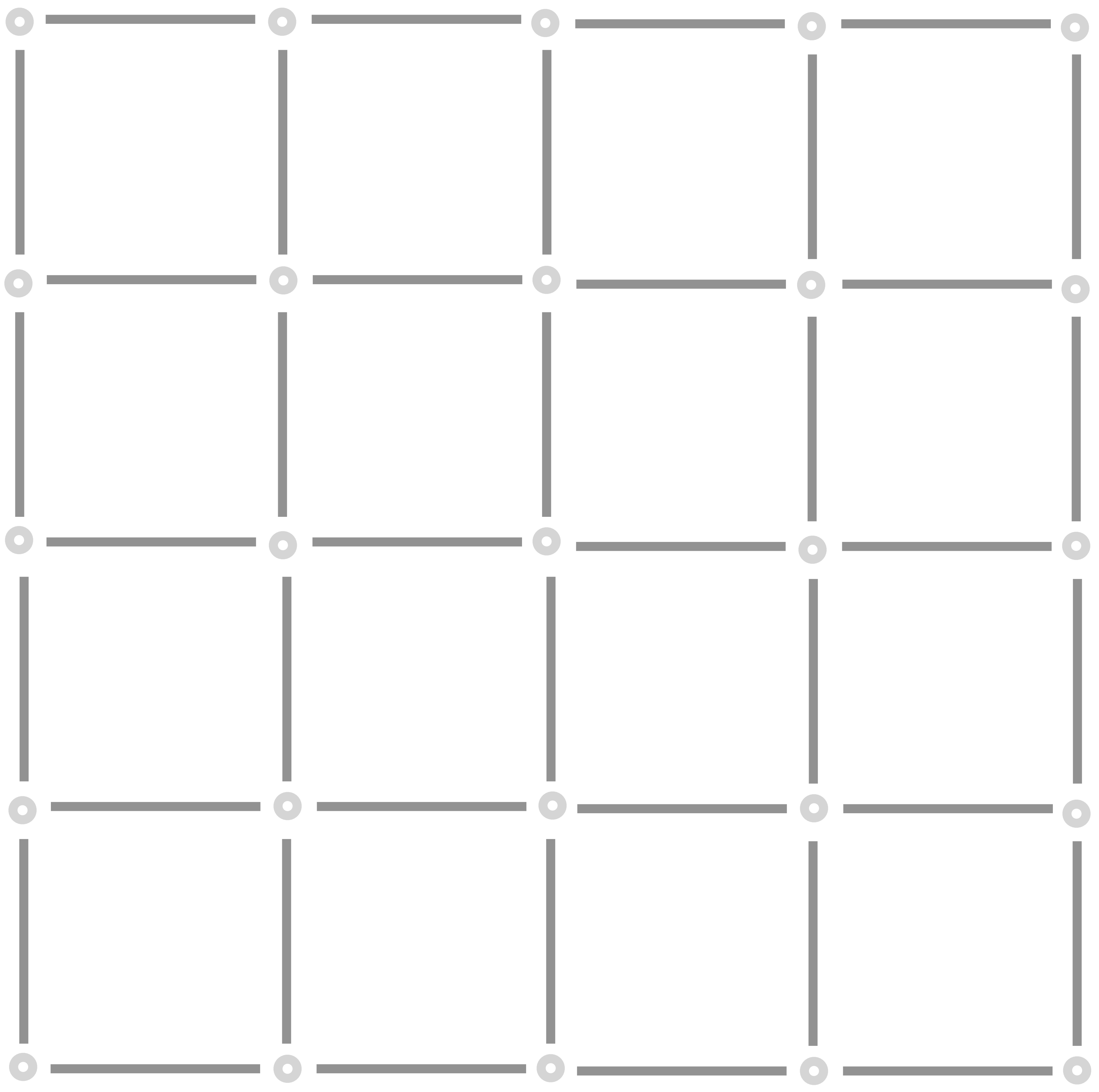
$$-\sum_{\boldsymbol{J}} p_{\boldsymbol{J}} \log p_{\boldsymbol{J}} + \sum_{\boldsymbol{J}} p_{\boldsymbol{J}} \log d_{\boldsymbol{J}} + \sum_{\boldsymbol{J}} p_{\boldsymbol{J}}\, S_{\text{vN}}(\hat{\rho}_{A,\boldsymbol{J}})$$

Classical | *Representation* | *Quantum*

S_A depends upon the extended Hilbert space **of the Kogut-Susskind encoding.**

***Gupta**, Mathew, Raychowdhury, and Davoudi (In preparation)*

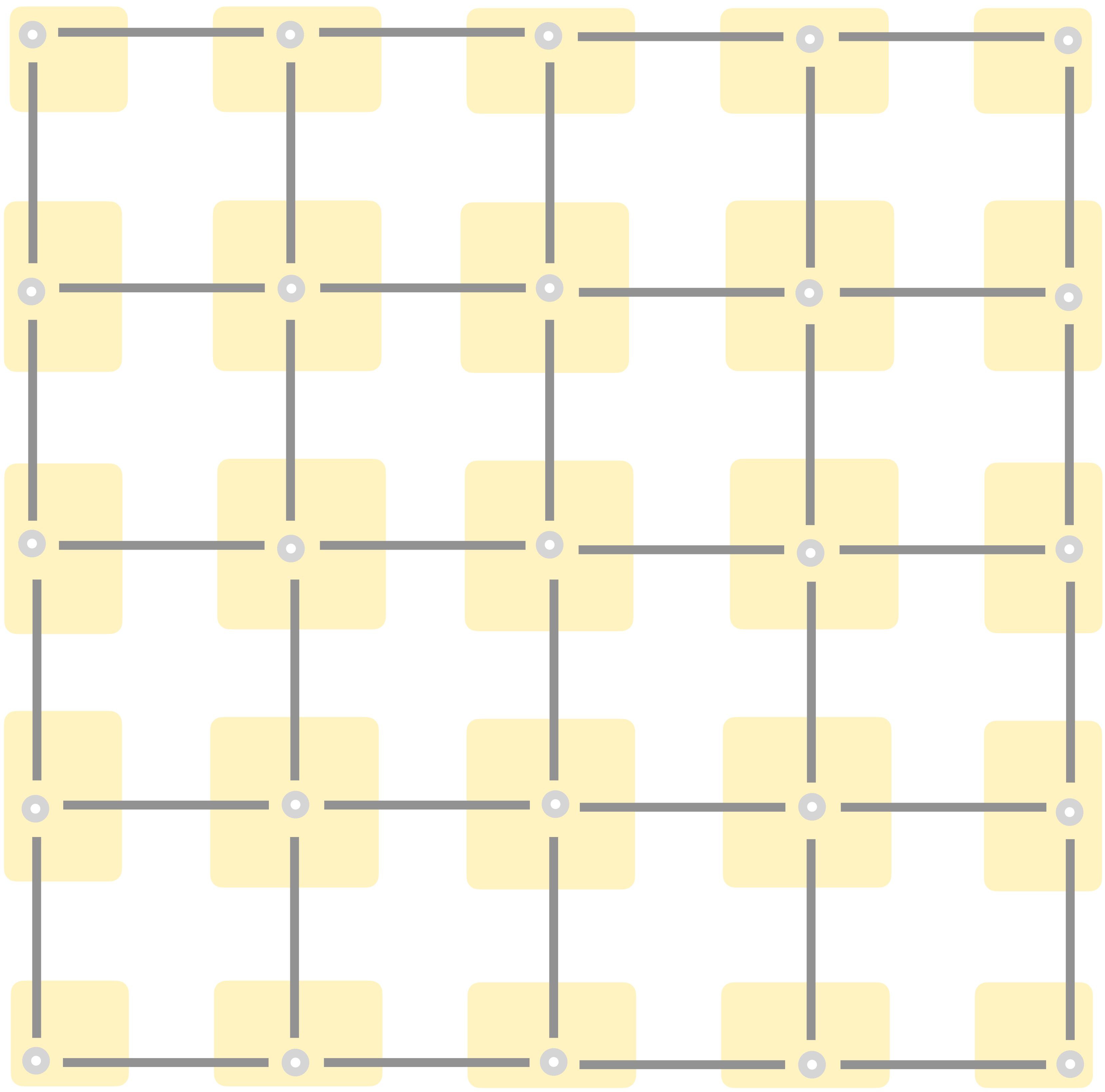
Kogut-Susskind encoding



$$\mathbb{H}^{\mathrm{enc,KS}} = \bigotimes_l \mathbb{H}_l$$

ENCODING
HILBERT
SPACE

Kogut-Susskind encoding



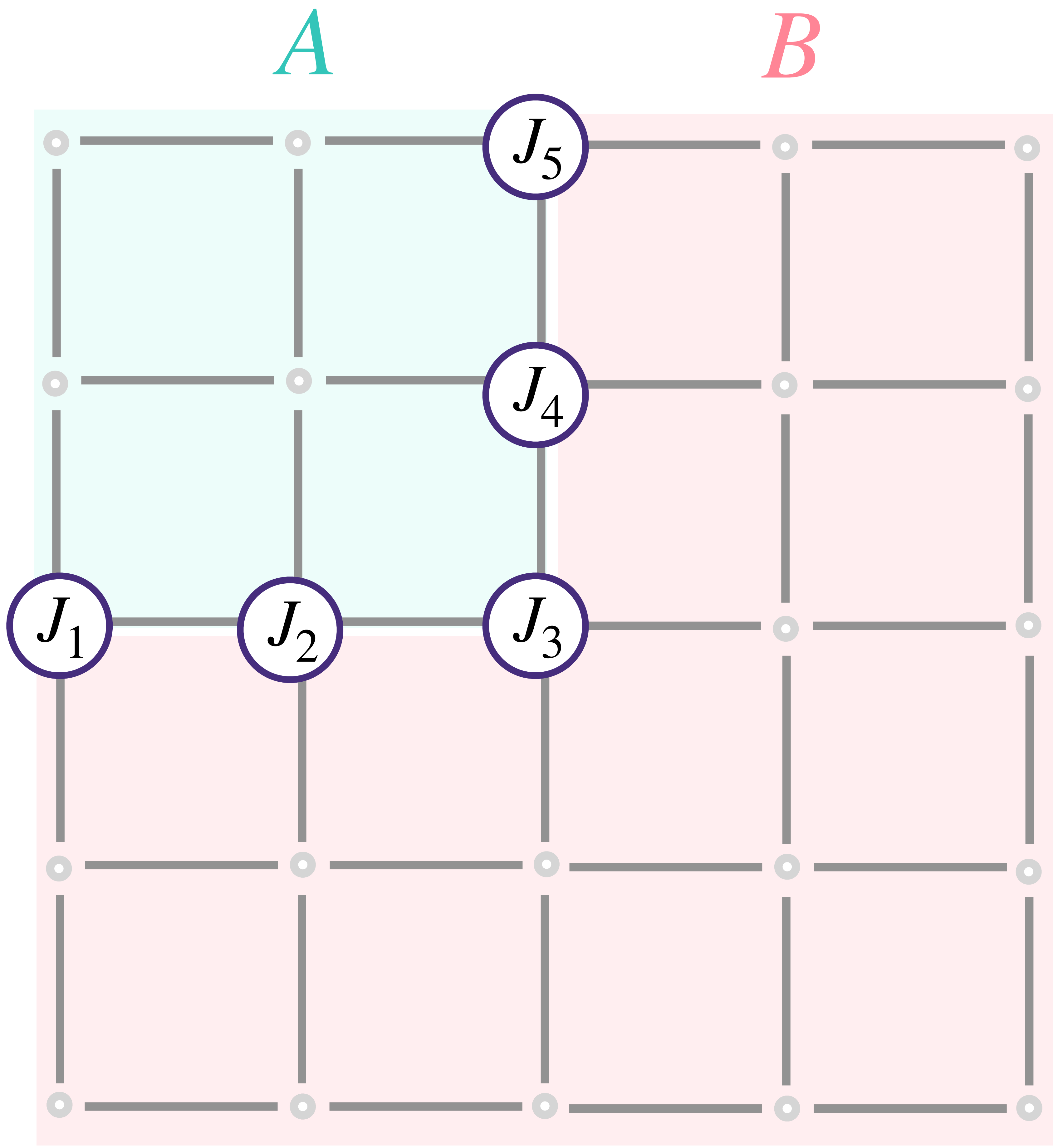
$$\mathbb{H}^{\text{enc,KS}} = \otimes_l \mathbb{H}_l$$

$$\mathbb{H}^{\text{phys,KS}}$$

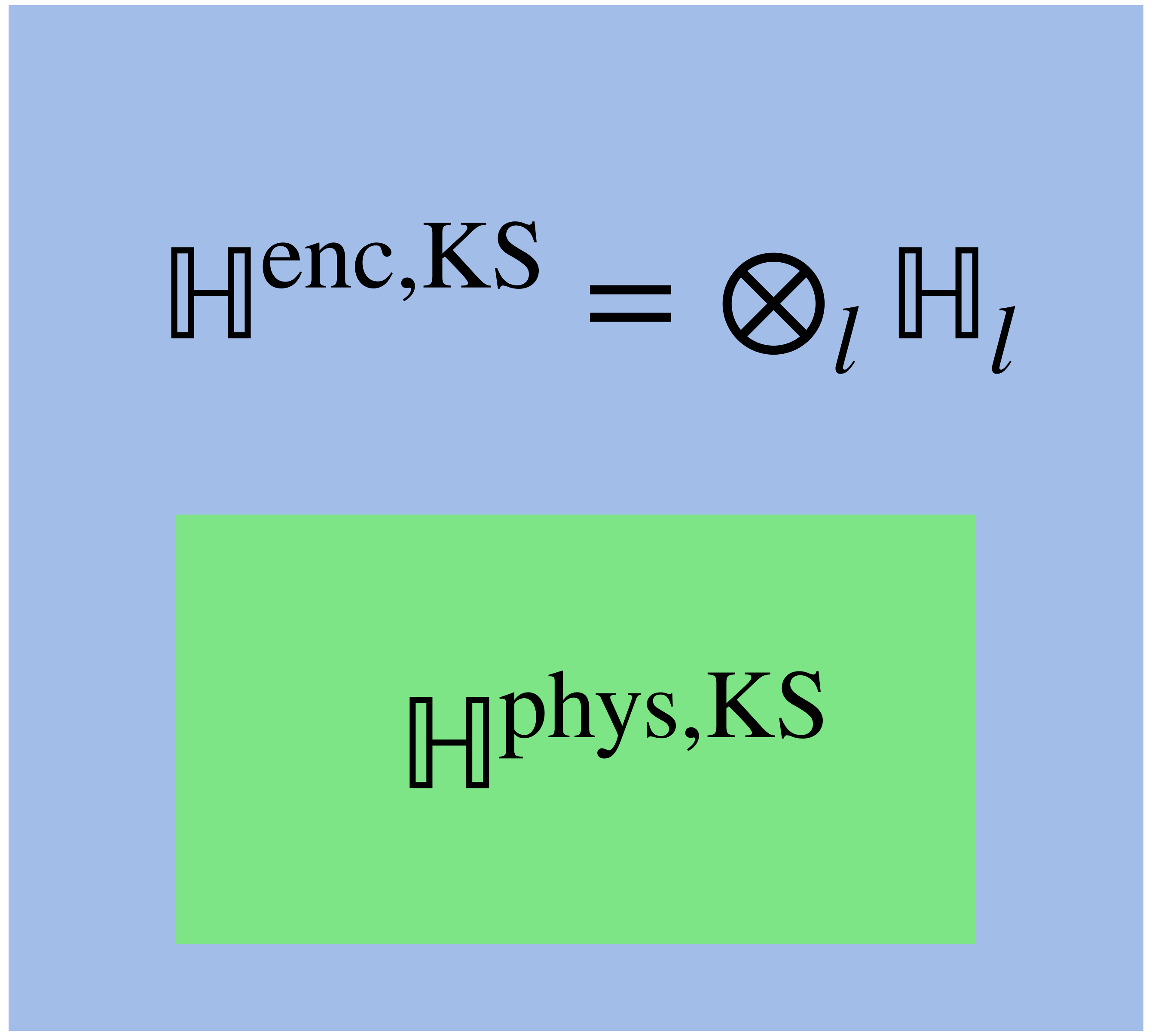
ENCODING
HILBERT
SPACE

$$\hat{G}^a(v) = \sum_{\substack{l \text{ incoming} \\ \text{at } v}} \hat{E}^a_R(l) - \sum_{\substack{l \text{ outgoing} \\ \text{from } v}} \hat{E}^a_L(l)$$

Kogut-Susskind encoding



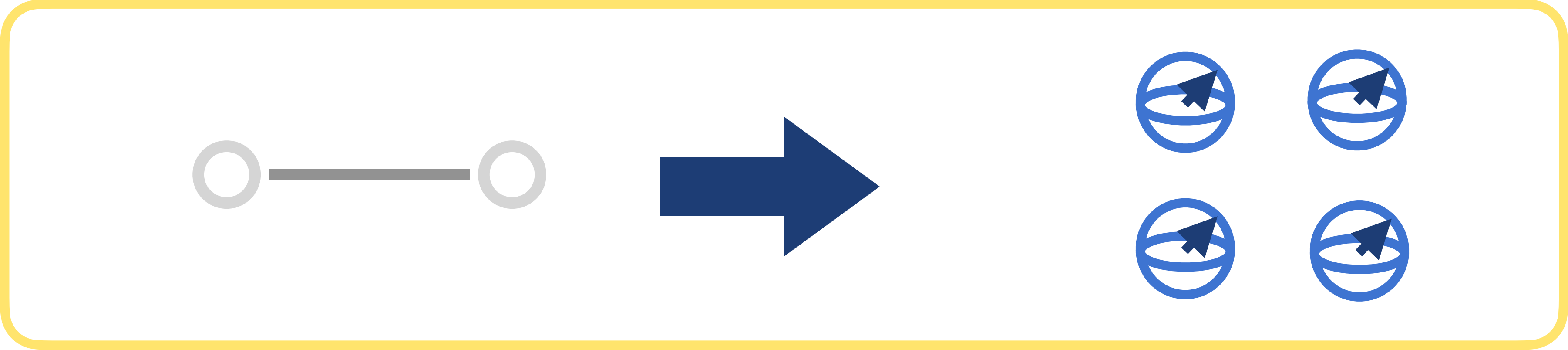
$$\mathbb{H}^{\text{enc,KS}} = \mathbb{H}_A^{\text{enc,KS}} \otimes \mathbb{H}_B^{\text{enc,KS}}$$



ENCODING HILBERT SPACE

$$\hat{G}^a(v) = \sum_{\substack{l \text{ incoming} \\ \text{at } v}} \hat{E}_R^a(l) - \sum_{\substack{l \text{ outgoing} \\ \text{from } v}} \hat{E}_L^a(l)$$

Why encoding entropy?



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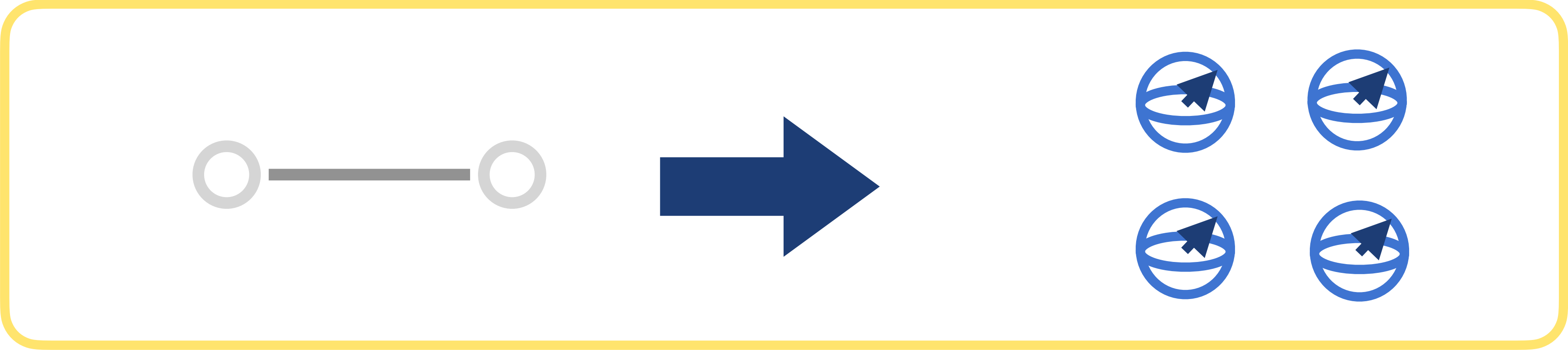
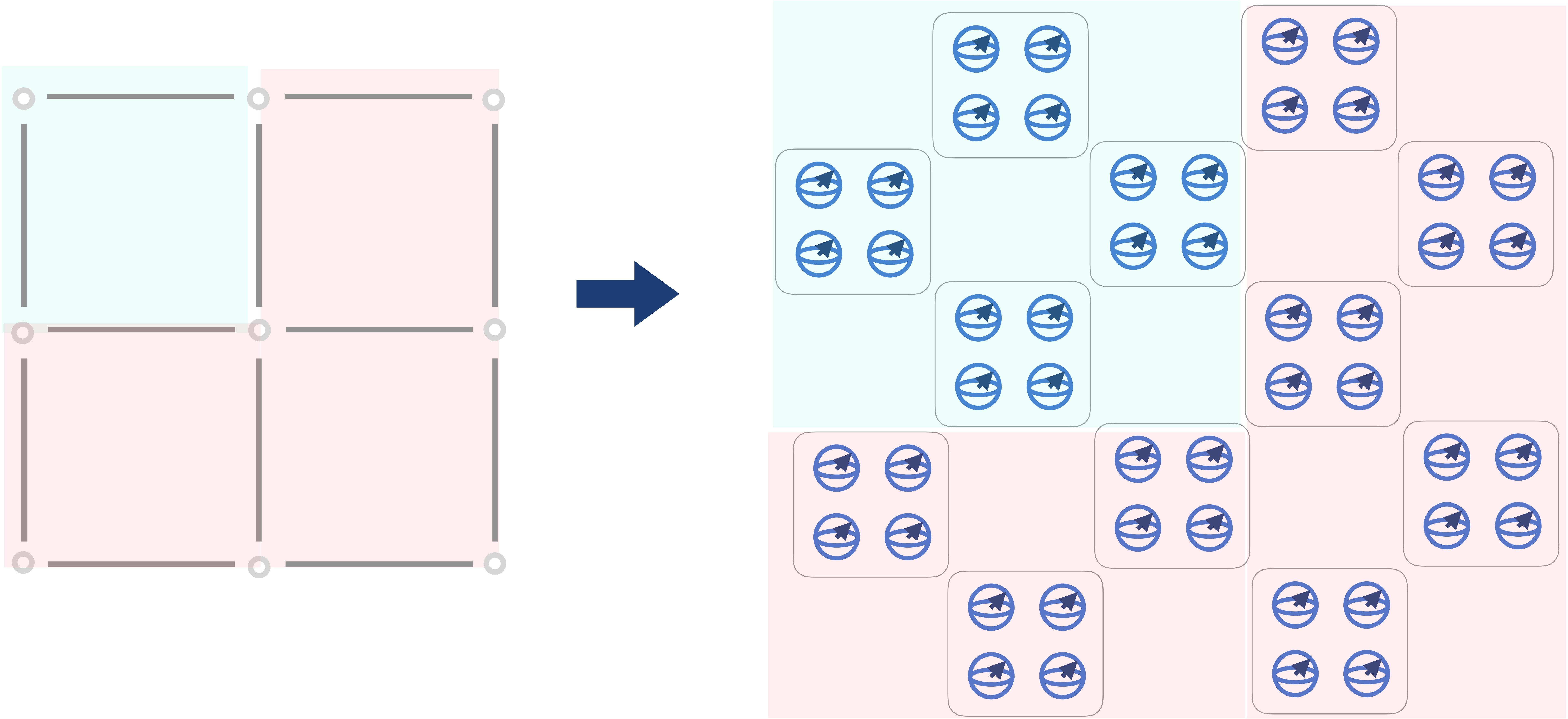
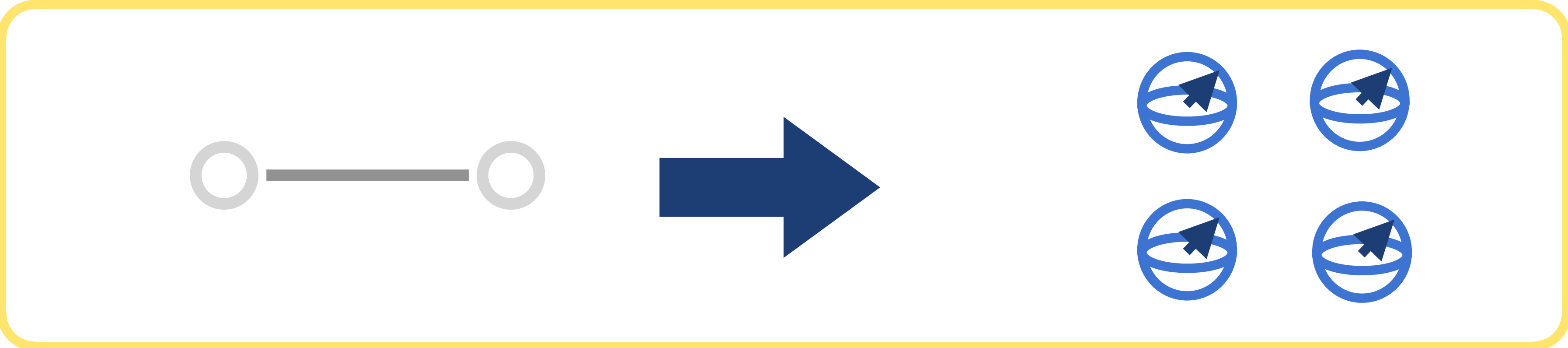
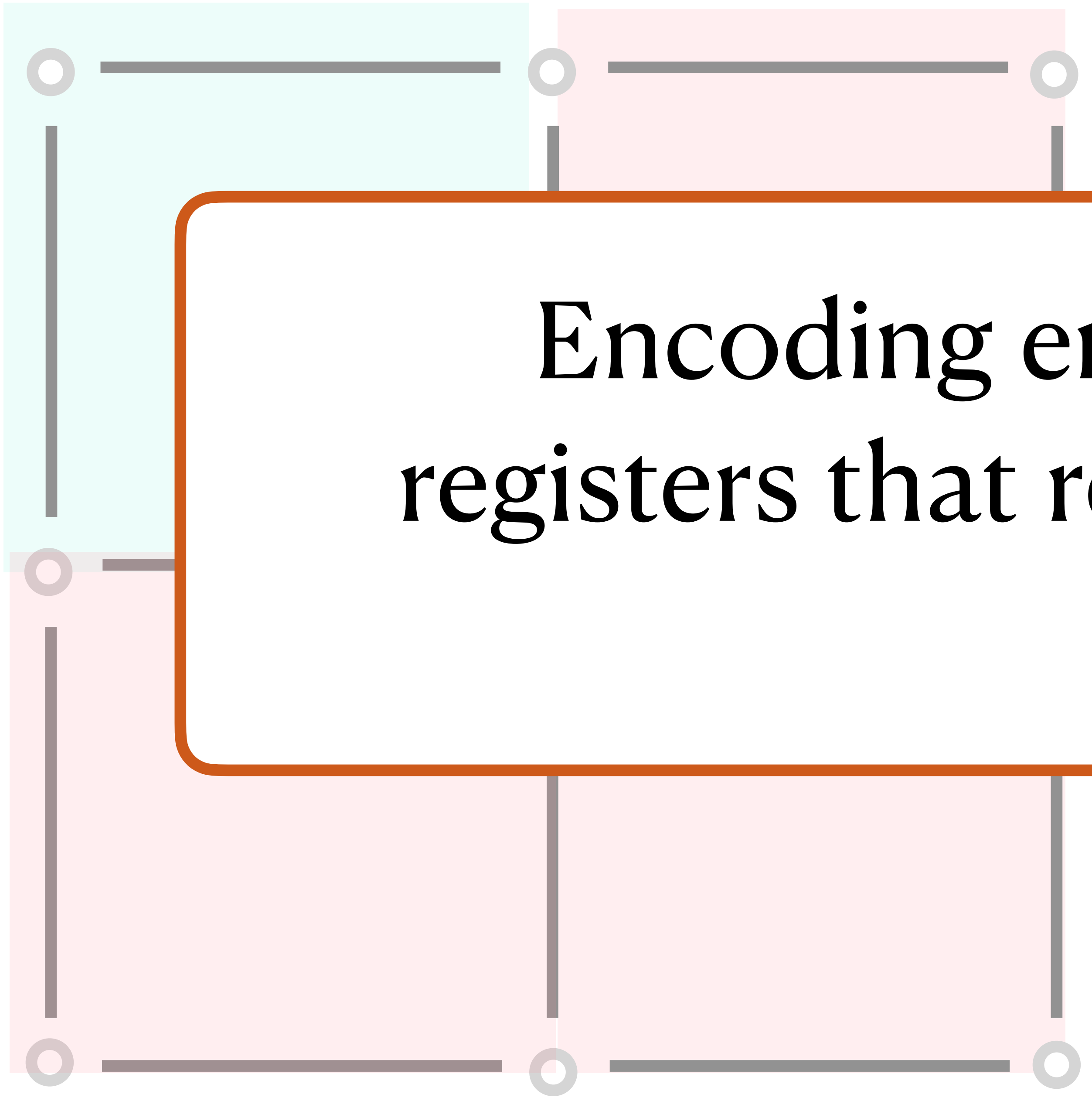
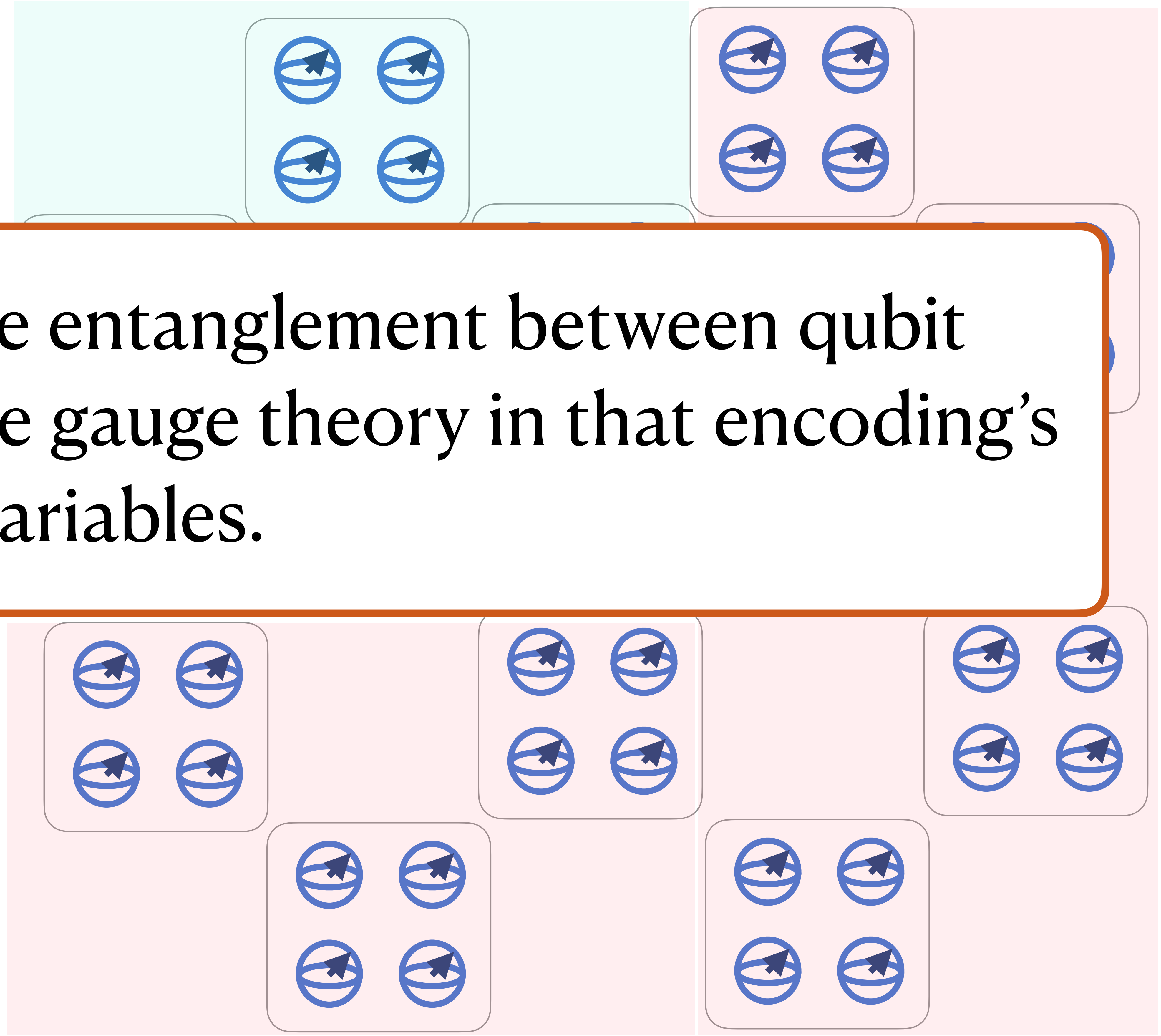




Why encoding entropy?

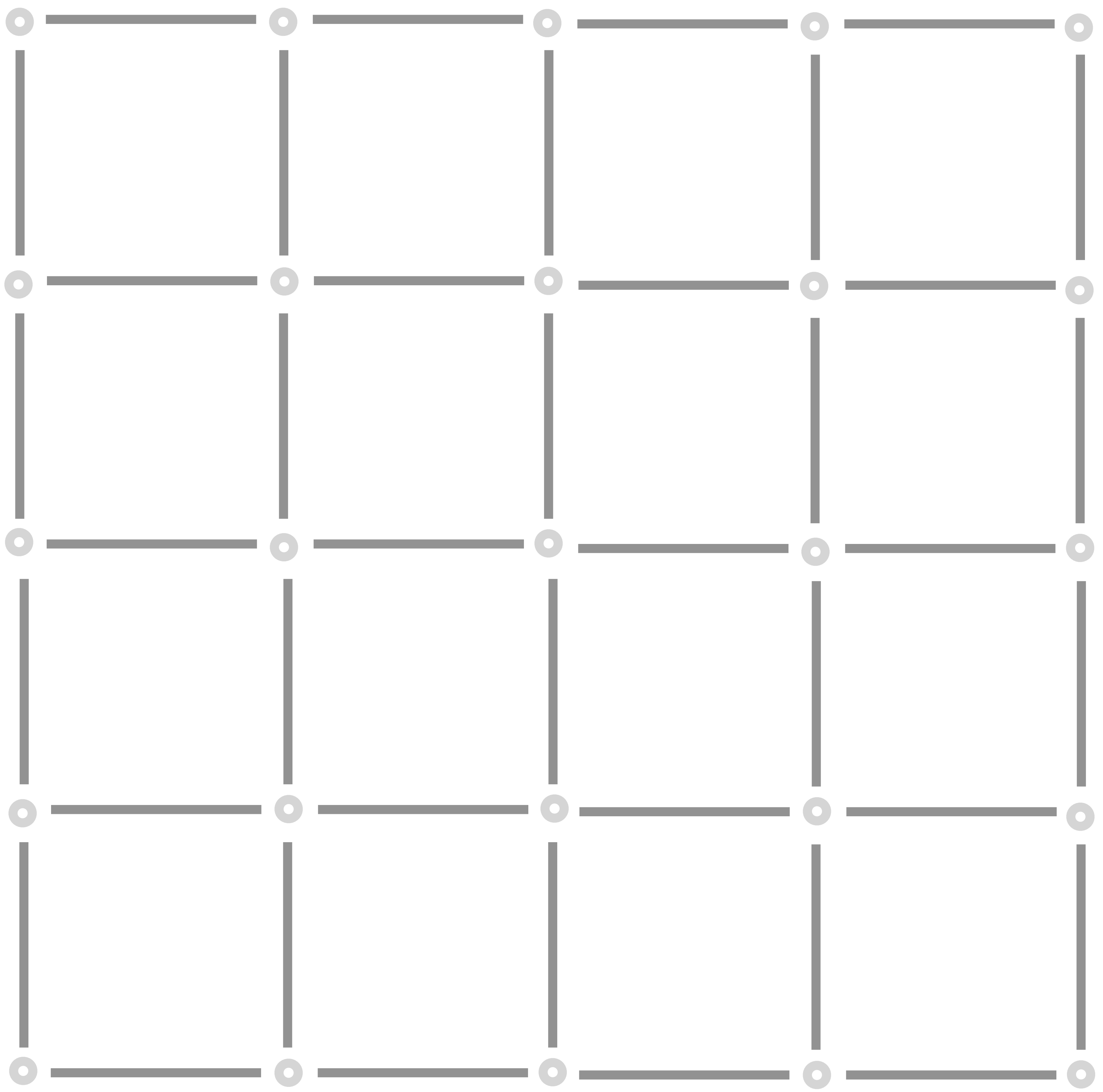






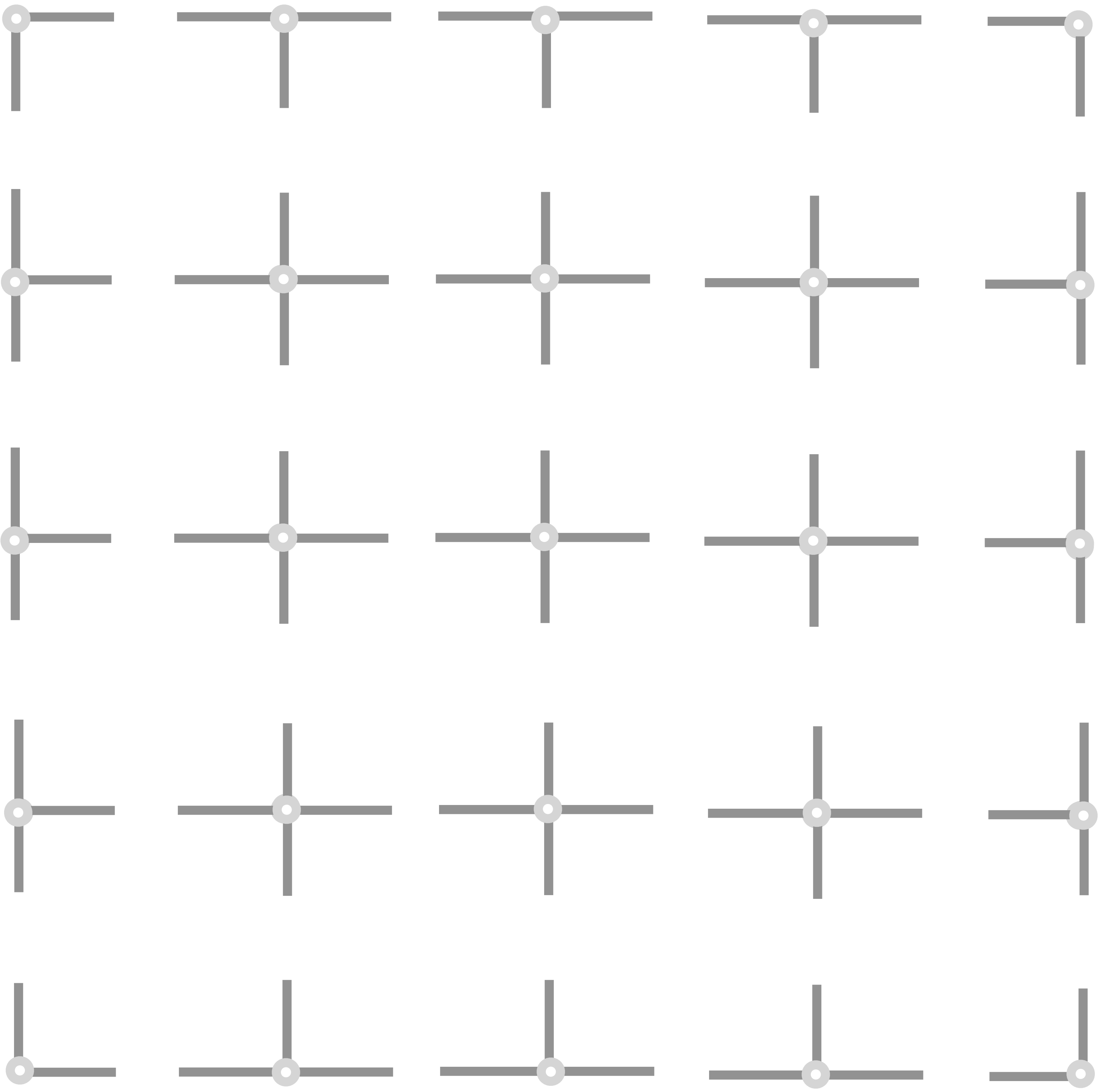
Encoding entropy is the entanglement between qubit registers that represent the gauge theory in that encoding's variables.

Dressed-site encoding

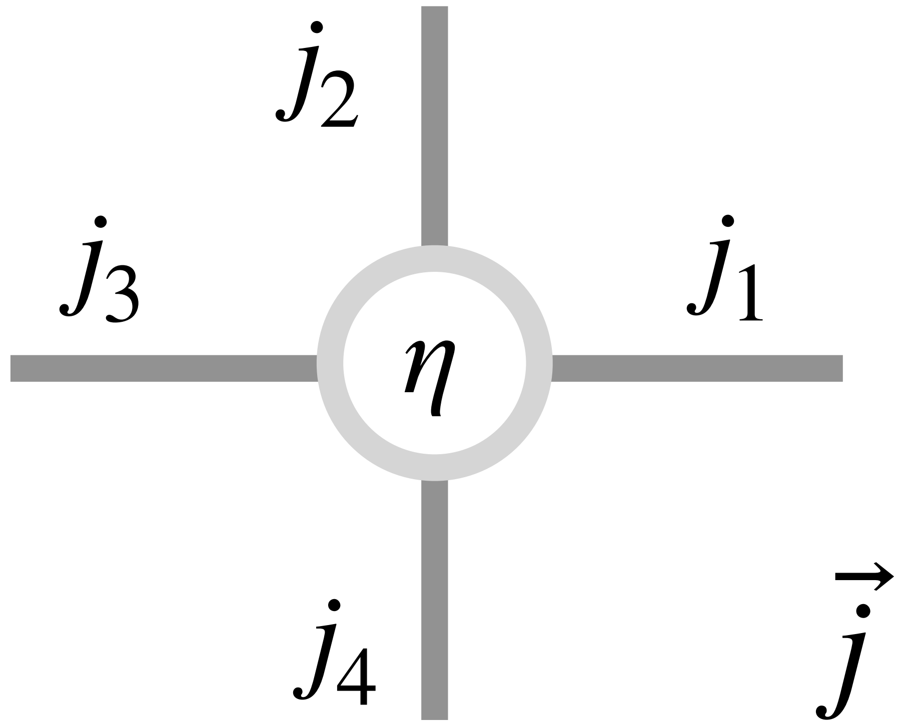


Raychowdhury and Stryker (2020); Cataldi, Magnifico, Silvi, and Montangero (2023)

Dressed-site encoding



DRESSED SITE v

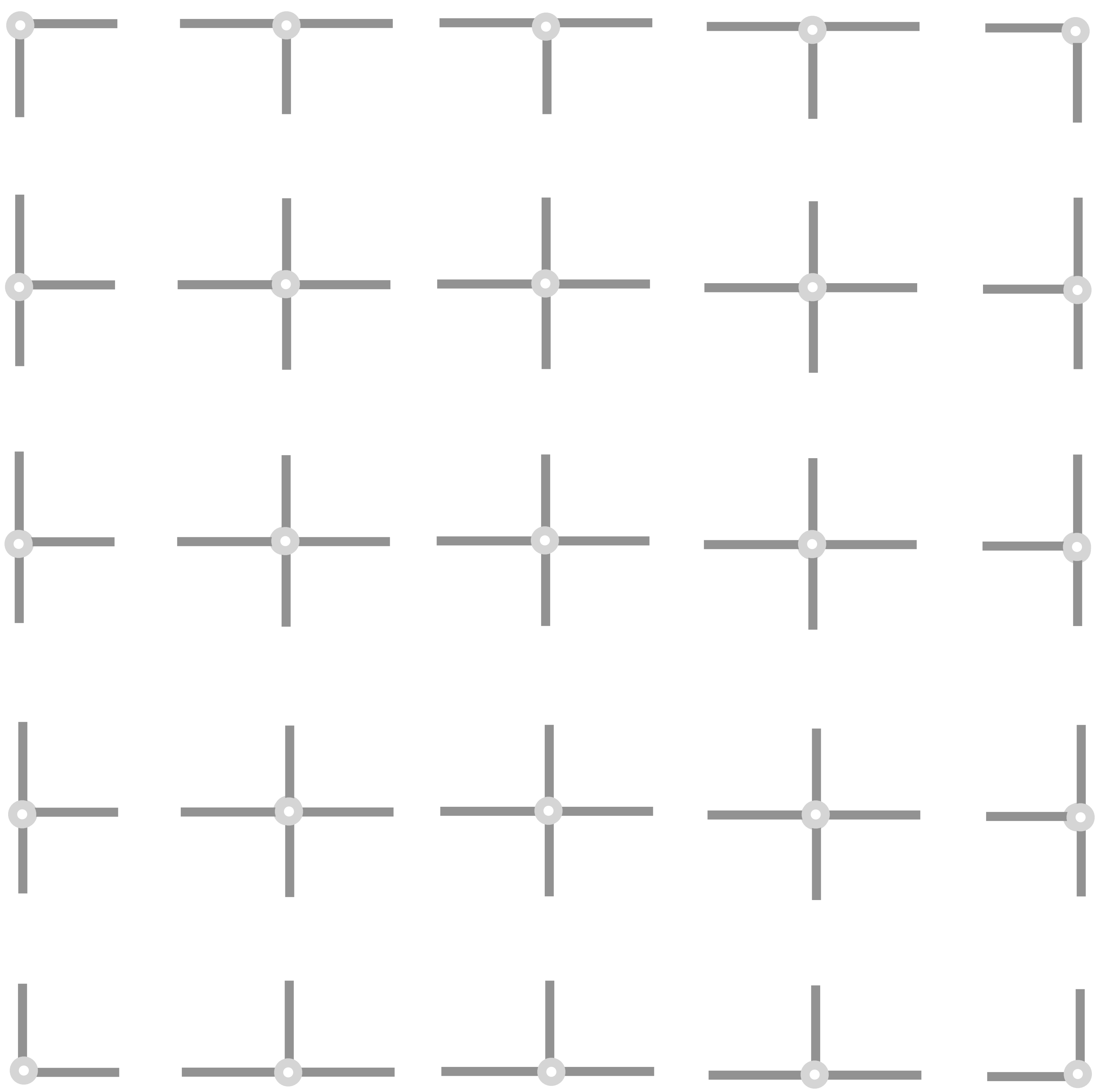


$\vec{j} = (j_1, j_2, j_3, j_4)$

$|\vec{j};\eta\rangle \qquad \mathbb{H}_v = \mathrm{span}\{\,|\vec{j};\eta\rangle\}$

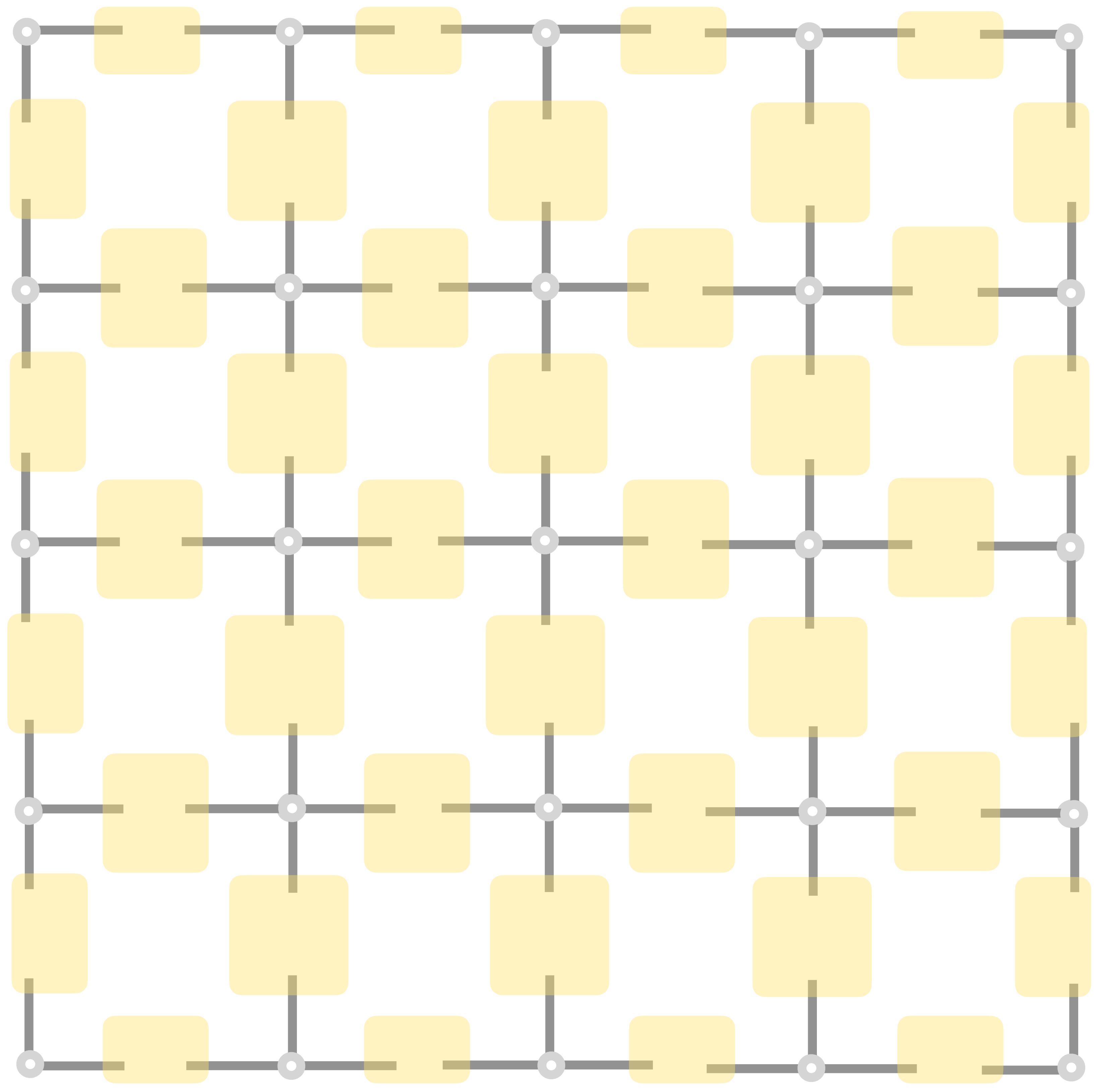
Raychowdhury and Stryker (2020); Cataldi, Magnifico, Silvi, and Montangero (2023)

Dressed-site encoding



$$\mathbb{H}^{\text{enc,DS}} = \bigotimes_v \mathbb{H}_v$$

Dressed-site encoding



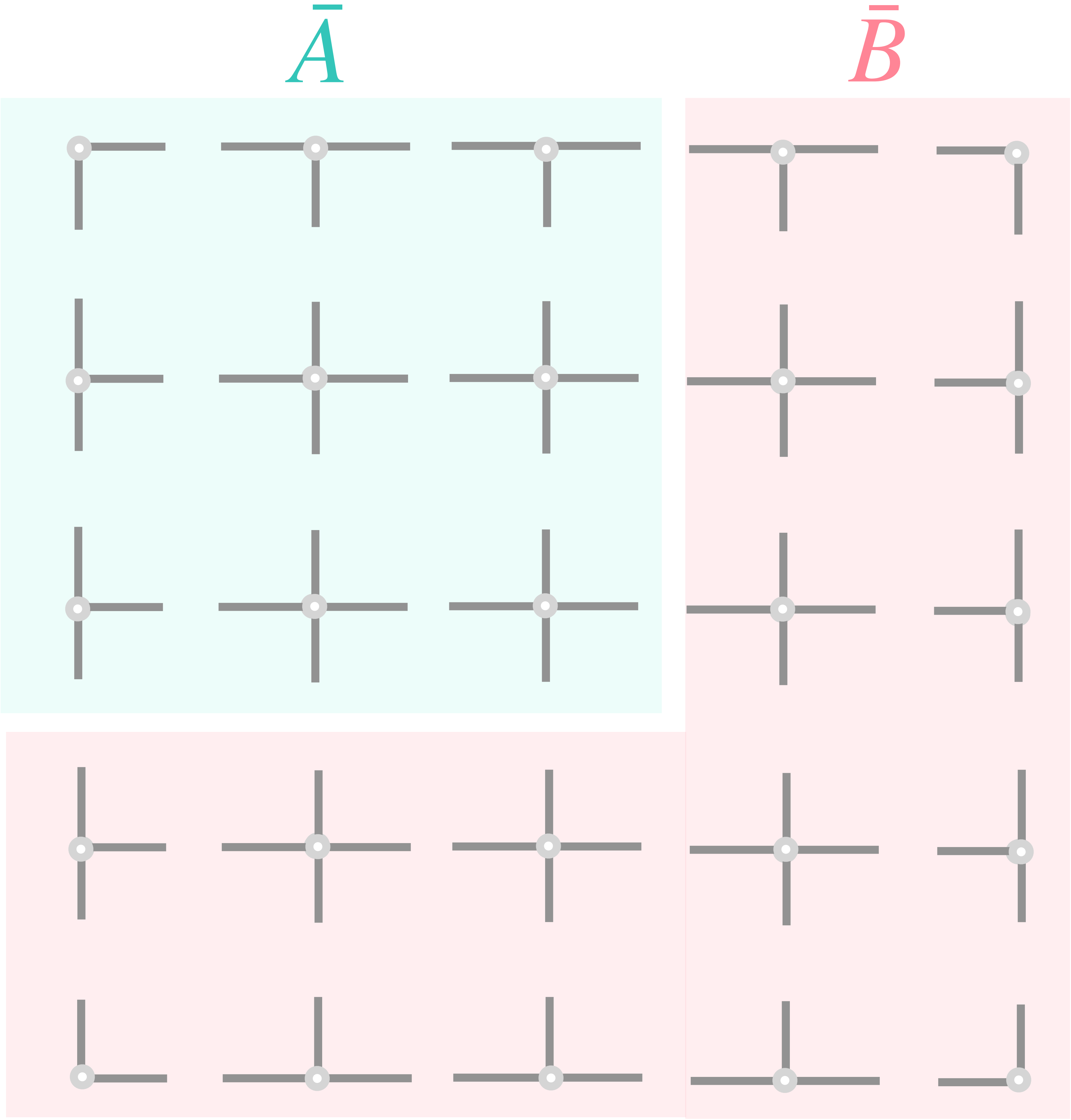
$$\mathbb{H}^{\mathrm{enc,DS}} = \bigotimes_v \mathbb{H}_v$$

$$\mathbb{H}^{\mathrm{phys,DS}}$$

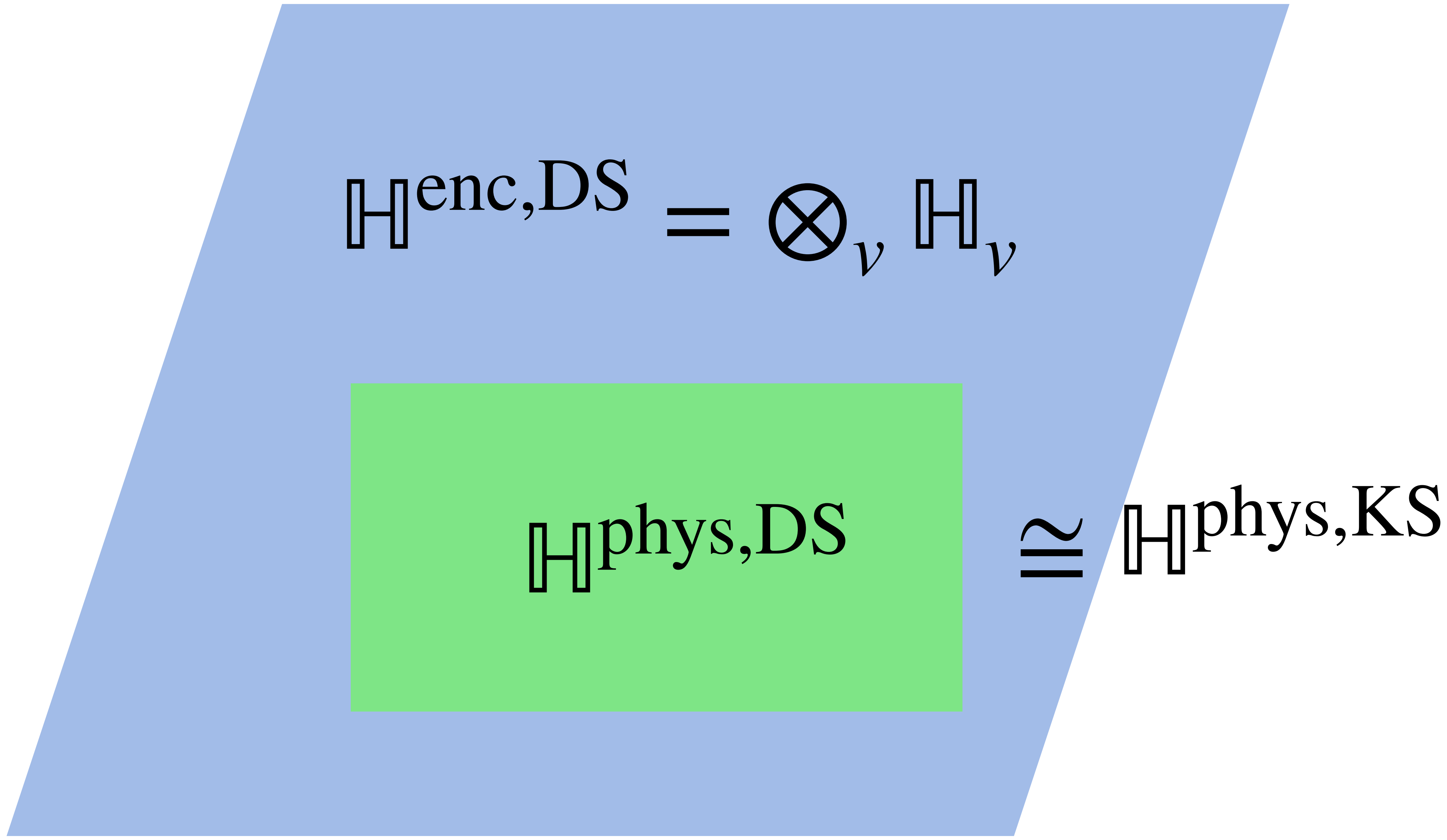
$$\cong \mathbb{H}^{\mathrm{phys,KS}}$$

$$j\,|_{\mathrm{left-half-link}} = j\,|_{\mathrm{right-half-link}}$$

Dressed-site encoding

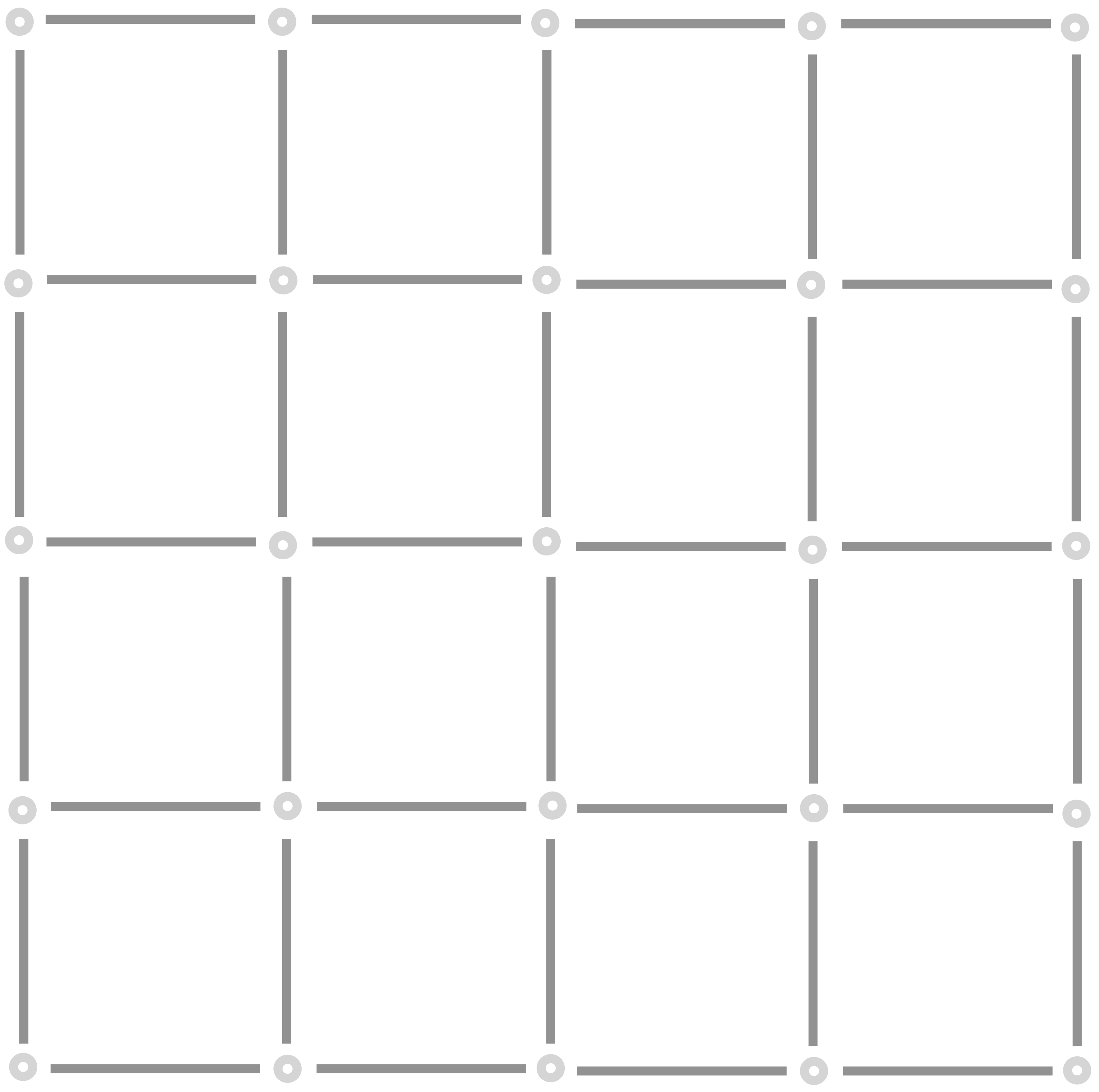


$$\mathbb{H}^{\text{enc,DS}} = \mathbb{H}^{\text{enc,DS}}_{\bar{A}} \otimes \mathbb{H}^{\text{enc,DS}}_{\bar{B}}$$



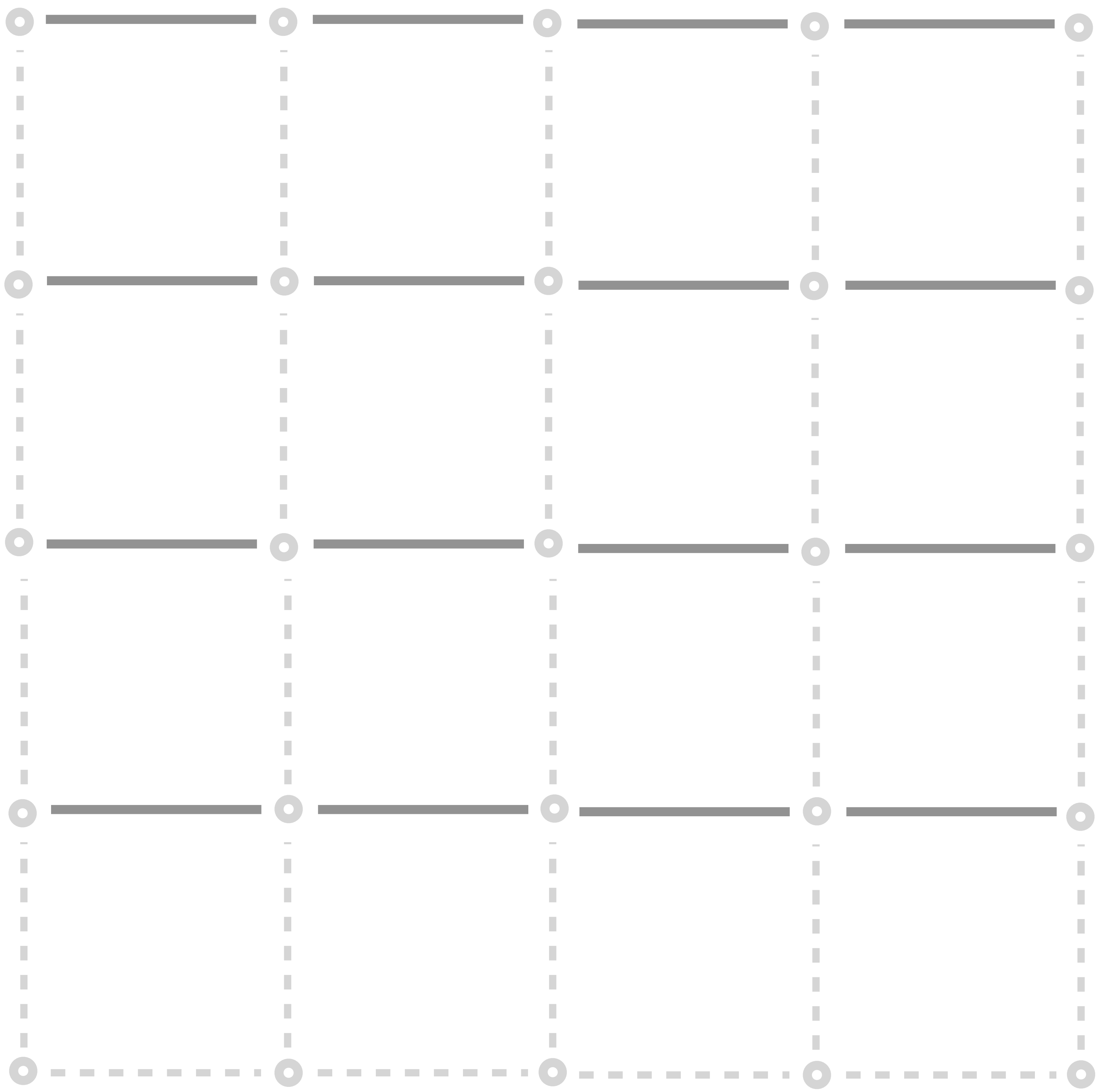
$$j|_{\text{left-half-link}} = j|_{\text{right-half-link}}$$

Maximal-tree gauge-fixed formulation



Bauer, D'Andrea, Freytsis, and Grabowska (2023); Burbano and Bauer (2024)

Maximal-tree gauge-fixed formulation



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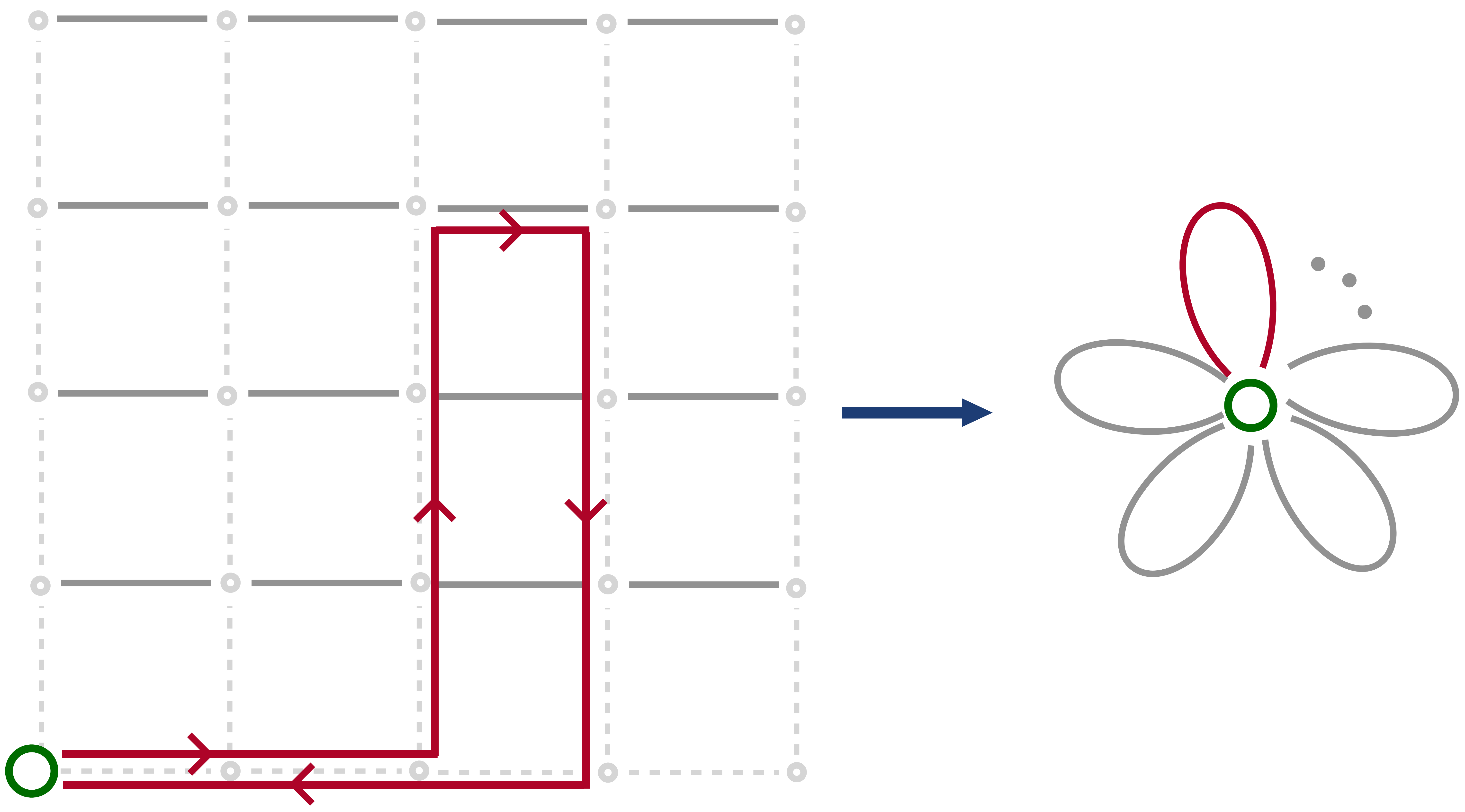

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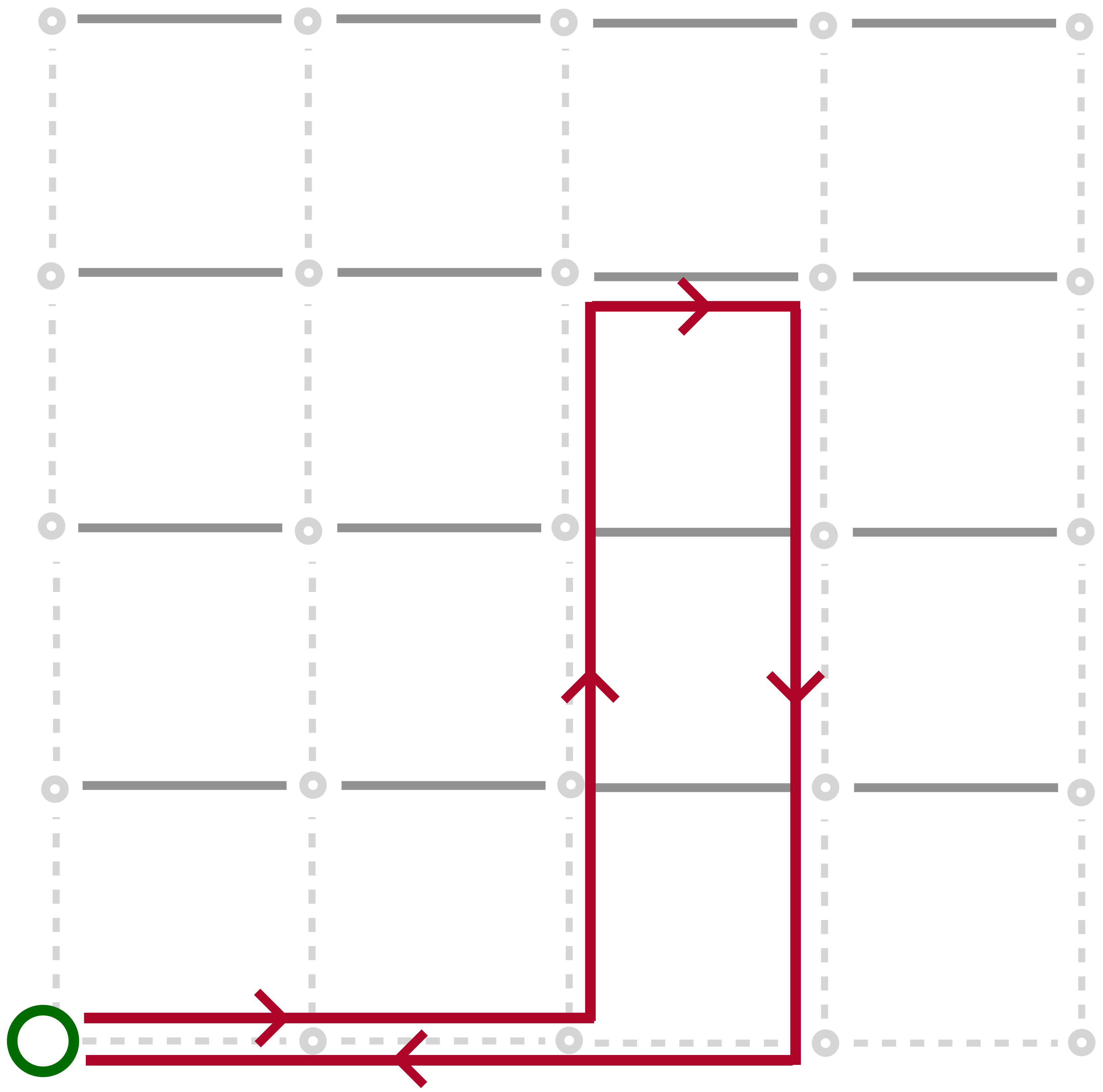
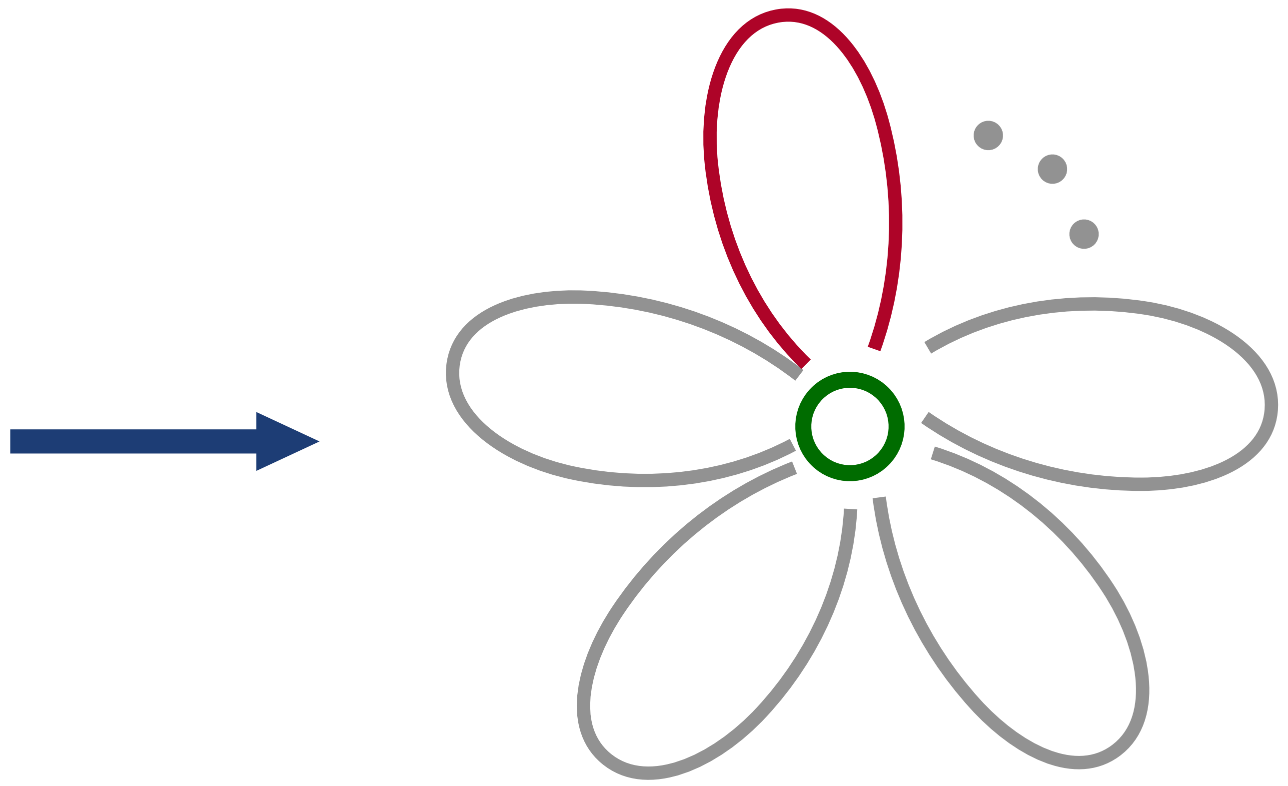
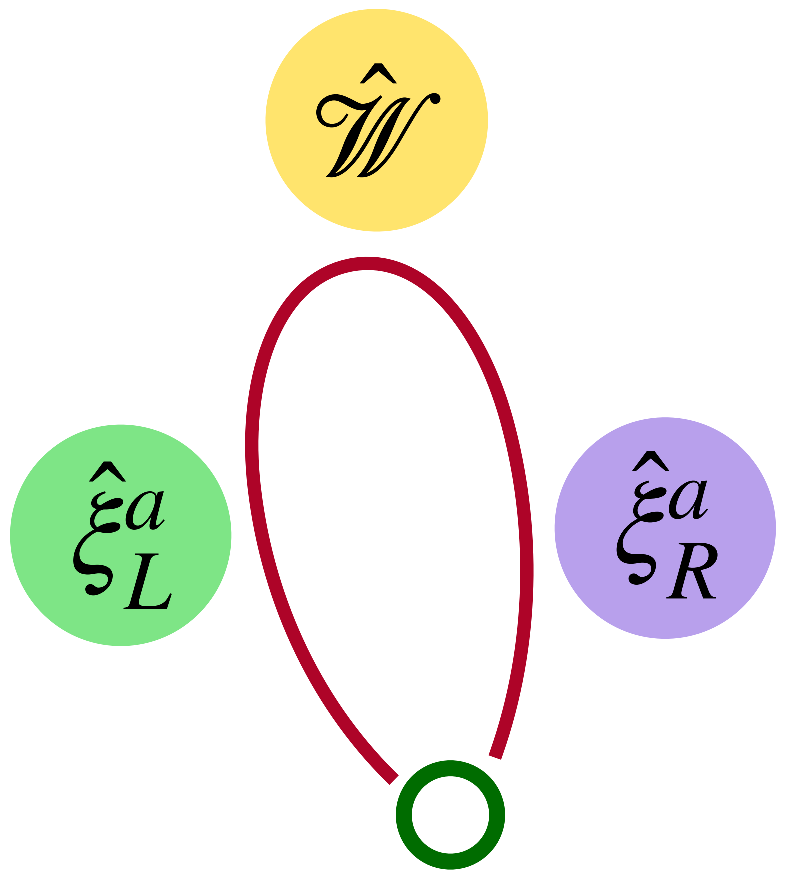




PETAL p

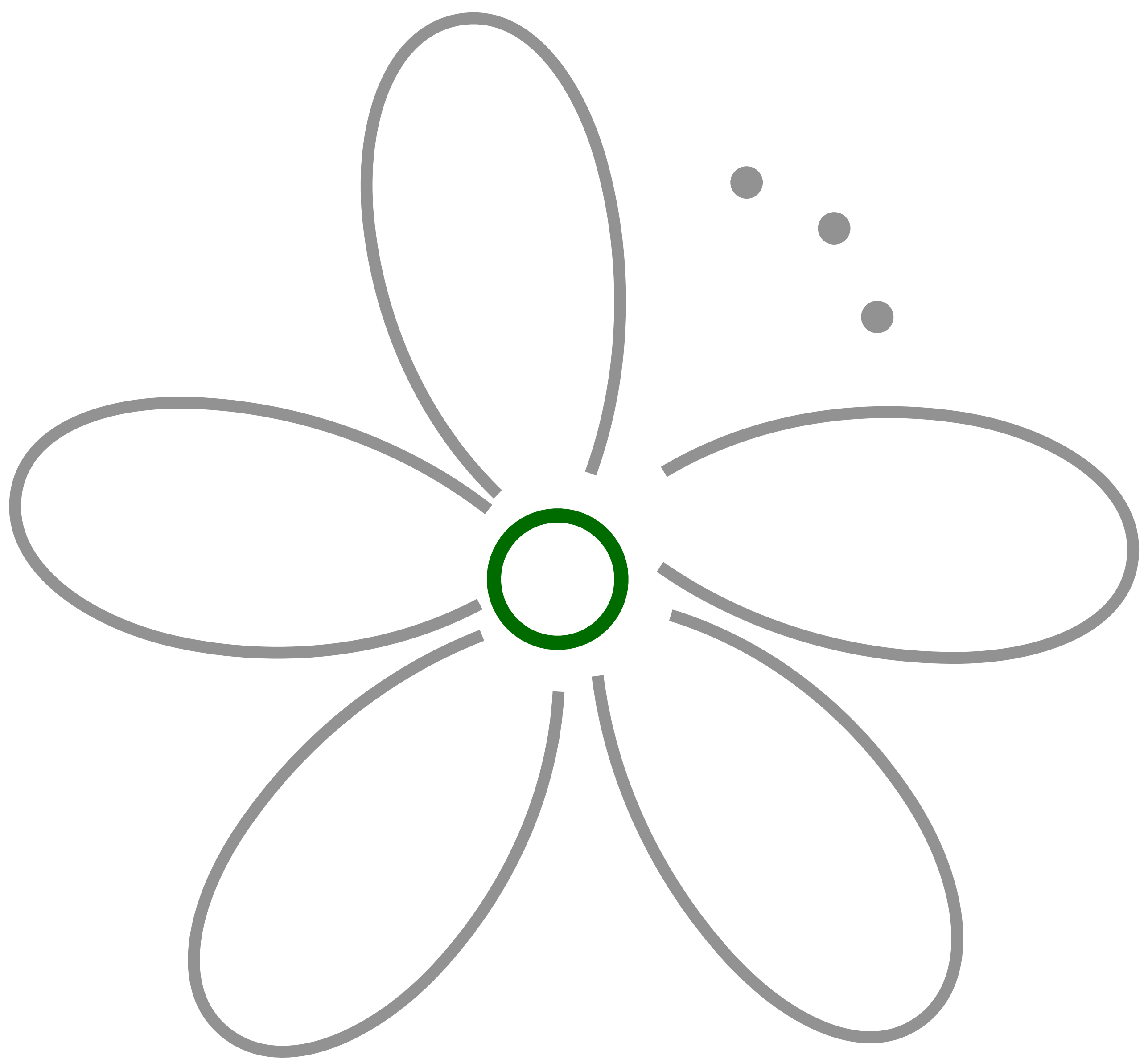


$$\mathbb{H}_p \cong L^2\left(SU(2)\right)$$

$$|j, m_L, m_R\rangle$$

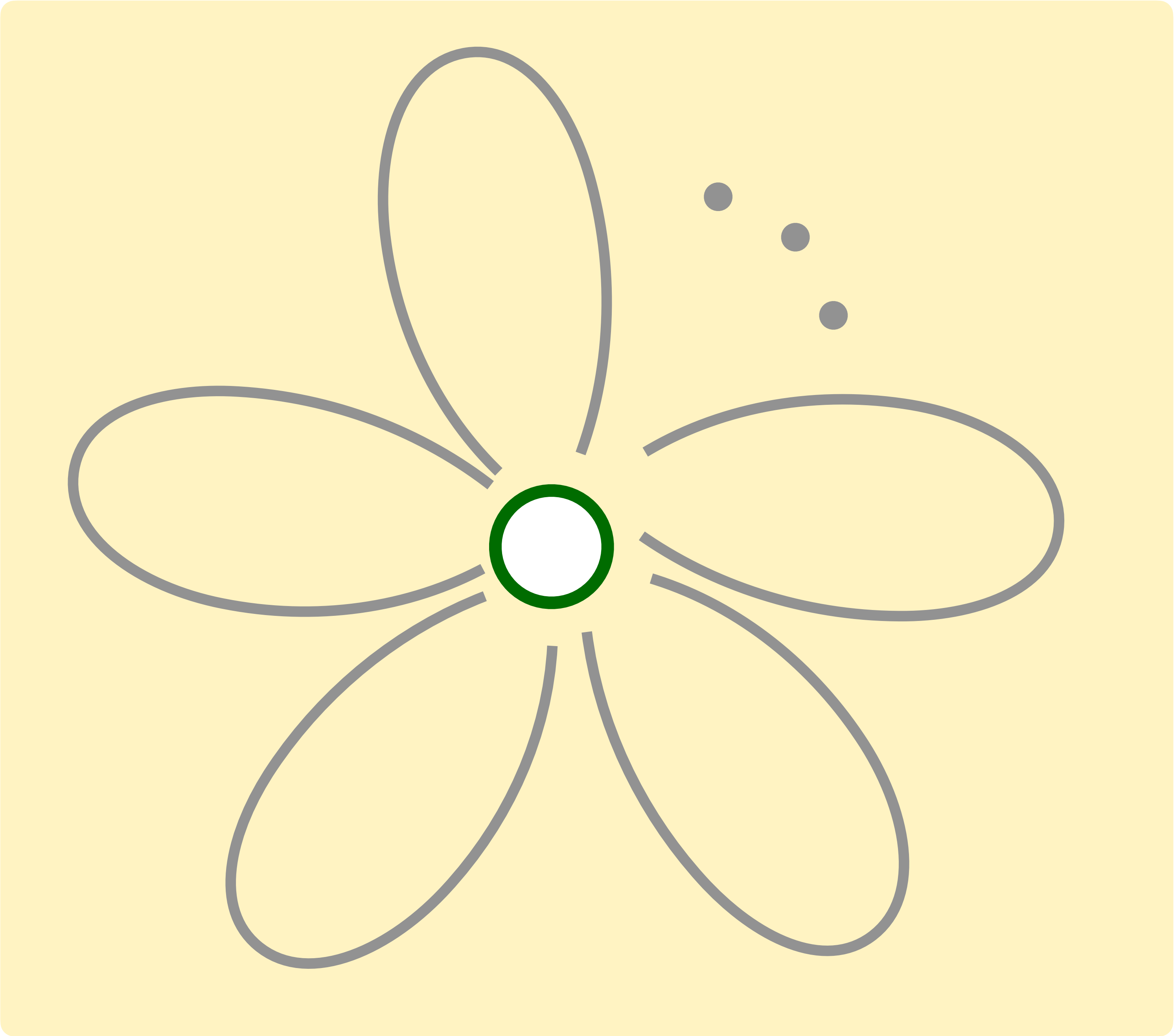
Bauer, D'Andrea, Freytsis, and Grabowska (2023), Burbano and Bauer (2024)

Maximal-tree gauge-fixed formulation



$$\mathbb{H}^{\text{enc,MT}} = \bigotimes_p \mathbb{H}_p$$

Maximal-tree gauge-fixed formulation

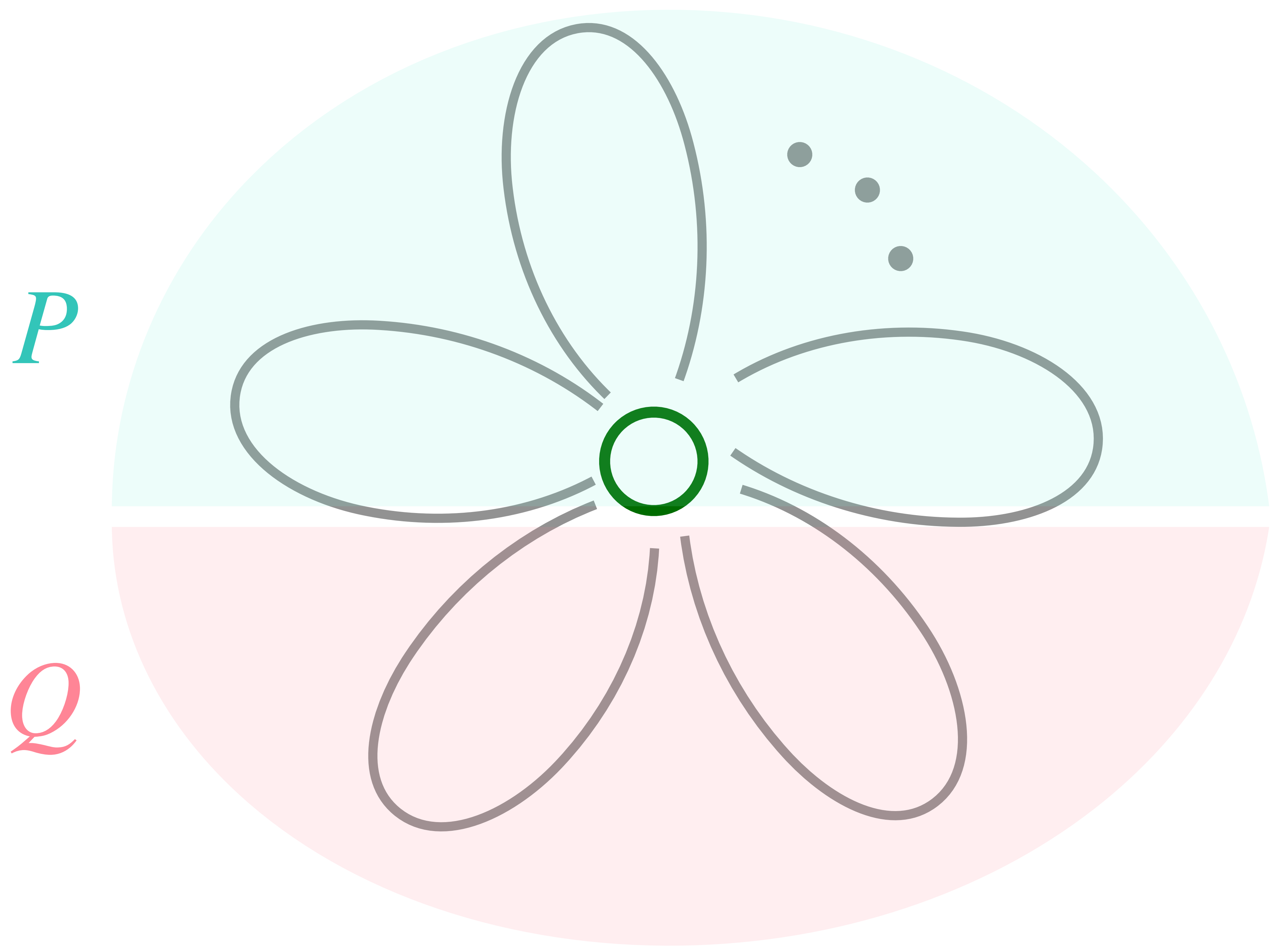


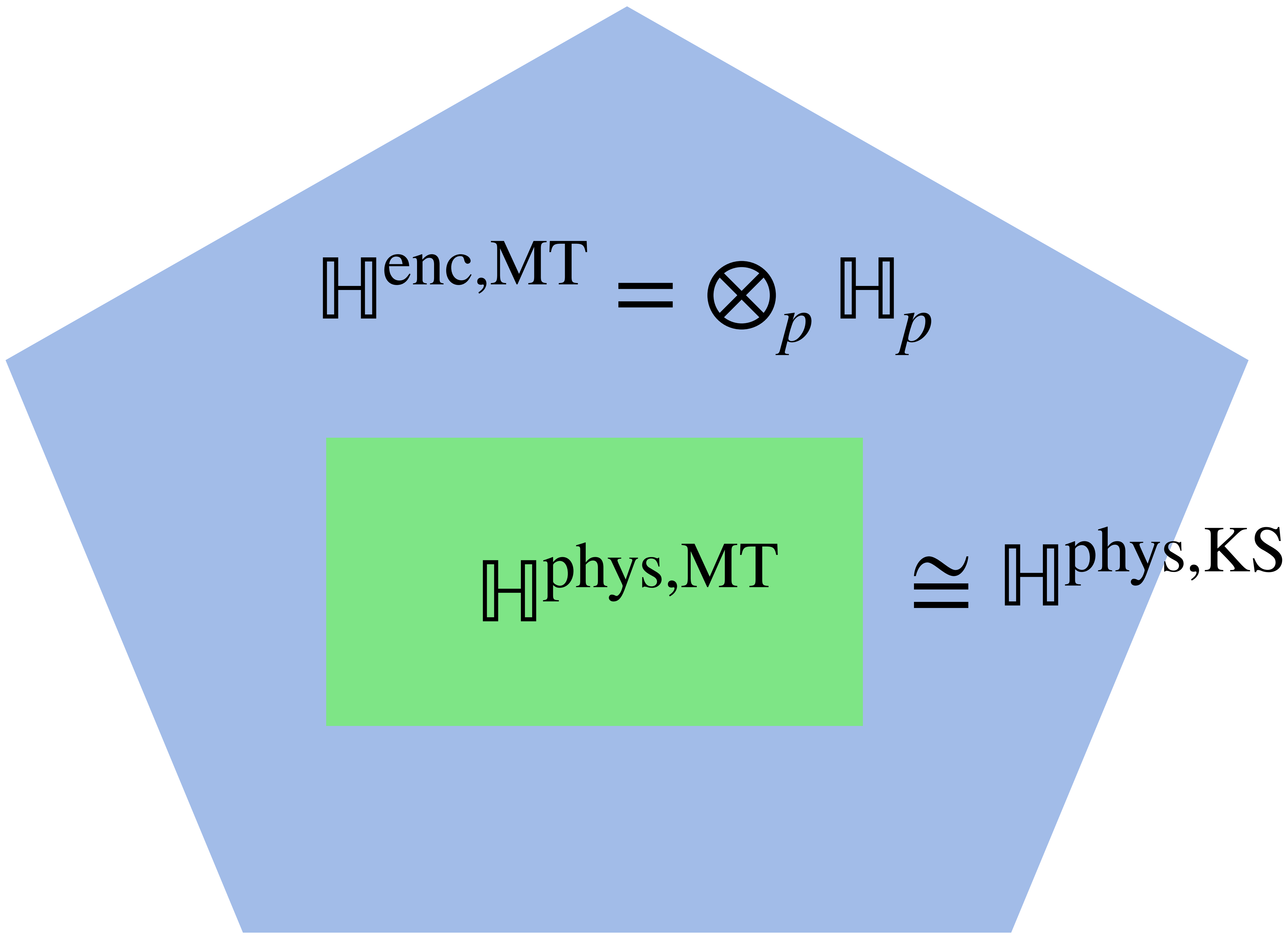
$$\mathbb{H}^{\text{enc,MT}} = \bigotimes_p \mathbb{H}_p$$

$$\mathbb{H}^{\text{phys,MT}} \cong \mathbb{H}^{\text{phys,KS}}$$

$$\hat{\mathcal{G}}^a = \sum_p \left(\hat{\xi}^a_R(p) - \hat{\xi}^a_L(p) \right)$$

Maximal-tree gauge-fixed formulation





$$\mathbb{H}^{\text{enc,MT}} = \mathbb{H}_P^{\text{enc,MT}} \otimes \mathbb{H}_Q^{\text{enc,MT}}$$

$$\hat{\mathcal{G}}^a = \sum_p \left(\hat{\xi}_R^a(p) - \hat{\xi}_L^a(p) \right)$$

Encoding

$\mathbb{H}^{\text{phys,KS}}$

$\mathbb{H}^{\text{phys,DS}}$

$\mathbb{H}^{\text{phys,MT}}$

Encoding

$$\mathbb{H}^{\text{enc,KS}} = \bigotimes_l \mathbb{H}_l$$

$$\mathbb{H}^{\text{phys,KS}}$$

$$\mathbb{H}^{\text{enc,DS}} = \bigotimes_v \mathbb{H}_v$$

$$\mathbb{H}^{\text{phys,DS}}$$

$$\mathbb{H}^{\text{enc,MT}} = \bigotimes_p \mathbb{H}_p$$

$$\mathbb{H}^{\text{phys,MT}}$$

Encoding

$$\mathbb{H}^{\text{enc,KS}} = \otimes_l \mathbb{H}_l$$

$$\mathbb{H}^{\text{phys,KS}}$$

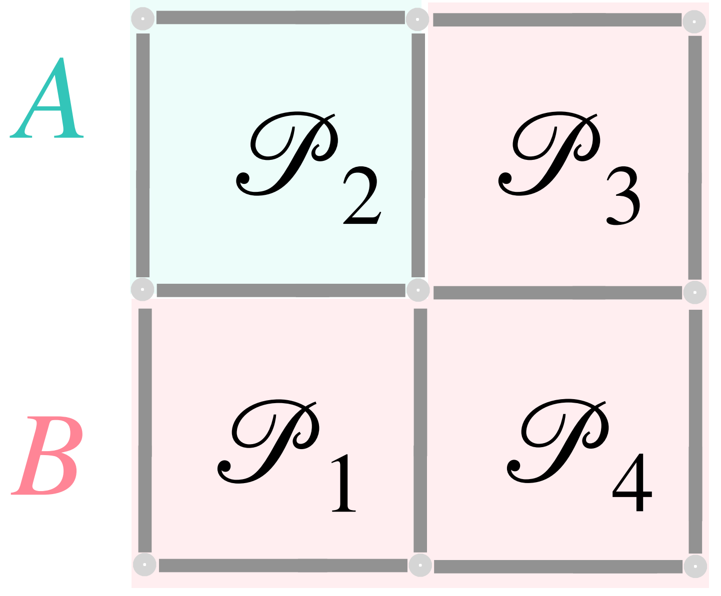
$$\mathbb{H}^{\text{enc,DS}} = \otimes_v \mathbb{H}_v$$

$$\mathbb{H}^{\text{phys,DS}}$$

$$\mathbb{H}^{\text{enc,MT}} = \otimes_p \mathbb{H}_p$$

$$\mathbb{H}^{\text{phys,MT}}$$

Encoding entropy



$$S_A^{\text{KS}}(\rho)$$

Encoding

$$\mathbb{H}^{\text{enc,KS}} = \otimes_l \mathbb{H}_l$$

$$\mathbb{H}^{\text{phys,KS}}$$

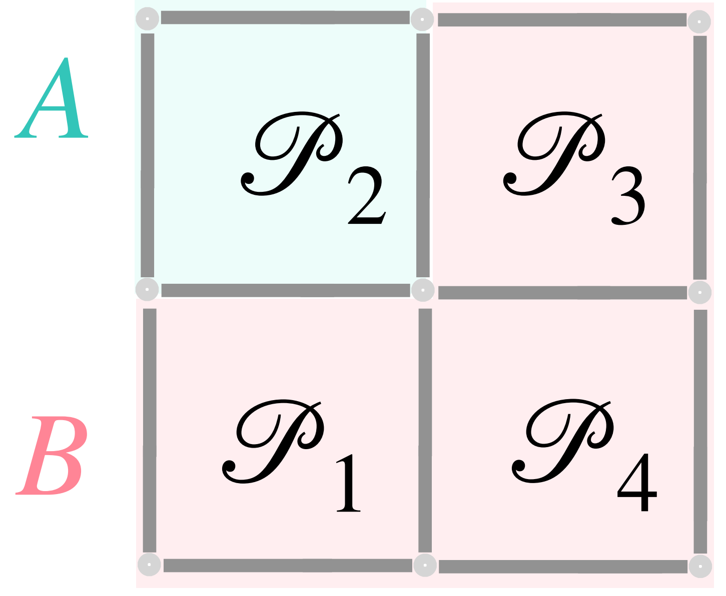
$$\mathbb{H}^{\text{enc,DS}} = \otimes_v \mathbb{H}_v$$

$$\mathbb{H}^{\text{phys,DS}}$$

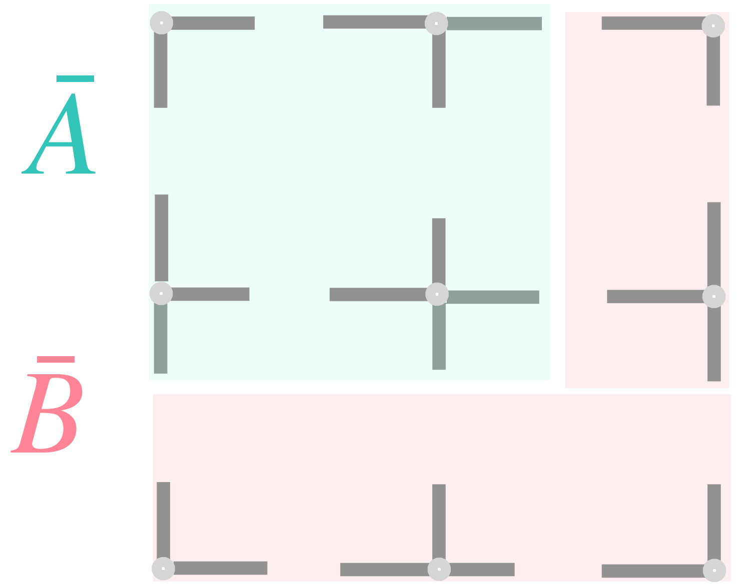
$$\mathbb{H}^{\text{enc,MT}} = \otimes_p \mathbb{H}_p$$

$$\mathbb{H}^{\text{phys,MT}}$$

Encoding entropy



$S_A^{\text{KS}}(\rho)$



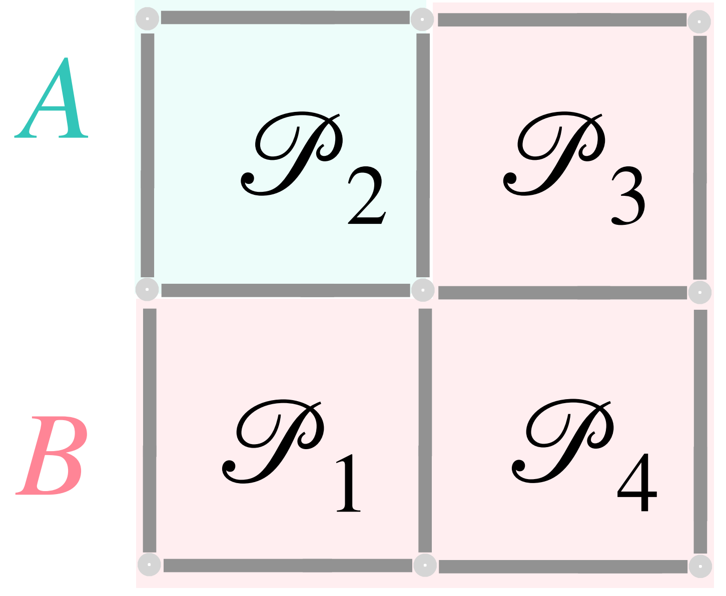
$S_{\bar{A}}^{\text{DS}}(\rho)$

Encoding

Encoding entropy

$$\mathbb{H}^{\mathrm{enc,KS}} = \otimes_l \mathbb{H}_l$$

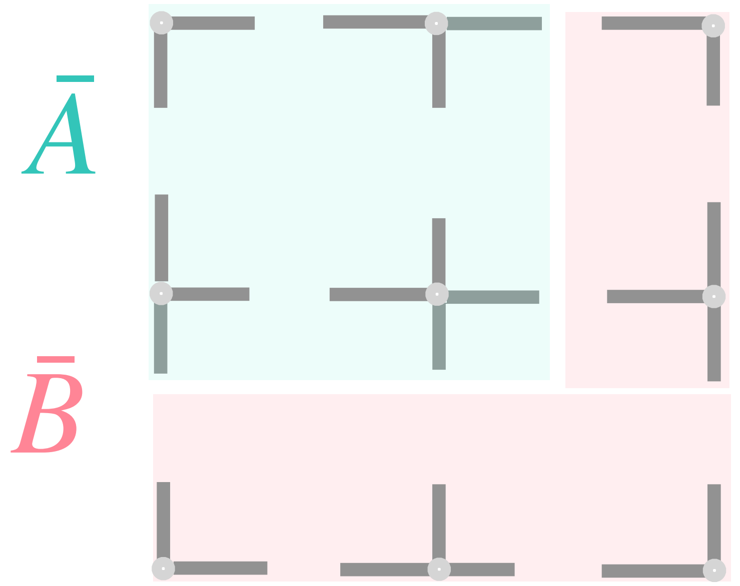
$$\mathbb{H}^{\mathrm{phys,KS}}$$



$S_A^{\mathrm{KS}}(\rho)$

$$\mathbb{H}^{\mathrm{enc,DS}} = \otimes_v \mathbb{H}_v$$

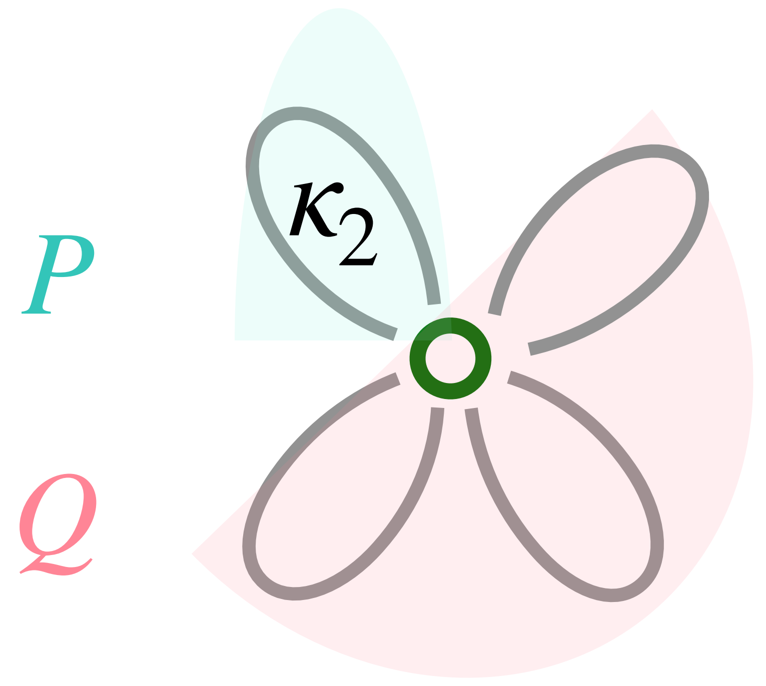
$$\mathbb{H}^{\mathrm{phys,DS}}$$



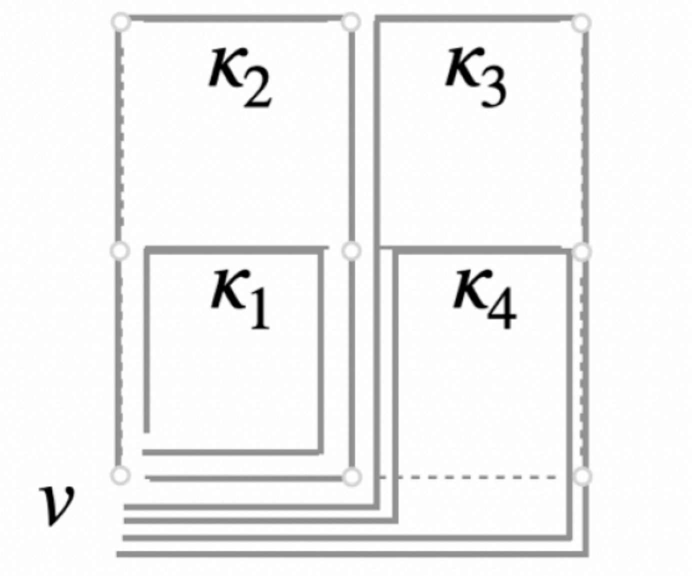
$S_{\bar{A}}^{\mathrm{DS}}(\rho)$

$$\mathbb{H}^{\mathrm{enc,MT}} = \otimes_p \mathbb{H}_p$$

$$\mathbb{H}^{\mathrm{phys,MT}}$$

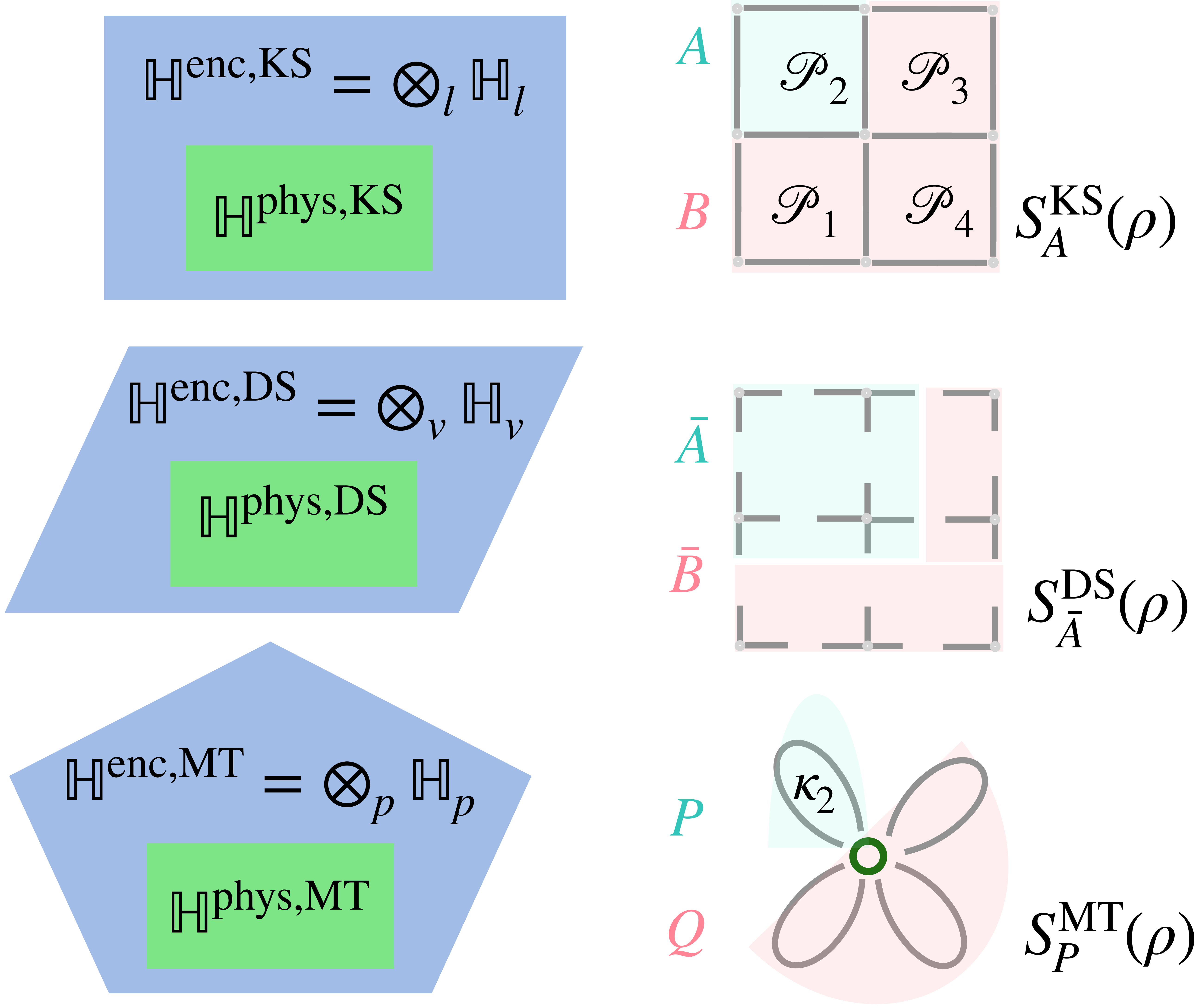


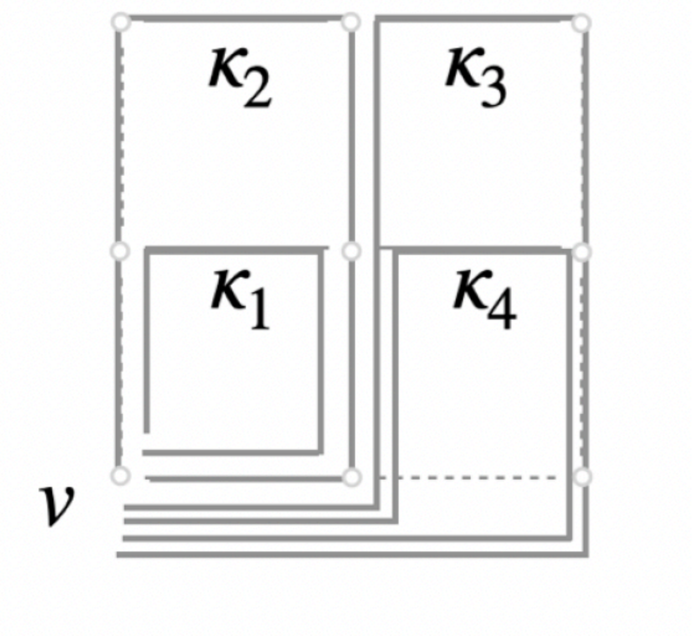
$S_P^{\mathrm{MT}}(\rho)$



Encoding | *Encoding entropy*

$$|\psi\rangle = \frac{1}{2}\sum_{i=1}^{4}(|\mathscr{P}_i\rangle)$$





Encoding

$$\mathbb{H}^{\mathrm{enc,KS}} = \otimes_l \mathbb{H}_l$$

$$\mathbb{H}^{\mathrm{phys,KS}}$$

$$\mathbb{H}^{\mathrm{enc,DS}} = \otimes_v \mathbb{H}_v$$

$$\mathbb{H}^{\mathrm{phys,DS}}$$

$$\mathbb{H}^{\mathrm{enc,MT}} = \otimes_p \mathbb{H}_p$$

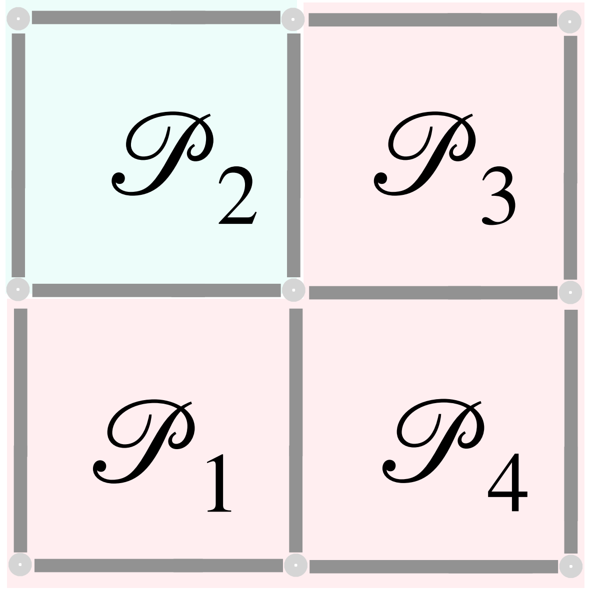
$$\mathbb{H}^{\mathrm{phys,MT}}$$

Encoding entropy

$$|\psi\rangle = \frac{1}{2}\sum_{i=1}^{4}(|\mathscr{P}_i\rangle)$$

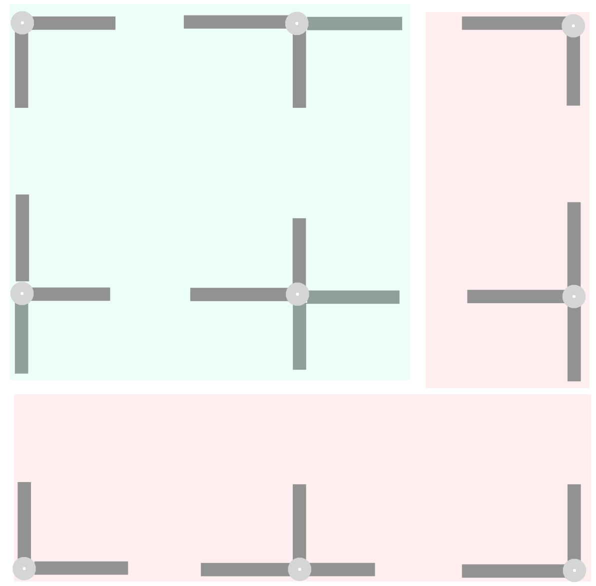
A

B



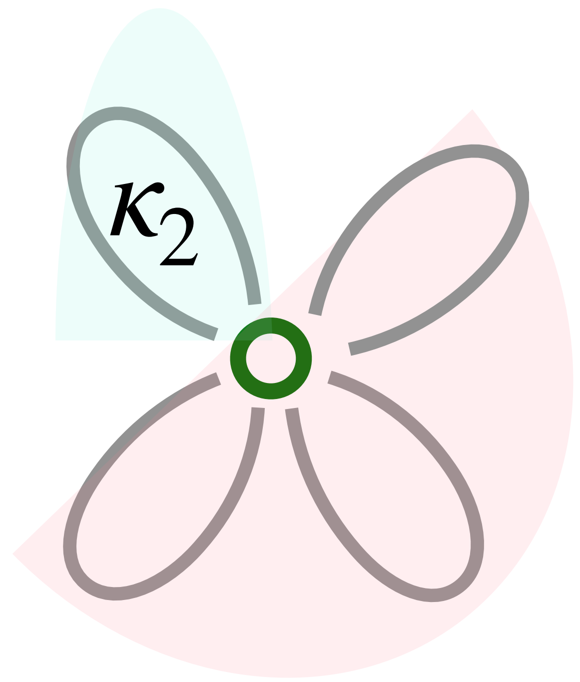
$\bar{A}$

$\bar{B}$

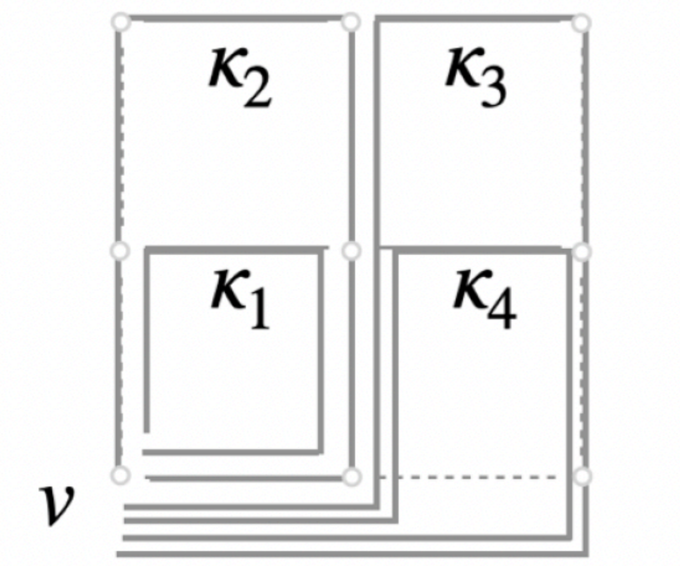


P

Q



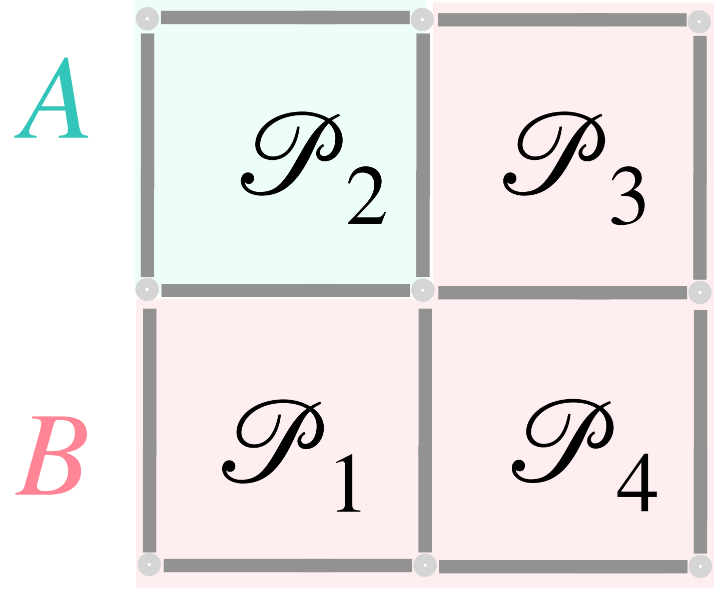
	Classical	Rep	Quantum
$S_A^{\mathrm{KS}}(\rho)$	1.5	1.0	0.5
$S_{\bar{A}}^{\mathrm{DS}}(\rho)$	2.0	0	0
$S_P^{\mathrm{MT}}(\rho)$	0.696	0.297	0.232



Encoding ***Encoding entropy*** ***Spatial correlations***

$$\mathbb{H}^{\text{enc,KS}} = \otimes_l \mathbb{H}_l$$

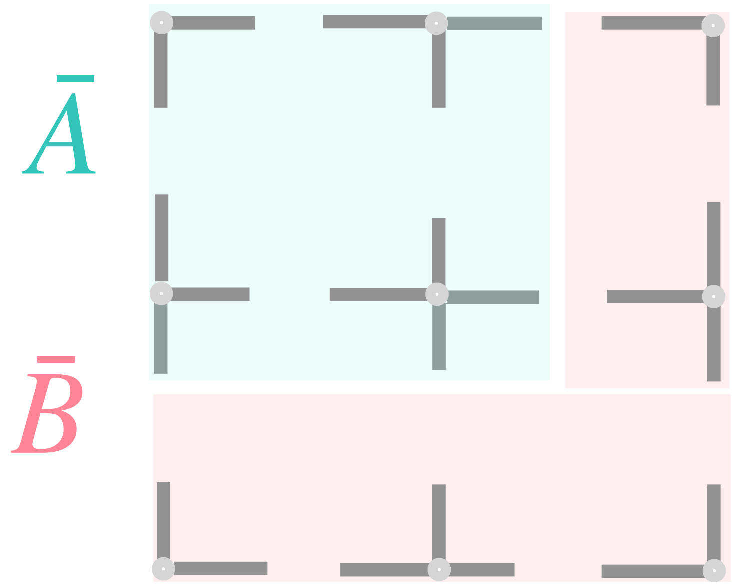
$$\mathbb{H}^{\text{phys,KS}}$$



$S_A^{\text{KS}}(\rho)$

$$\mathbb{H}^{\text{enc,DS}} = \otimes_v \mathbb{H}_v$$

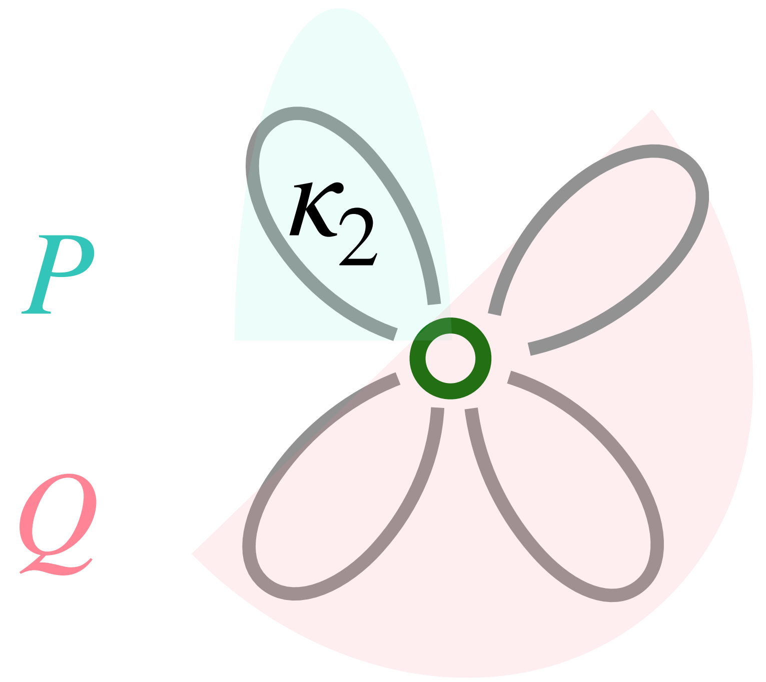
$$\mathbb{H}^{\text{phys,DS}}$$



$S_{\bar{A}}^{\text{DS}}(\rho)$

$$\mathbb{H}^{\text{enc,MT}} = \otimes_p \mathbb{H}_p$$

$$\mathbb{H}^{\text{phys,MT}}$$



$S_P^{\text{MT}}(\rho)$

Encoding

$$\mathbb{H}^{\text{enc,KS}} = \otimes_l \mathbb{H}_l$$

$$\mathbb{H}^{\text{phys,KS}}$$

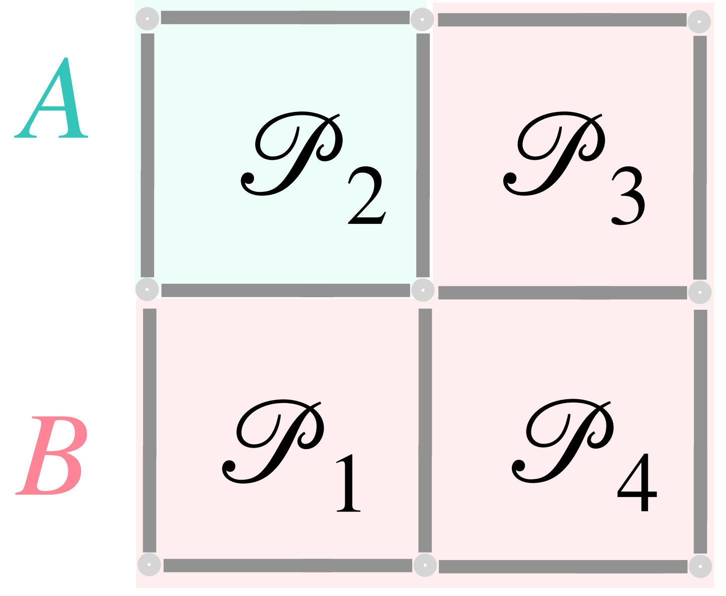
$$\mathbb{H}^{\text{enc,DS}} = \otimes_v \mathbb{H}_v$$

$$\mathbb{H}^{\text{phys,DS}}$$

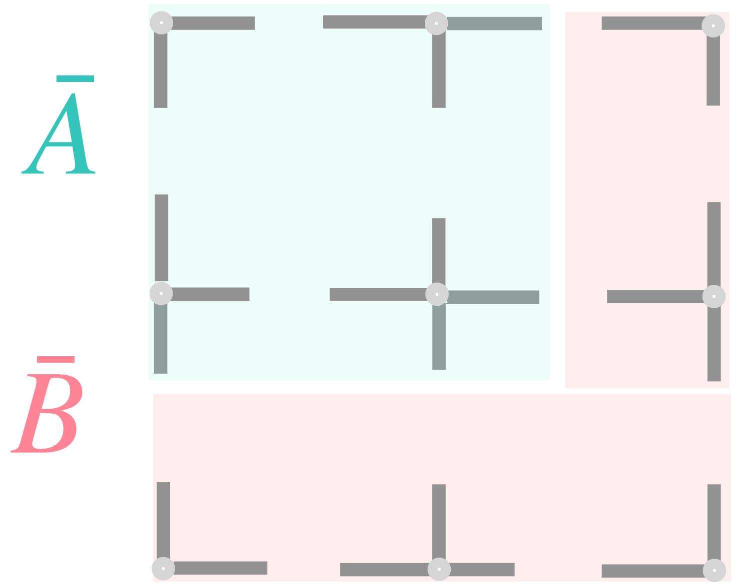
$$\mathbb{H}^{\text{enc,MT}} = \otimes_p \mathbb{H}_p$$

$$\mathbb{H}^{\text{phys,MT}}$$

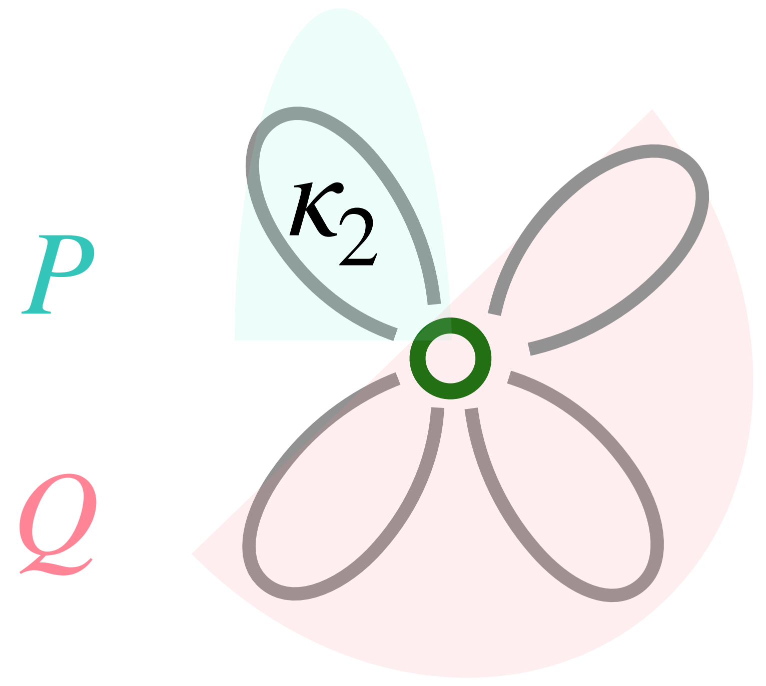
Encoding entropy



$S_A^{\text{KS}}(\rho)$



$S_{\bar{A}}^{\text{DS}}(\rho)$



$S_P^{\text{MT}}(\rho)$

Spatial correlations

Spatially local encodings.

But both resolve distinct boundary charges.

While not relevant for UV-finite/ relative entropies, boundary differences are crucial for simulation.

Casini, Huerta, and Rosabal (2013)

Upcoming work quantifies these distinctions.

Encoding

$$\mathbb{H}^{\text{enc,KS}} = \otimes_l \mathbb{H}_l$$

$$\mathbb{H}^{\text{phys,KS}}$$

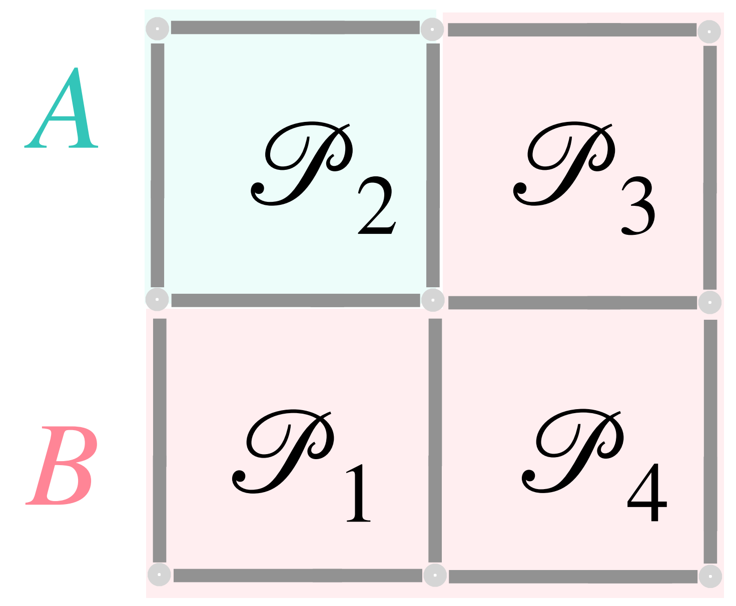
$$\mathbb{H}^{\text{enc,DS}} = \otimes_v \mathbb{H}_v$$

$$\mathbb{H}^{\text{phys,DS}}$$

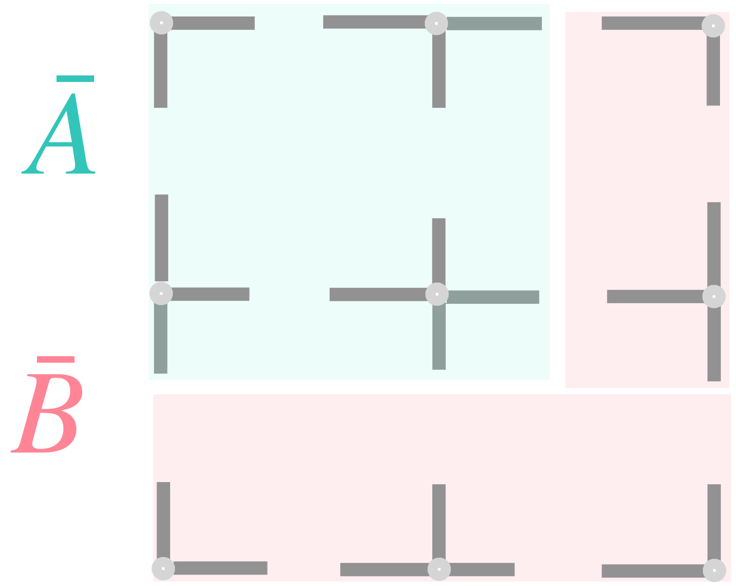
$$\mathbb{H}^{\text{enc,MT}} = \otimes_p \mathbb{H}_p$$

$$\mathbb{H}^{\text{phys,MT}}$$

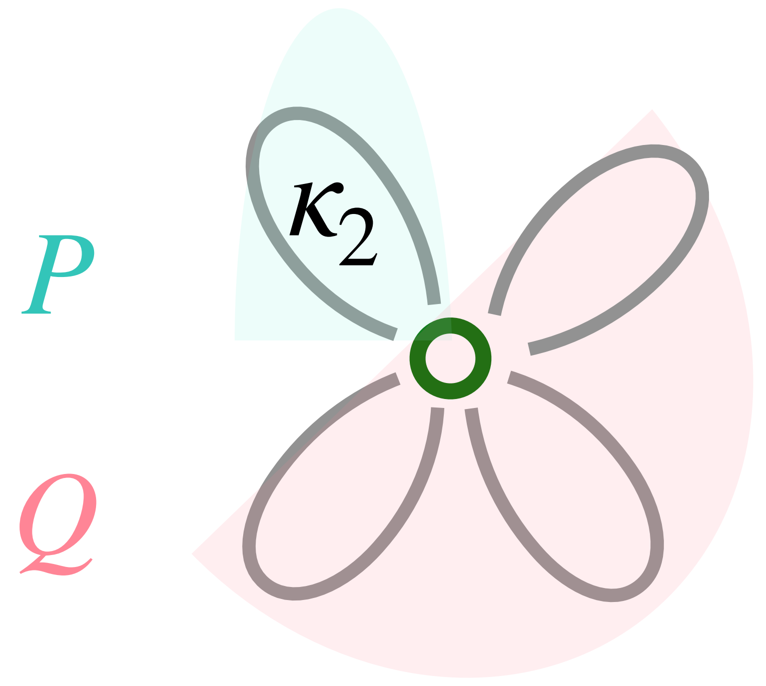
Encoding entropy



$S_A^{\text{KS}}(\rho)$



$S_{\bar{A}}^{\text{DS}}(\rho)$



$S_P^{\text{MT}}(\rho)$ $\rightarrow$

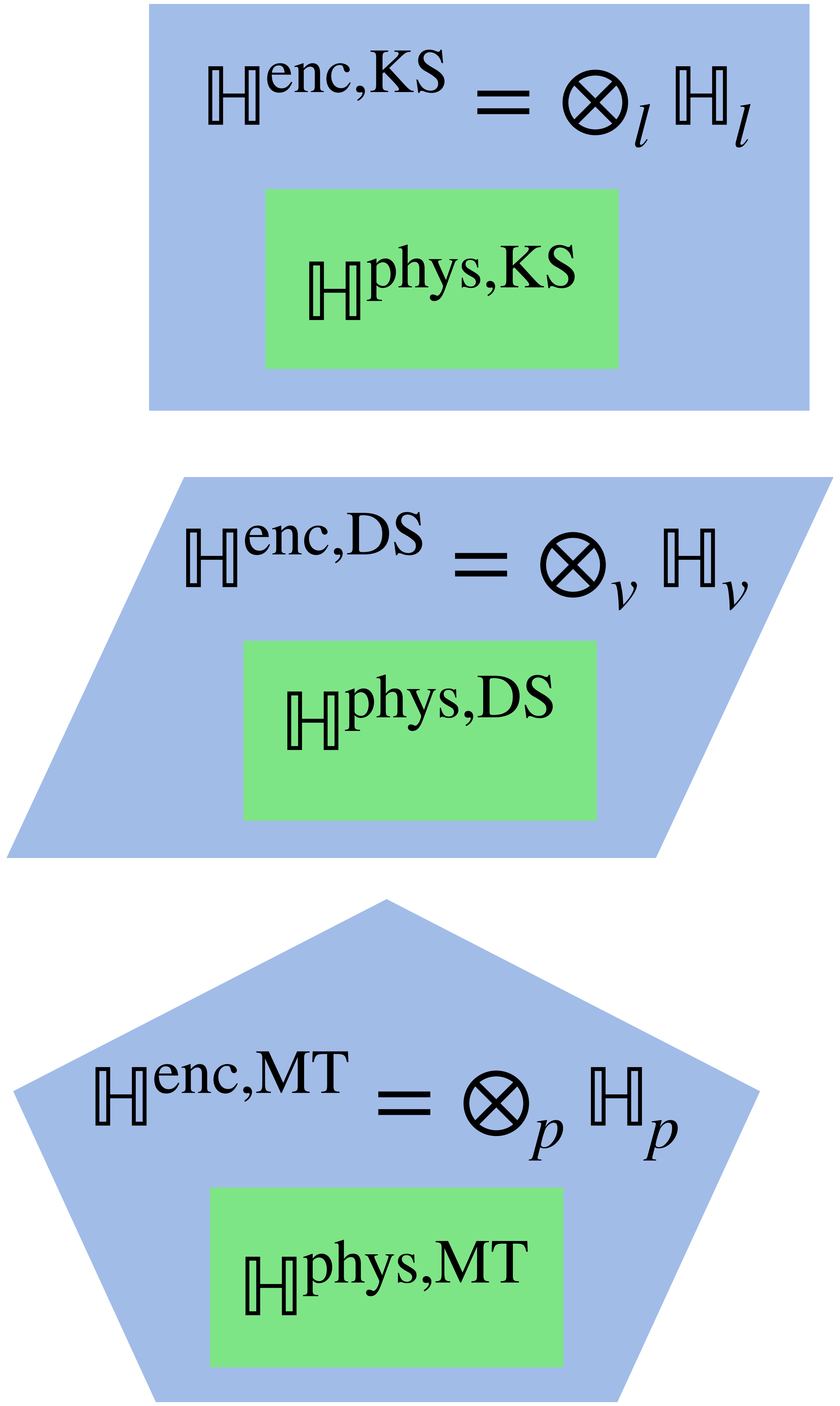
Spatial correlations

Spatially nonlocal encoding.

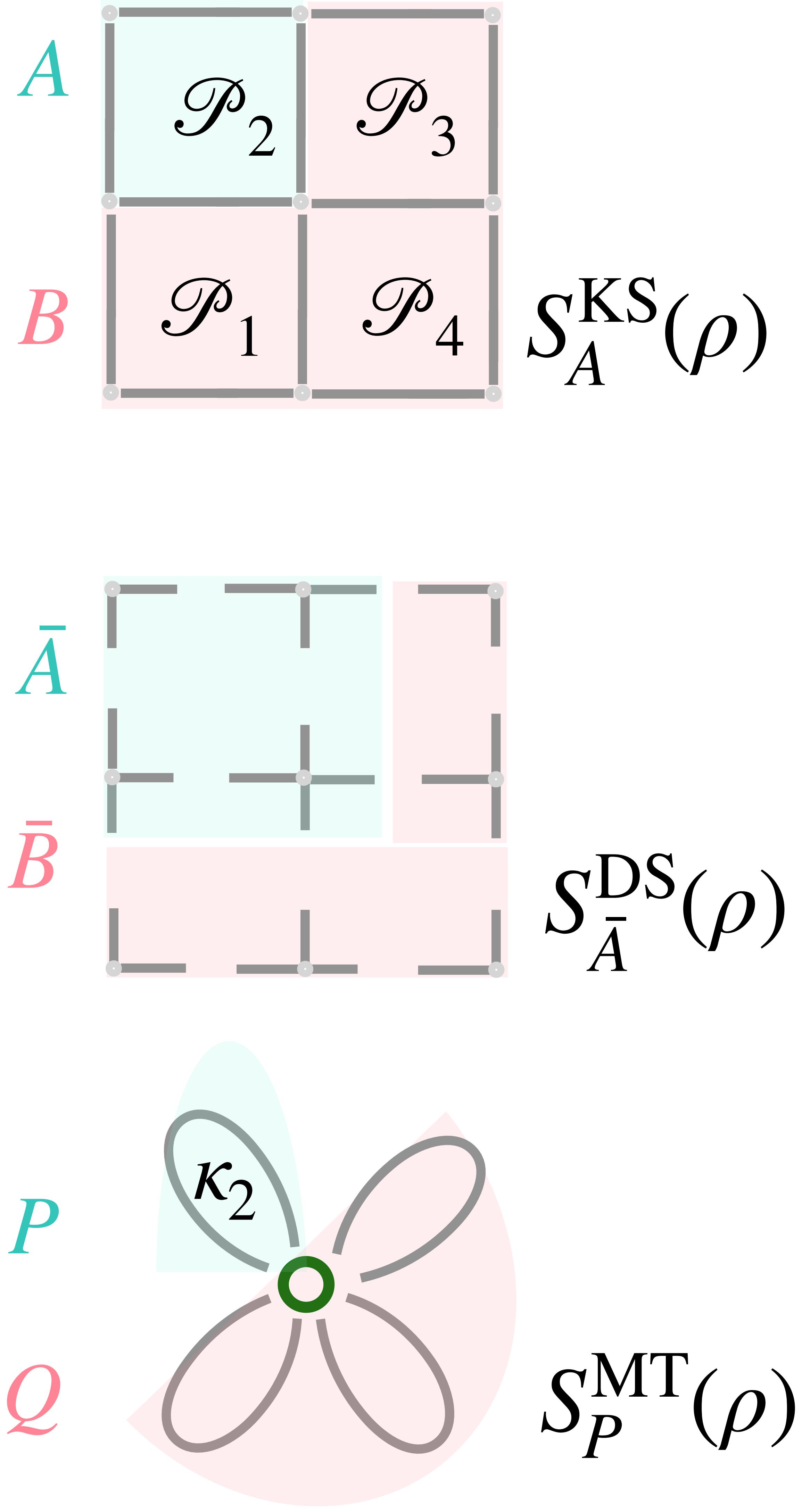
No bipartition of the MT petals corresponds to a spatial bipartition.

We demonstrate an encoding-nonlocal algebraic procedure for extracting spatial entanglement from the MT encoding.

Encoding



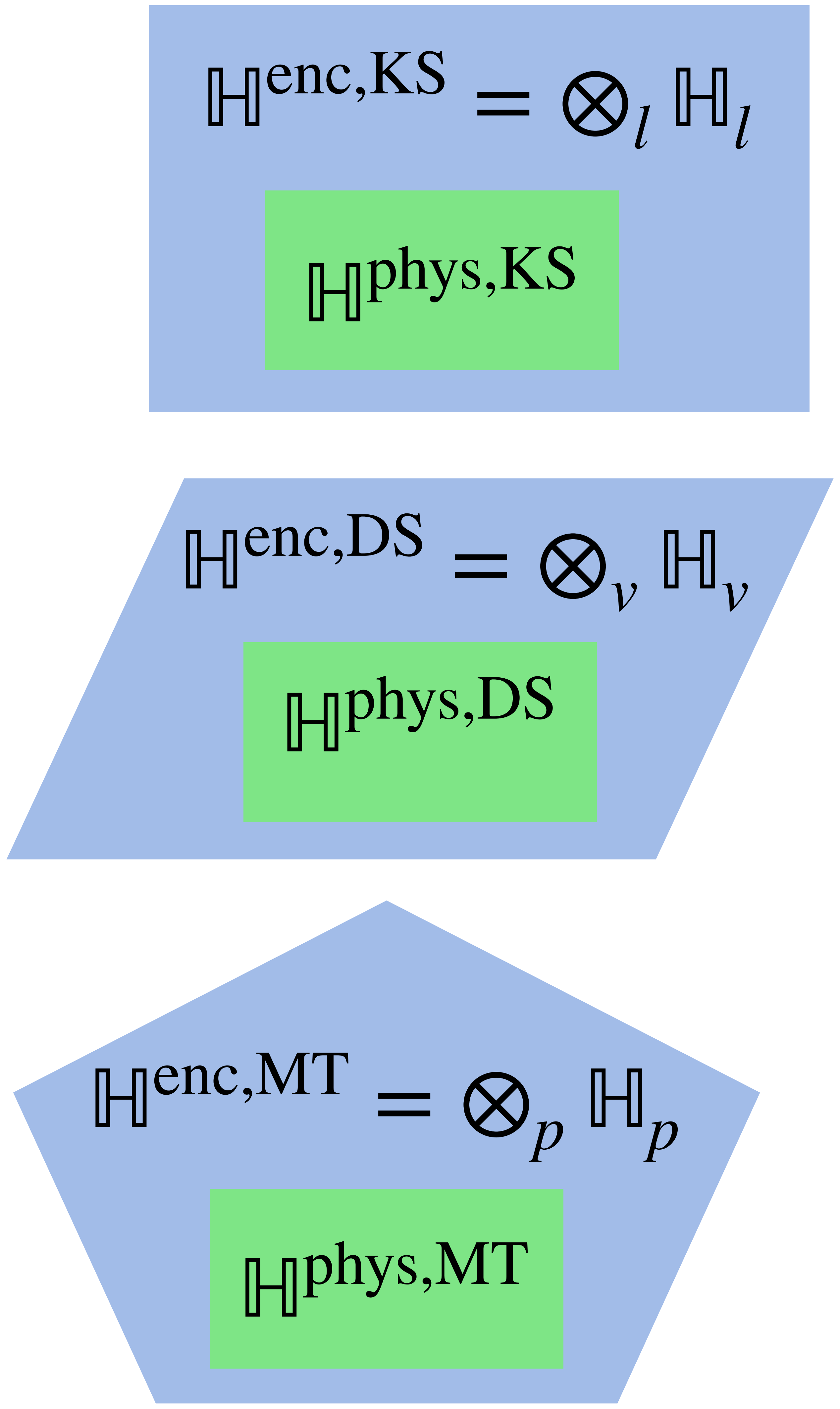
Encoding entropy



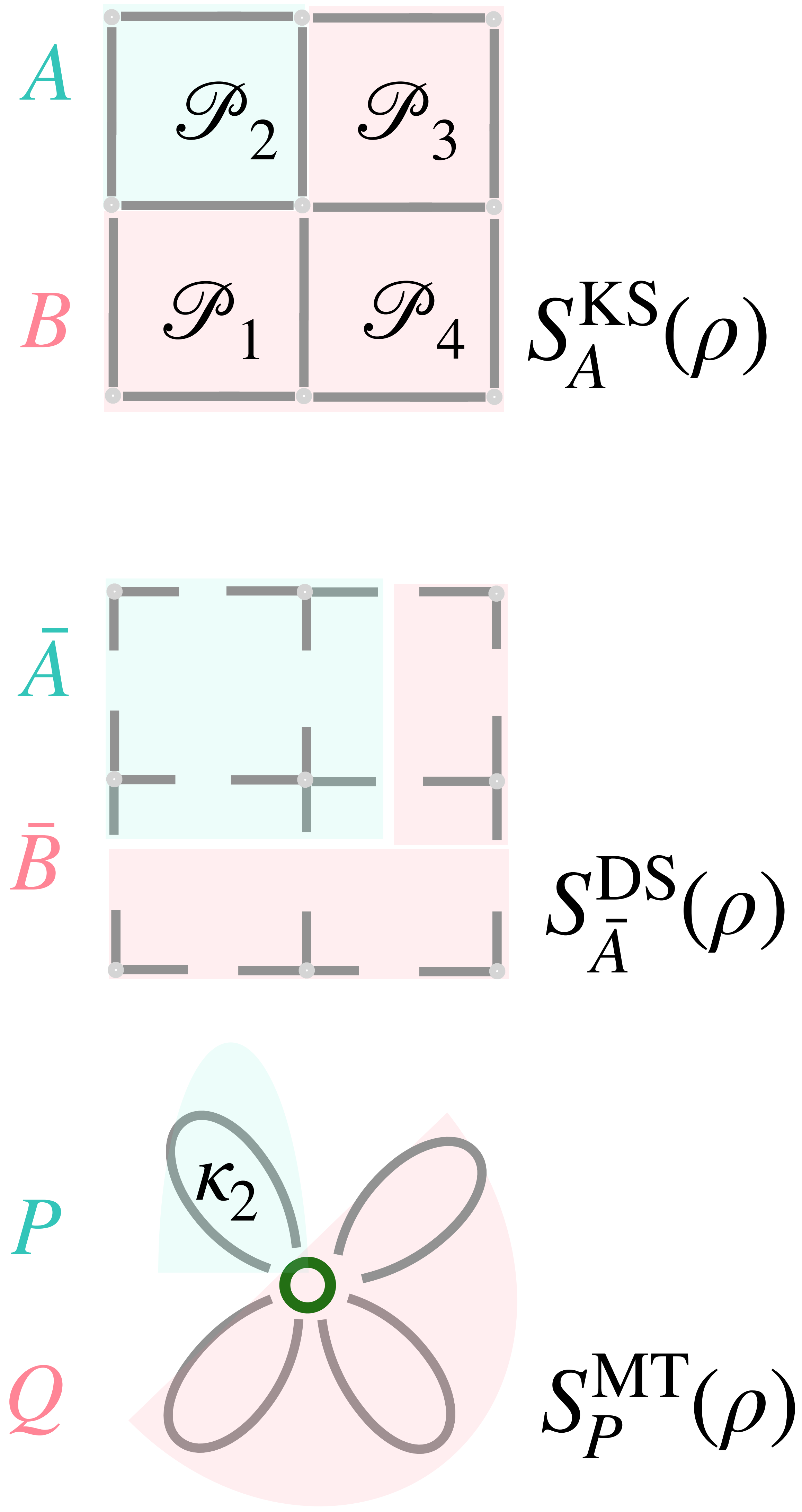
Spatial correlations

In several commonly used (1+1)D formulations, the encoding entropy is already spatial.

Encoding



Encoding entropy



Future work

- Relating encoding entropy to concrete simulation-resource requirements
- Characterizing eigenstate encoding entropies across encodings and coupling regimes.
- Extending to other encodings.