$\mathrm{I} = 2$ pion scattering length in the large-N limit

UNIVERSITÄT BERN

Claudio Bonanno

Albert Einstein Center for Fundamental Physics

Institute for Theoretical Physics – University of Bern

✉ claudio.bonanno@unibe.ch

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Work in collaboration with:

M. García Pérez, A. González-Arroyo, K.-I. Ishikawa, M. Okawa, F. Romero-López

The large-N limit of QCD

$N \to \infty \qquad g^2 N = \lambda = \text{fixed} \qquad N_\mathrm{f} = \text{fixed}$

't Hooft **72** (1974) 461–473

- Preserves chiral symmetry breaking, confinement, asymptotic freedom
- Path integral can be expanded in series of $1/N$

$$\langle \mathcal{A} \rangle_N = A_\infty \left[1 + A_1 \frac{1}{N} + A_2 \frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)\right]$$

- MAIN IDEA: $1/N = 1/3 \approx$ **small**
  - $1/N$ expansion bridges $N = \infty$ and $N = 3$
  - Fundamental theoretical framework with predictive power for QCD

Meson scattering in large-N QCD

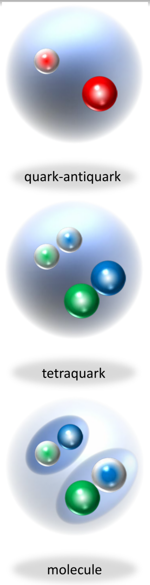

Figure from: [2509.08648]

- Interesting theoretical problem
- Can shed light on open problems in hadron physics
  - **Tetraquark candidates** (e.g., T_{cc}^{+}) Weinberg PRL **110** (2013) 261601
    $QQ\bar{q}\bar{q}$ can form tetraquark if $\frac{m}{M} \ll \frac{1}{N}$ Allaman et al. JHEP **11** (2024) 034
  - **Light scalar mesons** (e.g., σ) Peláez PRL **92** (2004) 102001
    Large-N pole behavior depends on nature of resonance
    Peláez Phys. Rept. **658** (2016) 1
- Actively studied in recent years using χPT, bootstrap and lattice
  Cheung et al. [2605.21582] — Albert et al. [2606.19420] — Bocchia et al. [2606.14676]
  DeGrand [2409.02242] — Baeza-Ballesteros et al. JHEP **06** (2022) 049; JHEP **08** (2025) 110
- Scattering at large N hard with traditional lattice approach
- EGUCHI–KAWAI REDUCTION: lattice framework for $N \to \infty$ QCD
- **This talk** $\to$ clean setup: isospin-2 $\pi\pi$ scattering

Eguchi–Kawai reduction

Main idea $\rightarrow$ large-N volume independence Eguchi–Kawai PRL **48** (1982) 1063

$N \rightarrow \infty \quad \rightarrow \quad$ lattice gauge theory becomes independent of the volume

$\rightarrow$ Possibility of studying the large-N limit on a **1-point lattice**

Reducing $V = 1$ allows to reach much larger values of N

$\rightarrow$ Up to $N \gtrsim \mathcal{O}(1000)$, while $N \lesssim \mathcal{O}(10)$ (traditional approach)

Large-N volume independence requires unbroken center symmetry
With periodic boundary conditions (PBC) $\rightarrow$ spontaneous breaking for $L < L_c$

1-point box with **twisted boundary conditions** $\rightarrow$ center symmetry preserved
$\Longrightarrow$ **Twisted Eguchi–Kawai** (TEK) formulation

González-Arroyo & Okawa PRD **27** (1983) 2397; JHEP **07** (2010) 043; JHEP **12** (2014) 106

TEK formulation of the gluon action

- $U_\mu(x + aL\hat{\nu}) = \Gamma_\nu U_\mu(x)\Gamma_\nu^\dagger$ — Twisted Boundary Conditions (TBC)
- $U_\mu(x) \longrightarrow U_\mu(x)\Gamma_\mu$ — Change of variable via twist eater Γ_μ
- $U_\mu(x) \longrightarrow U_\mu$ — 1 point $\Longrightarrow$ only $d = 4$ $\mathrm{SU}(N)$ matrices

$$\mathcal{S}_{\mathrm{TEK}}[U] = -Nb \sum_{\nu \neq \mu} z_{\nu\mu} \mathrm{Tr}\left\{U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right\}$$

TEK Wilson gauge action

$b = 1/\lambda_0 = \beta/(2N^2)$

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▶ **Twist factor** $z_{\nu\mu} = z_{\mu\nu}^* \in \mathbb{Z}_N$

$z_{\nu\mu} \neq 1 \rightarrow$ TBC

$z_{\nu\mu} = 1 \rightarrow$ PBC

$$\Gamma_\mu \Gamma_\nu = z_{\nu\mu} \Gamma_\nu \Gamma_\mu$$

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- $z_{\nu\mu} = \exp\left\{2\pi \mathrm{i} \dfrac{k(N)}{\sqrt{N}} \epsilon_{\nu\mu}\right\}$ $\quad \boxed{\sqrt{N} \equiv L \in \mathbb{N}}$

- $k \in \mathbb{N}$ $\quad k(N)/\sqrt{N} \sim$ fixed JPA **50** (2017) 26, 265401

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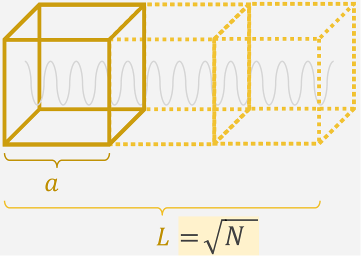
- $k \in \mathbb{N}$ $\quad k(N)/\sqrt{N} \sim$ fixed JPA **50** (2017) 26, 265401
- $(z_{\nu\mu})^L = 1 \Longrightarrow U_\mu$ effectively L-periodic

$\Longrightarrow$ **Effective size** $L = \sqrt{N}$



Cartoon from P. Butti Lattice 23 parallel talk

TEK formulation of the Dirac operator

gluons $\sim \mathcal{O}(N^2)$ # quarks $\sim \mathcal{O}(N_{\mathrm{f}}N)$

$\longrightarrow$ quarks suppressed at large N compared to gluons

Quenched Lattice TEK Dirac operator in **Wilson formulation**:

González-Arroyo & Okawa PLB **755** (2016) 132–137 [1510.05428]

$$D_{\mathrm{TEK}} = \frac{1}{2\kappa} - \frac{1}{2}\sum_{\mu}\left[(1+\gamma_\mu)\mathcal{W}_\mu + (1-\gamma_\mu)\mathcal{W}_\mu^\dagger\right]$$

$\mathcal{W}_\mu = U_\mu \otimes \Gamma_\mu^* \longleftarrow$ twist-eater

$\kappa \longleftarrow$ Wilson hopping parameter

TEK Dirac operator in momentum space:

$$\widetilde{D}_{\mathrm{TEK}}(p) = \frac{1}{2\kappa} - \frac{1}{2}\sum_{\mu}\left[(1+\gamma_\mu)\mathcal{W}_\mu \mathrm{e}^{\mathrm{i}ap_\mu} + (1-\gamma_\mu)\mathcal{W}_\mu^\dagger \mathrm{e}^{-\mathrm{i}ap_\mu}\right]$$

$$\boxed{ap_\mu \propto \frac{1}{\sqrt{N}} = \frac{1}{L}}$$

$\pi\pi$ scattering at large N

- $N \longrightarrow \infty$ = free pions $\longrightarrow$ $a_0 \underset{N\to\infty}{\sim} \boxed{\overline{a}_0}\,\dfrac{1}{N}$

Target $\longrightarrow$ $\boxed{\overline{a}_0 = N a_0 \sim \mathcal{O}(N^0)}$

- Finite-volume **energy shift**: $\Delta E = E_{\pi\pi} - 2m_\pi \sim \mathcal{O}(1/N)$

- **Lüscher formula**: ΔE (finite volume) $\leftrightarrow$ a_0 (infinite volume)

Lüscher Commun. Math. Phys. **105** (1986) 153; NPB **354** (1991) 531

$$\Delta E = \frac{4\pi a_0}{m_\pi \ell^3}\left[1 + \mathcal{O}\left(\frac{a_0}{\ell}\right)\right] \qquad a_0/\ell \ll 1$$

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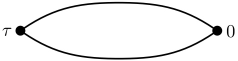
$$N\ell^3 \Delta E = \frac{4\pi \bar{a}_0}{m_\pi}\left[1 + \mathcal{O}\left(\frac{\bar{a}_0}{N\ell}\right)\right] \qquad \bar{a}_0/(N\ell) \ll 1$$

- Goal: compute $N\ell^3 \Delta E \equiv \Delta\varepsilon \sim \mathcal{O}(N^0), \mathcal{O}(\ell^0)$

Calculation of I = 2 $\Delta\varepsilon$ at large N — extended lattice

$$O_\pi(\tau) = \sum_{\vec{x}} \overline{\psi}(\vec{x},\tau)\gamma_5\psi(\vec{x},\tau)$$

$$G_\pi(\tau) = \quad \langle \mathrm{Tr}\,\{O_\pi(\tau)O_\pi(0)\}\rangle \quad = \langle D(\tau)\rangle$$



D

Calculation of I = 2 $\Delta\varepsilon$ at large N — extended lattice

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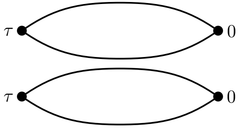
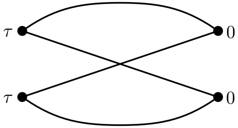
$$G_\pi(\tau) = \langle \mathrm{Tr}\{\mathcal{O}_\pi(\tau)\mathcal{O}_\pi(0)\}\rangle = \langle D(\tau)\rangle$$
$$G_{\pi\pi}(\tau) = \langle \mathrm{Tr}\{\mathcal{O}_{\pi\pi}(\tau)\mathcal{O}_{\pi\pi}(0)\}\rangle = 2\langle D^2(\tau)\rangle - \langle C(\tau)\rangle$$



D^2



C

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$$
\begin{aligned}
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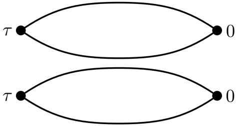
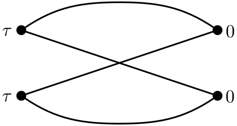
$$



$\boxed{D^2 \sim \mathcal{O}[(N\ell^3)^2]}$



$\boxed{C \sim \mathcal{O}(N\ell^3)}$

Large-N factorization $\rightarrow \langle D^2\rangle = \langle D\rangle^2 + \mathcal{O}(1/N^2)$

Calculation of I = 2 $\Delta\varepsilon$ at large N — extended lattice

$$\mathcal{O}_\pi(\tau) = \sum_{\vec{x}} \overline{\psi}(\vec{x},\tau)\gamma_5\psi(\vec{x},\tau) \qquad\qquad \mathcal{O}_{\pi\pi}(\tau) \equiv \mathcal{O}_\pi(\tau)\mathcal{O}_\pi(\tau)$$

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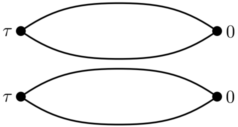

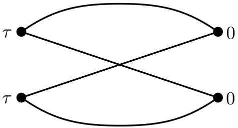

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Large-N factorization $\longrightarrow \langle D^2\rangle = \langle D\rangle^2 + \mathcal{O}(1/N^2)$

- $G_{\pi\pi}(\tau) = 2G_\pi^2(\tau) + \mathcal{O}(1/N) \underset{\tau\to\infty}{\sim} 2A_\pi^2 \exp\{-2m_\pi\tau\}$ (LO)

Calculation of I = 2 $\Delta\varepsilon$ at large N — extended lattice

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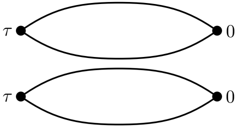

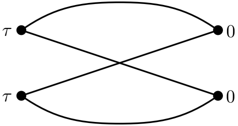

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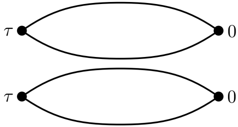

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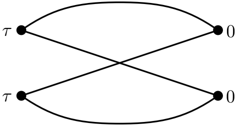

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- $G_{\pi\pi}(\tau) \underset{\tau\to\infty}{\sim} 2G_\pi^2(\tau)(1-\Delta E\tau) + \mathcal{O}(1/N^2)$ (NLO)

Calculation of I = 2 $\Delta\varepsilon$ at large N — 1-point lattice

$$G_{\pi\pi} = 2\langle D\rangle^2 - \langle C\rangle + \mathcal{O}(1/N^2) \qquad G_{\pi} = \langle D\rangle$$

- $$\frac{G_{\pi\pi}(\tau)}{2G_{\pi}^2(\tau)} \underset{\tau\to\infty}{\sim} 1 - \Delta E\tau + \mathcal{O}\left(\frac{1}{N^2}\right)$$
- $$\frac{G_{\pi\pi}(\tau)}{2G_{\pi}^2(\tau)} = 1 - \frac{\langle C(\tau)\rangle}{2\langle D(\tau)\rangle^2} + \mathcal{O}\left(\frac{1}{N^2}\right) = 1 - \frac{\langle \overline{C}(\tau)\rangle}{2\langle \overline{D}(\tau)\rangle^2 N\ell^3} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\boxed{D \underset{\substack{N\to\infty\\ \ell\to\infty}}{\sim} \overline{D}N\ell^3 \qquad C \underset{\substack{N\to\infty\\ \ell\to\infty}}{\sim} \overline{C}N\ell^3}$$

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$$\Longrightarrow \boxed{\frac{\langle \overline{C}(\tau)\rangle}{2\langle \overline{D}(\tau)\rangle^2} \underset{\tau\to\infty}{\sim} \left(N\ell^3\Delta E\right)\tau = \Delta\varepsilon\,\tau} \quad \longleftarrow$$

$\mathcal{O}(N^0), \mathcal{O}(\ell^0)$
can be employed in TEK

Calculation of I = 2 $\Delta\varepsilon$ at large N — 1-point lattice

$$G_{\pi\pi} = 2\langle D\rangle^2 - \langle C\rangle + \mathcal{O}(1/N^2) \qquad\qquad G_\pi = \langle D\rangle$$

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$\mathcal{O}(N^0), \mathcal{O}(\ell^0)$
can be employed in TEK

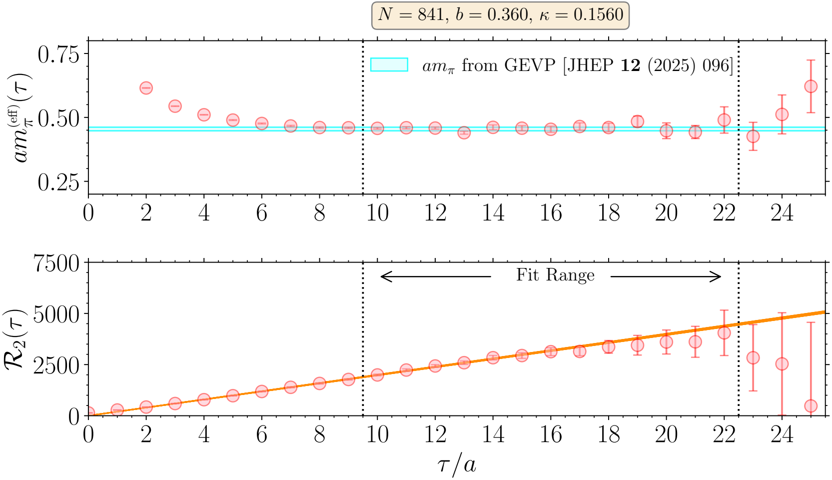
- $\langle \overline{D}(\tau)\rangle \to \sum_{p_0} e^{ip_0\tau} \langle \mathrm{Tr}\{\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(p_0)\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(0)\}\rangle$

$$\boxed{ap_0 \propto \frac{1}{\sqrt{N}} = \frac{1}{L}}$$

- $\langle \overline{C}(\tau)\rangle \to \sum_{p_0,q_0,k_0} e^{i(p_0-q_0+k_0)\tau} \langle \mathrm{Tr}\{\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(p_0)\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(q_0)\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(k_0)\gamma_5 \widetilde{D}^{-1}_{\mathrm{TEK}}(0)\}\rangle$

Calculation of I = 2 $\Delta\varepsilon$ at large N — 1-point lattice

$$\mathcal{R}_2(\tau) \equiv \frac{\langle \overline{C}(\tau) \rangle}{2\,\langle \overline{D}(\tau) \rangle^2} \underset{\tau\to\infty}{\sim} \left(N\ell^3 \Delta E\right)\tau = \Delta\varepsilon\,\tau$$

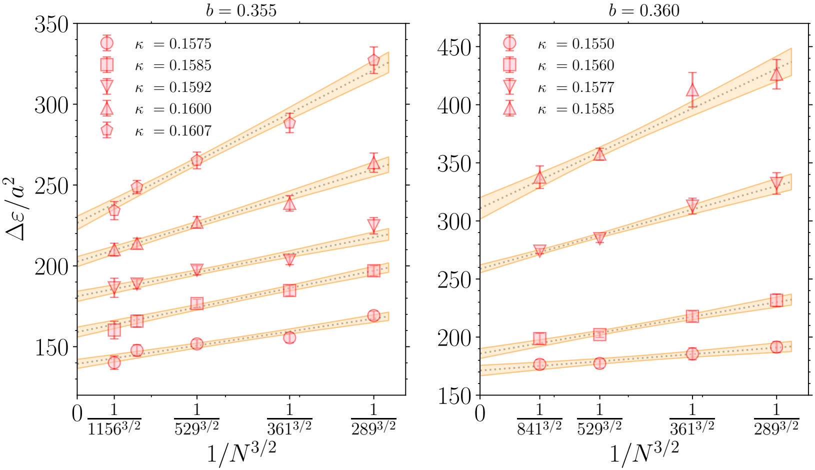


Large-N limit of the energy shift

Lüscher formula: $$\Delta\varepsilon = \frac{4\pi}{m_\pi}\overline{a}_0\left[1 + \mathcal{O}\left(\frac{1}{N\ell}\right)\right]$$

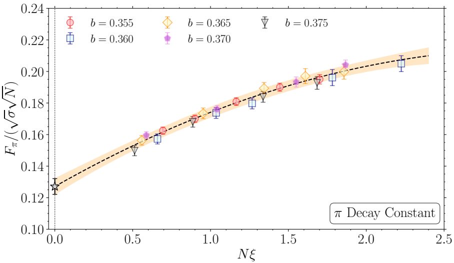
Large-N limit of the energy shift

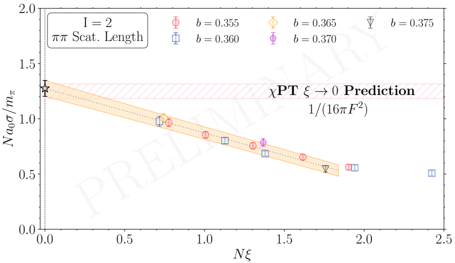
Lüscher formula: $$\Delta\varepsilon = \frac{4\pi}{m_\pi}\overline{a}_0\left[1+\mathcal{O}\left(\frac{1}{N^{3/2}}\right)\right] \qquad \ell = a\sqrt{N}$$



$$F_\pi = \frac{1}{\sqrt{2} m_\pi} \langle 0 | A_0 | \pi \rangle \ \text{versus}\ m_\pi^2$$

$$\frac{a_0}{m_\pi} \ \text{versus}\ m_\pi^2$$





No chiral log at large N

Y units $\longrightarrow$ string tension σ

$$F_\pi = F \left[1 + \bar{\ell}_4 \, \xi + O(\xi^2) \right] \qquad \frac{a_0}{m_\pi} = \frac{1}{16\pi F^2} \left[1 - (16\pi)^2 L_{\pi\pi} \, \xi + O(\xi^2) \right]$$

$$\xi \equiv \frac{m_\pi^2}{16\pi^2 F^2} \sim O\left(\frac{1}{N}\right) \qquad F \sim O(\sqrt{N}) \qquad \bar{\ell}_4, L_{\pi\pi} \sim O(N) \qquad L_{\pi\pi} = \frac{\bar{\ell}_1 + 2\bar{\ell}_2}{192\pi^2}$$

$F_\pi \longrightarrow$ **CB** et al. JHEP **12** (2025) 096 $\qquad a_0 \longrightarrow$ Work In Progress

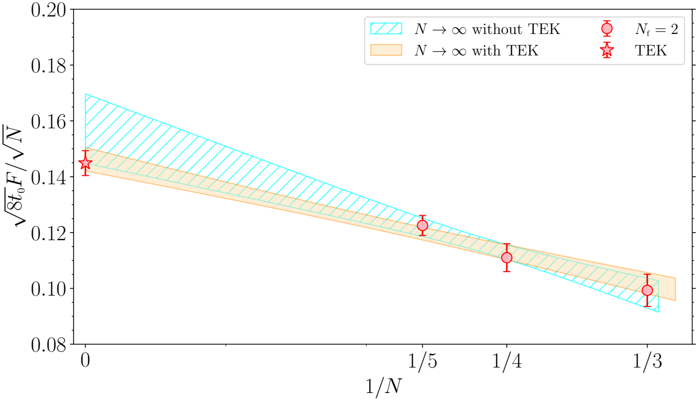
The $1/N$ expansion from the lattice

- N-dependence in TEK $\longrightarrow$ effective-volume dependence (e.g., $\Delta\varepsilon \rightarrow \overline{a}_0$)
- $\Longrightarrow$ TEK only agrees with standard theory for $N = \infty$

- Leading order in $1/N$ ($N = \infty$) $\quad\longrightarrow\quad$ TEK
- Sub-leading terms in $1/N$ $\quad\longrightarrow\quad$ TEK + standard finite-N results
- $\Longrightarrow$ Large-N **INTERPOLATION** as opposed to large-N extrapolation

$$\langle \mathcal{A} \rangle_N = \underbrace{A_\infty}_{\text{TEK}} \underbrace{\left[1 + A_1 \frac{1}{N} + A_2 \frac{1}{N^2} + O\left(\frac{1}{N^3}\right)\right]}_{\text{TEK + standard method}}$$

$1/N$ expansion of F



$$\boxed{\frac{a_0}{m_\pi} \underset{m_\pi \to 0}{\longrightarrow} \frac{1}{16\pi F^2}}$$

$N = 3$, $N_f = 2$ result:

$F = 86.6(6)$ MeV

[FLAG 21]

TEK + $N_f = 2$ finite-N data (standard method) DeGrand et al. PRD **108** (2023) 094516

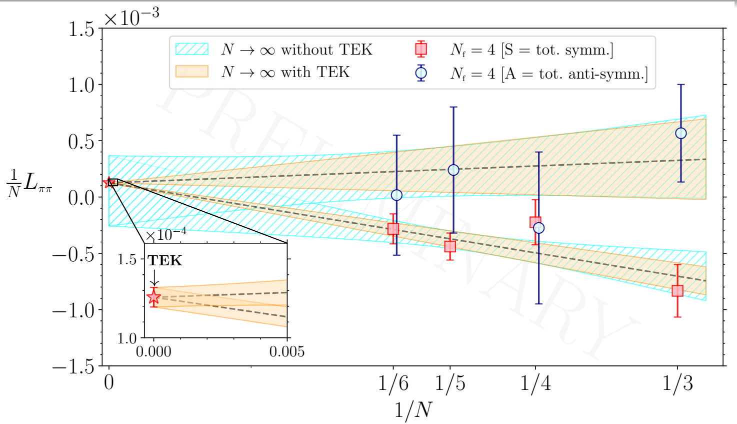
$$\frac{\sqrt{3}}{\sqrt{N}}F = [122(4)\ \text{MeV}]\left[1 - 0.46(7)\frac{N_f}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right]$$

Conversion of $N = \infty$ result:

$\sqrt{8t_0} \simeq 0.409$ fm [FLAG 24]

TEK: $\mathcal{O}(N^0) \to \times 3$ error reduction $\quad \mathcal{O}(1/N) \to \times 3$ error reduction

$1/N$ expansion of $L_{\pi\pi}$



$L_{\pi\pi}$: sub-leading term in chiral expansion of a_0

TEK + $N_f = 4$ finite-N data (standard method) Baeza-Ballesteros et al. JHEP **06** (2022) 049

$$\frac{L_{\pi\pi}}{N} \times 10^4 = 1.126(62) \left\{ 1 + [-1.9(1.1) \pm 12.3(4.3) N_f] \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \right\}$$

A → +
S → −

TEK: $\mathcal{O}(N^0) \to \times 50$ error reduction $\mathcal{O}(1/N) \to \times 35$ error reduction

Conclusions

- **TEK** + **traditional methods** $\longrightarrow$ precision large-N QCD
  $N = \infty \longrightarrow$ TEK
  $1/N$ expansion $\longrightarrow$ TEK + finite-N data

- Present results $\longrightarrow$ first steps of a new research program

- **Impact**: TEK large-N results already employed as "experimental data" in recent χPT and bootstrap studies of large-N QCD Cheung et al. [2605.21582] — Albert et al. [2606.19420]

Future outlooks

- Systematic study of scattering amplitudes and form factors in large-N QCD via TEK
- Future targets: $\mathrm{I} = 0$ $\pi\pi$ scattering, $\rho\rho$ and $\rho\pi$ scattering, ...

Extra Slides

$[Na_0/m_\pi](\xi = 0)$: lattice artifacts

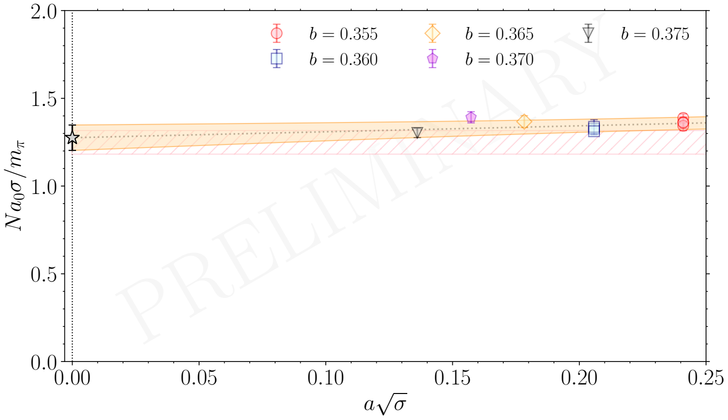
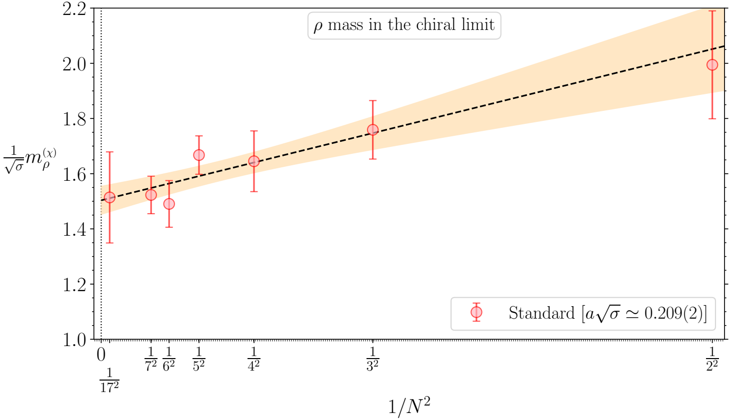


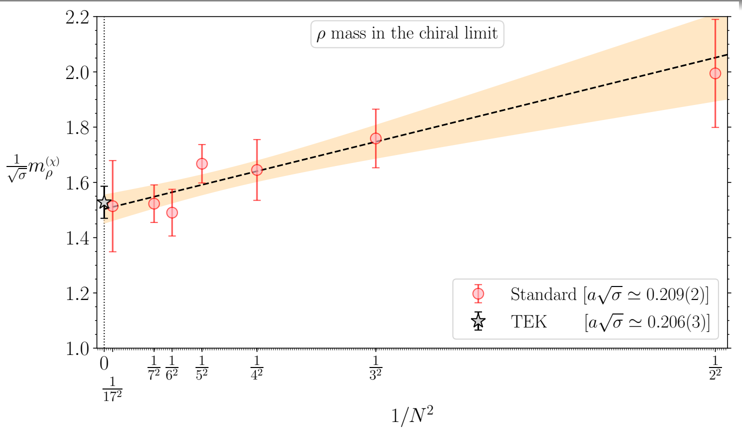
Figure refers to the chiral limit.

Comparison with: Bali et al. JHEP **06** (2013) 071



- Previous state of the art (standard method): 1 lattice spacing, **no continuum limit**

Comparison with: Bali et al. JHEP **06** (2013) 071

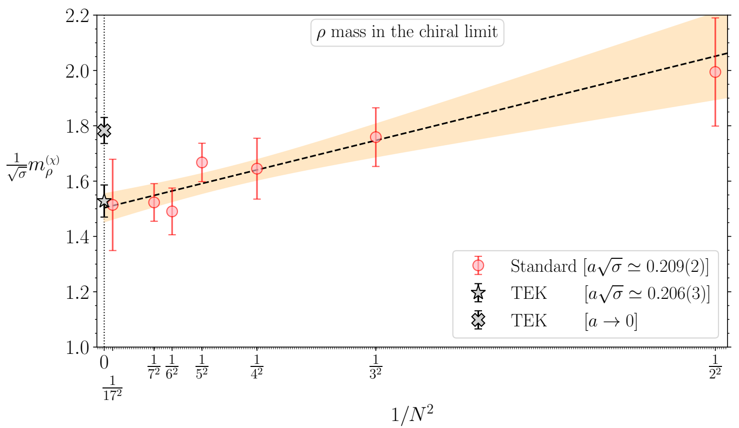


- Previous state of the art (standard method): 1 lattice spacing, **no continuum limit**

- TEK result **at same lattice spacing** reproduces standard method at cheaper cost

$N^2 = 841^2 = 29^4$ TEK $\sim$ 1.2 $\times$ the cost of $N^2 L^4 = 3^2 \times 16^4 \implies$ TEK $\times$ 27 cheaper than $N = 17$

Comparison with: Bali et al. JHEP **06** (2013) 071



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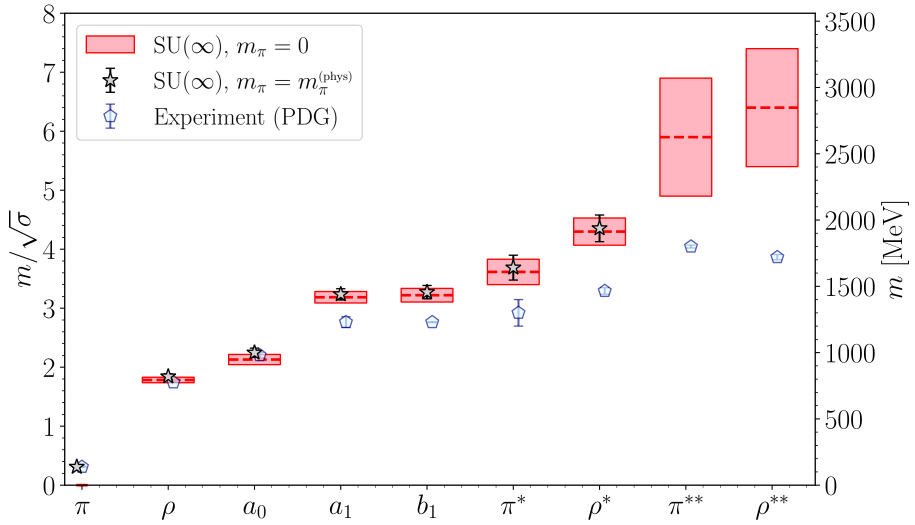
- 5 lattice spacings (3 finer) $\longrightarrow$ **significant improvement** of state of the art

Low-lying meson spectrum at large N

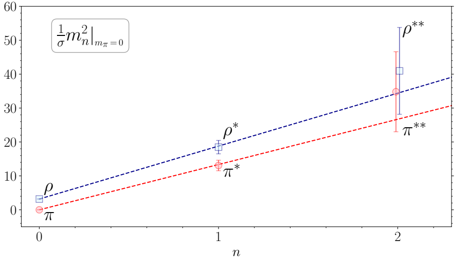
Phys. point convention: $m_\pi^{(\text{phys})} = 135$ MeV PDG $\sqrt{\sigma}\big|_{\text{phys}} = 445$ MeV PLB **854** (2024) 138754

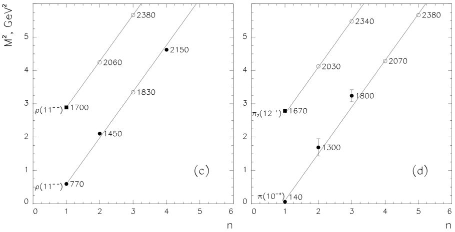


Comparison with PDG gives idea of magnitude of higher-order corrections in $1/N$

Large-N radial Regge trajectories for π and ρ

Radial Regge trajectory $\longrightarrow$ $$\left.\frac{m_n^2}{\sigma}\right|_{m_\pi=0} = \left.\frac{m_0^2}{\sigma}\right|_{m_\pi=0} + \frac{\mu_r^2}{\sigma} n$$ ($m_n = n^{\text{th}}$ excited state mass)





Systematics of $q\bar{q}$ states in the (n, M^2) and (J, M^2) planes

A. V. Anisovich, V. V. Anisovich, and A. V. Sarantsev Phys. Rev. D **62**, 051502(R) – **Published 25 July, 2000**

$$\frac{\mu_r}{\sqrt{\sigma}} \simeq 2.65$$

Exp. data
PRD 62 (2000) 051502

$$\frac{\mu_r}{\sqrt{\sigma}} \simeq \sqrt{4\pi} \simeq 3.55$$

Large-N eff. model
PLB 323 (1994) 41

$$\frac{\mu_r}{\sqrt{\sigma}} = 3.65(21)$$

TEK
π channel

$$\frac{\mu_r}{\sqrt{\sigma}} = 3.95(24)$$

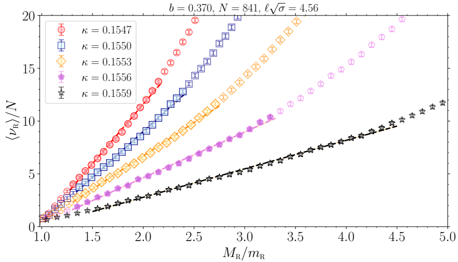
TEK
ρ channel

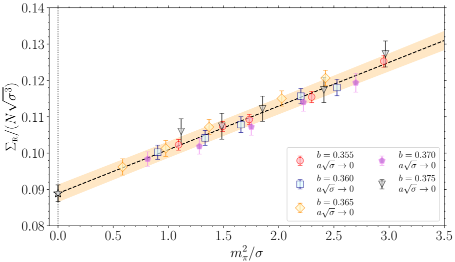
$$\mu_r(N=\infty) \simeq 1.94(11) \text{ GeV} \qquad \mu_r(\text{exp}) \simeq 1.18 \text{ GeV}$$

$\Sigma = -\langle\overline{\psi}\psi\rangle \sim \mathcal{O}(N)$ → order parameter for chiral symmetry breaking

$$\Sigma_{\mathrm{R}}^{(\mathrm{eff})} = \frac{\pi}{2}\frac{\langle\nu_{\mathrm{R}}(M_{\mathrm{R}},m_{\mathrm{R}})\rangle}{V\sqrt{M_{\mathrm{R}}-m_{\mathrm{R}}}} = \Sigma_{\mathrm{R}}\left[1+\mathcal{O}(m_{\mathrm{R}})\right]$$

Giusti–Lüscher method
JHEP **03** (2009) 013





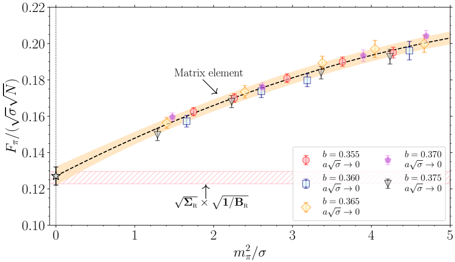
Chiral condensate Σ_{R}/N extracted from slope of Dirac mode number $\langle\nu\rangle_{\mathrm{R}}$
→ Non-perturbative renormalization: Z_{P} ($\overline{\mathrm{MS}}$, $\mu = 2$ GeV)

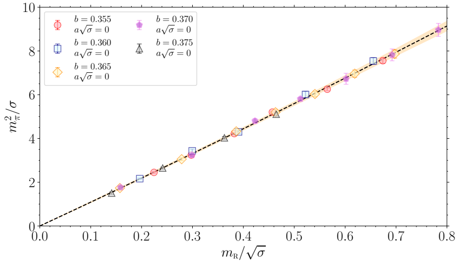
$$\left(\frac{1}{N}\Sigma_{\mathrm{R}}\right)_{N=\infty} = [247(4)\ \mathrm{MeV}]^3 \qquad \left(\frac{1}{N}\Sigma_{\mathrm{R}}\right)^{(\mathrm{FLAG})}_{N=3,N_{\mathrm{f}}=2} = [184(7)\ \mathrm{MeV}]^3$$

MeV conversion of TEK results: $\sqrt{8t_0} \simeq 0.409$ fm [FLAG 24]

$$F_\pi = \frac{1}{\sqrt{2}m_\pi} \langle 0 | A_0 | \pi(\vec{p} = \vec{0}) \rangle \sim \mathcal{O}(\sqrt{N})$$

m_π^2 versus quark mass m_{R}





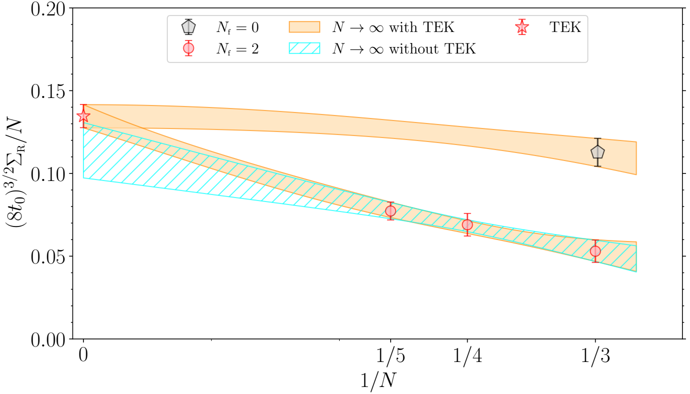
$$\left(\sqrt{\frac{3}{N}}F\right)_{N=\infty} = 122(4)\text{ MeV} \qquad \left(\sqrt{\frac{3}{N}}F\right)^{\text{(FLAG)}}_{N=3,N_f=2} = 86.6(6)\text{ MeV}$$

$$\left(\frac{\bar{\ell}_3}{N}\right)_{N=\infty} = -0.034(27) \qquad \left(\frac{\bar{\ell}_3}{N}\right)^{\text{(FLAG)}}_{N=3,N_f=2} = 1.14(27)$$

$$\left(\frac{\bar{\ell}_4}{N}\right)_{N=\infty} = 0.448(54) \qquad \left(\frac{\bar{\ell}_4}{N}\right)^{\text{(FLAG)}}_{N=3,N_f=2} = 1.467(93)$$

$1/N$ expansion of Σ_R/N

$N = 3$ quenched $N_f = 0$ result $\longrightarrow$ only affected by $\mathcal{O}(1/N^2)$ corrections
$\longrightarrow$ much closer to $N = \infty$ than unquenched $N_f = 2$ ones

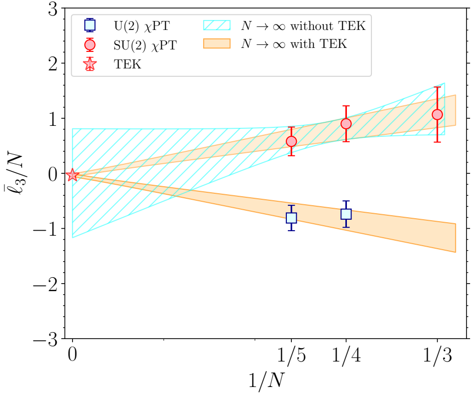


$$\left(\sqrt{8t_0}\right)^3 \frac{\Sigma_R}{N} = 0.1346(70)\left[1 - 1.27(33)\frac{N_f}{N} + \left[-1.38(70) + 0.91(56)N_f^2\right]\frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)\right]$$

$1/N$ expansion of $\bar{\ell}_3/N$

$\mathrm{SU}(N_\mathrm{f})$ vs $\mathrm{U}(N_\mathrm{f})$ χPT: NLO LECs differ for $\mathcal{O}(1/N)$ terms $\implies$ same $N \to \infty$ limit

$\mathrm{U}(N_\mathrm{f})$ has also light η' $\implies$ should converge faster towards the large-N limit



$$M_\pi^2 = M^2 \left[1 - \frac{M^2}{32\pi^2 F^2}\bar{\ell}_3 + \mathcal{O}(M^4)\right]$$

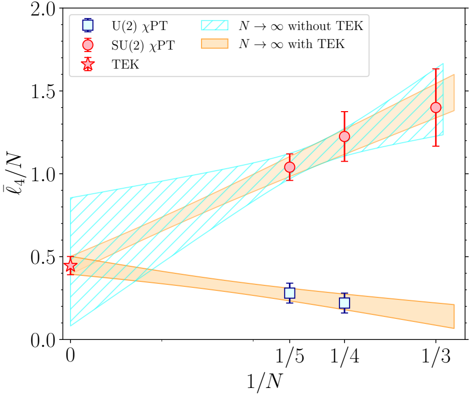
$$M^2 \equiv N_\mathrm{f} \frac{\Sigma_\mathrm{R}}{F^2} m_\mathrm{R} \qquad \bar{\ell}_3 \sim \mathcal{O}(N)$$

$$F \equiv \lim_{m_\mathrm{R}\to 0} F_\pi \sim \mathcal{O}\left(\sqrt{N}\right)$$

$$\frac{\bar{\ell}_3}{N} = \left[-0.034(27) + 1.69(40)\frac{N_\mathrm{f}}{N}\right]_{\mathrm{SU}(2)} \qquad \frac{\bar{\ell}_3}{N} = \left[-0.034(27) - 1.63(37)\frac{N_\mathrm{f}}{N}\right]_{\mathrm{U}(2)}$$

$\mathrm{SU}(N_f)$ vs $\mathrm{U}(N_f)$ χPT: NLO LECs differ for $\mathcal{O}(1/N)$ terms $\implies$ same $N \to \infty$ limit

$\mathrm{U}(N_f)$ has also light η' $\implies$ should converge faster towards the large-N limit

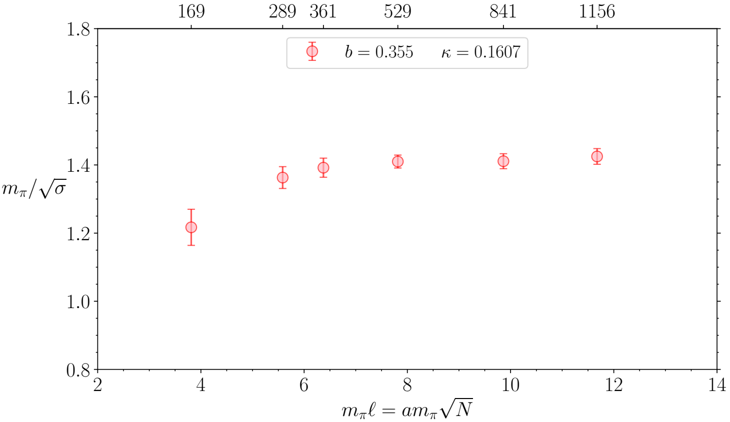


$$F_\pi = F\left[1 + \frac{M^2}{16\pi^2 F^2}\bar{\ell}_4 + \mathcal{O}(M^4)\right]$$

$$M^2 \equiv N_f \frac{\Sigma_R}{F^2} m_R \qquad \bar{\ell}_4 \sim \mathcal{O}(N)$$

$$F \equiv \lim_{m_R \to 0} F_\pi \sim \mathcal{O}\left(\sqrt{N}\right)$$

$$\frac{\bar{\ell}_4}{N} = \left[0.448(54) + 1.49(19)\frac{N_f}{N}\right]_{\mathrm{SU}(2)} \qquad \frac{\bar{\ell}_4}{N} = \left[0.448(54) - 0.44(15)\frac{N_f}{N}\right]_{\mathrm{U}(2)}$$



$N = 169 \implies \ell = a\sqrt{N} \approx 1.5$ fm $\qquad N = 1156 \implies \ell = a\sqrt{N} \approx 3.8$ fm