Renormalization and Improvement Conditions from Flowed Fermion Fields

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$$H_{\mathrm{ew}} = Z\Big[O_1 + \sum_{j=2} r_j O_j\Big] \qquad O_1 = \overline{d}_L \gamma_\mu u_L \overline{\nu}_L \gamma_\mu \tau_L,\ O_2 = \ldots,\ \ldots$$

- Determination of relative renormalization constants r_j in a mixed action setup[1]
- Presently we determine m_{cr} and $\dfrac{Z_V}{Z_A}$
- Possible methods use for example PCAC relation[2]
- We will use global Ward identities from spurionic non-anomalous $\mathrm{U}(1)_A$ symmetry and probes at positive flow time

[1] Di Carlo et al., "First-principle evaluation of inclusive hadronic τ decays in QCD+QED".

[2] Lüscher et al., "Non-perturbative determination of the axial current normalization constant in 0(a) improved lattice QCD".

$$S = S_G + \sum_x a^4 \overline{\chi}(x) \left(D_{0,\mathrm{SW}} + m_0 + i\gamma_5\mu_0\right) \chi(x) + \sum_x a^4 \overline{c}(x) \left(D_{0,\mathrm{SW}} + m_0 + i\gamma_5\mu_0\right) c(x)$$

- Spurionic non-anomalous $\mathrm{U}(1)_A$ transformation

$$\overline{\chi} \longrightarrow \overline{\chi} e^{-i\gamma_5 \frac{\omega}{2}} \qquad \overline{c} \longrightarrow \overline{c} e^{-i\gamma_5 \frac{\omega}{2}} \qquad \alpha \longrightarrow \alpha + \omega$$

$$\chi \longrightarrow e^{-i\gamma_5 \frac{\omega}{2}} \chi \qquad c \longrightarrow e^{-i\gamma_5 \frac{\omega}{2}} c \qquad \cot(\alpha) = \frac{m_r}{\mu_r}$$

[3] Lüscher, "Chiral symmetry and the Yang–Mills gradient flow".

[4] Shindler, "Chiral Ward identities, automatic improvement and the gradient flow".

Condition for m_{cr}

$$S = S_G + \sum_x a^4 \overline{\chi}(x) \left(D_{0,\mathrm{SW}} + m_0 + i\gamma_5\mu_0\right) \chi(x) + \sum_x a^4 \overline{c}(x) \left(D_{0,\mathrm{SW}} + m_0 + i\gamma_5\mu_0\right) c(x)$$

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$$\chi \longrightarrow e^{-i\gamma_5 \frac{\omega}{2}} \chi \qquad c \longrightarrow e^{-i\gamma_5 \frac{\omega}{2}} c \qquad \cot(\alpha) = \frac{m_r}{\mu_r}$$

- The fields at positive flow time $\overline{\chi}_t$ and χ_t transform in the same way since the flow commutes with $\mathrm{U}(1)_A$[3,4]

$$\overline{\psi}_t := \sqrt{Z_\chi}\, \overline{\chi} K_t^\dagger e^{i\gamma_5 \frac{\alpha}{2}} \qquad \psi_t := \sqrt{Z_\chi}\, e^{i\gamma_5 \frac{\alpha}{2}} K_t \chi$$

[3] Lüscher, "Chiral symmetry and the Yang–Mills gradient flow".

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Condition for m_{cr}

- Pseudo-scalar density

$$\mathcal{O}(a) = \langle \psi_t i\gamma_5 \psi_t \rangle_{(m_0,\mu_0)} = -Z_\chi \sin(\alpha)\langle \chi_t \chi_t \rangle_{(m_0,\mu_0)} + Z_\chi \cos(\alpha)\langle \chi_t i\gamma_5 \chi_t \rangle_{(m_0,\mu_0)}$$

- In the continuum due to parity and $\mathrm{U}(1)_A$: $\langle \psi_t i\gamma_5 \psi_t \rangle_{(M_r,\alpha)} = \langle \psi_t i\gamma_5 \psi_t \rangle_{(M_r,0)} = 0$

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$$\cot(\alpha) = \frac{\langle \chi_t \chi_t \rangle_{(m_0,\mu_0)}}{\langle \chi_t i\gamma_5 \chi_t \rangle_{(m_0,\mu_0)}} + \mathcal{O}(a)$$

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$$\mathcal{O}(a) + \frac{Z_m}{Z_\mu} \frac{m_0 - m_{\mathrm{cr}}}{\mu_0} = \frac{m_r}{\mu_r} = \cot(\alpha) = \frac{\langle \chi_t \chi_t \rangle_{(m_0,\mu_0)}}{\langle \chi_t i\gamma_5 \chi_t \rangle_{(m_0,\mu_0)}} + \mathcal{O}(a)$$

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- $\mathcal{O}(a)$-improvement such that $\langle \psi_t i\gamma_5 \psi_t \rangle_{(m_0,\mu_0)} = \mathcal{O}(a^2)$

Condition for m_{cr}

$$\overline{\psi}_t \coloneqq Z_\chi^{\frac{1}{2}}(\overline{\chi} + \sqrt{c_{\mathrm{fl}}}\overline{\eta})K_t^\dagger e^{i\frac{\gamma_5}{2}(\alpha + \check{b}_\chi a\mu_0)} \qquad \psi_t \coloneqq Z_\chi^{\frac{1}{2}} e^{i\frac{\gamma_5}{2}(\alpha + \check{b}_\chi a\mu_0)} K_t(\chi + \sqrt{c_{\mathrm{fl}}}\eta)$$

- With this definition[5][6] one has $\langle \psi_t i\gamma_5 \psi_t \rangle_{(m_0,\mu_0)} = \mathcal{O}(a^2)$

$$\begin{bmatrix} \cot(\tilde{\alpha})\langle \overline{\chi}_{t_1} i\gamma_5 \chi_{t_1}\rangle_{(m_0,\mu_0)} + c_{\mathrm{fl}}\langle \overline{\eta}_{t_1}\eta_{t_1}\rangle_{(m_0,\mu_0)} = \langle \overline{\chi}_{t_1}\chi_{t_1}\rangle_{(m_0,\mu_0)} \\ \cot(\tilde{\alpha})\langle \overline{\chi}_{t_2} i\gamma_5 \chi_{t_2}\rangle_{(m_0,\mu_0)} + c_{\mathrm{fl}}\langle \overline{\eta}_{t_2}\eta_{t_2}\rangle_{(m_0,\mu_0)} = \langle \overline{\chi}_{t_2}\chi_{t_2}\rangle_{(m_0,\mu_0)} \end{bmatrix} \quad \tilde{\alpha} = \alpha + a\check{b}_\chi \mu_0$$

[5] Lüscher, "Chiral symmetry and the Yang–Mills gradient flow".

[6] Shindler, "Chiral Ward identities, automatic improvement and the gradient flow".

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- Use $t_1 = t$ and $t_2 = t + \delta t$ and determine $\cot(\tilde{\alpha})$, c_{fl} as a function of t

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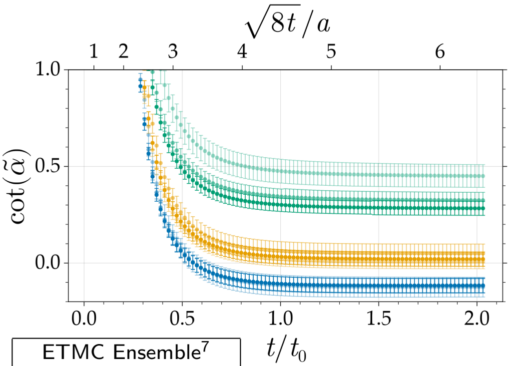
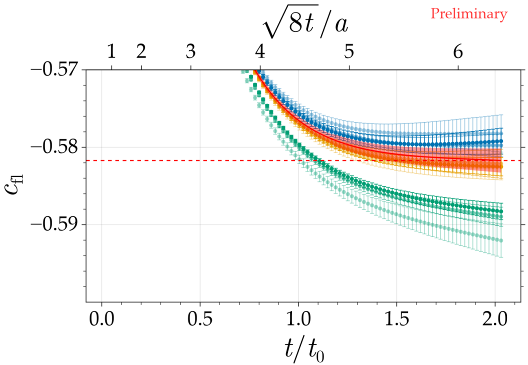
[6] Shindler, "Chiral Ward identities, automatic improvement and the gradient flow".

$\cot(\tilde{\alpha})$ and c_{fl}





ETMC Ensemble[7]	
$N_T \times N_L^3$	96×48^3
a	0.09 fm
M_π	174 MeV
LM_π	3.8
N_{cnfg} used	47

$\hat{m} = -0.4302$
- $\hat{\mu} = 0.000777$
- $\hat{\mu} = 0.002197$
- $\hat{\mu} = 0.0026$

$\hat{m} = -0.429$
- $\hat{\mu} = 0.0026$
- $\hat{\mu} = 0.0007385$
- $\hat{\mu} = 0.002089$

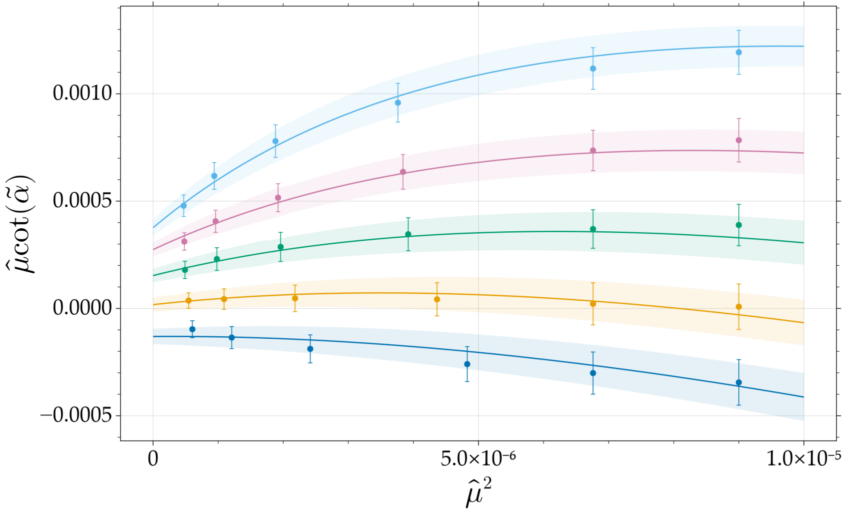
$\hat{m} = -0.4266$
- $\hat{\mu} = 0.0006928$
- $\hat{\mu} = 0.00196$
- $\hat{\mu} = 0.0026$

$\hat{m} = \hat{m}_{\text{cr}}$
- $\hat{\mu} = 0$

[7] Alexandrou et al., *Strange and charm quark contributions to the muon anomalous magnetic moment in lattice QCD with twisted-mass fermions.*

$$\tilde{\alpha} = \operatorname{arccot}\left(\frac{Z_m}{Z_\mu}\frac{m_q + N_1 a m_q^2 + N_2 a \mu_0^2}{\mu_0 + D_1 a \delta m_q \mu_0}\right) + \check{b}_\chi a \mu_0 \qquad m_q = m_0 - m_{\rm cr}$$

6 parameters
30 data points



$$m_{\rm cr} = -0.42918(28)$$

$$\frac{Z_m}{Z_\mu} = 0.120(4)$$

Condition for $\frac{Z_V}{Z_A}$

- For each flavor there is a spurionic non-anomalous $\mathrm{U}(1)_A$ transformation
- Consider the currents

$$V_\mu^{ud}(x) = \overline{\chi}^u(x)\gamma_\mu\chi^d(x) \qquad \text{and} \qquad A_\mu^{ud}(x) = \overline{\chi}^u(x)\gamma_\mu i\gamma_5\chi^d(x)$$

[8] Frezzotti et al., "Reducing cutoff effects in maximally twisted LQCD close to the chiral limit".

[9] Shindler, "Chiral Ward identities, automatic improvement and the gradient flow".

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- Using $\mathrm{U}(1)_A$ derive at maximal twist the Identity

$$Z_V Z_\chi\left[\langle V_\mu^{ud}(x)V_{t,\mu}^{du}(y)\rangle_{(\hat{\mu}_u,\hat{\mu}_d)} + \langle V_\mu^{ud}(x)V_{t,\mu}^{du}(y)\rangle_{(-\hat{\mu}_u,\hat{\mu}_d)}\right] =$$

$$- Z_A Z_\chi\left[\langle A_\mu^{ud}(x)A_{t,\mu}^{du}(y)\rangle_{(\hat{\mu}_u,\hat{\mu}_d)} + \langle A_\mu^{ud}(x)A_{t,\mu}^{du}(y)\rangle_{(-\hat{\mu}_u,\hat{\mu}_d)}\right] + \mathcal{O}(a^2)$$

- Uses automatic $\mathcal{O}(a)$-improvement at maximal twist[8][9]

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[9] Shindler, "Chiral Ward identities, automatic improvement and the gradient flow".

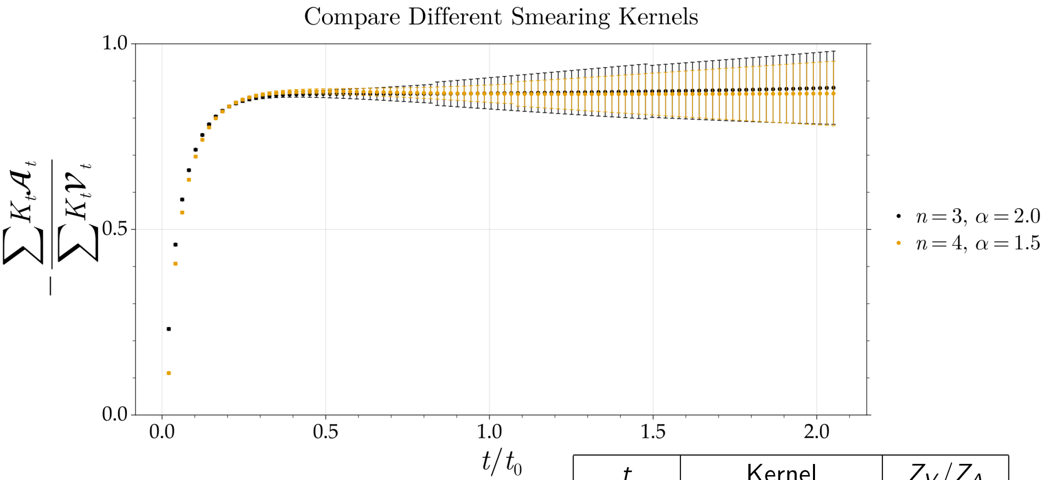
Condition for $\frac{Z_V}{Z_A}$

$$\mathcal{V}_t(x,y) := \langle V_\mu^{ud}(x) V_{t,\mu}^{du}(y)\rangle_{(\hat{\mu}_u,\hat{\mu}_d)} + \langle V_\mu^{ud}(x) V_{t,\mu}^{du}(y)\rangle_{(-\hat{\mu}_u,\hat{\mu}_d)}$$
$$\mathcal{A}_t(x,y) := \langle A_\mu^{ud}(x) A_{t,\mu}^{du}(y)\rangle_{(\hat{\mu}_u,\hat{\mu}_d)} + \langle A_\mu^{ud}(x) A_{t,\mu}^{du}(y)\rangle_{(-\hat{\mu}_u,\hat{\mu}_d)}$$

- Smearing kernels $K_t(x,y) = e^{\frac{|x-y|^2}{4\alpha t}} |x-y|^{2n}$

$$-\frac{\sum_{x,y} K_t(x,y)\mathcal{A}_t(x,y)}{\sum_{x,y} K_t(x,y)\mathcal{V}_t(x,y)} = \frac{Z_V}{Z_A} + \mathcal{O}(a^2)$$

Condition for $\frac{Z_V}{Z_A}$



t	Kernel	Z_V/Z_A
≈ 0.51	$n=3,\ \alpha=2.0$	$0.87(1)$
≈ 0.51	$n=4,\ \alpha=1.5$	$0.872(7)$

Conclusion

- Introduced conditions based on global $\mathrm{U}(1)_A$ WI
- Preliminarily m_{cr} and $\frac{Z_V}{Z_A}$ can be determined
- Currently working on:
  - Quote a reliable error for m_{cr} and $\frac{Z_V}{Z_A}$
  - Determine $\frac{Z_P}{Z_S}$
  - Extend to H_{ew}

- Fermion flow equation[10]

$$\partial_t \chi_t = \Delta \chi_t \qquad \text{with} \qquad \chi_t\big|_{t=0} = \chi$$
$$\partial_t \overline{\chi}_t = \overline{\chi}_t \overleftarrow{\Delta} \qquad \text{with} \qquad \overline{\chi}_t\big|_{t=0} = \overline{\chi}$$
$$\Delta = D_\mu D_\mu$$

- Field η_t:

$$\langle \overline{\eta}_t \eta_t \rangle \propto \mathrm{tr}\left(K_t^\dagger K_t\right)$$

[10] Lüscher, "Chiral symmetry and the Yang–Mills gradient flow".

Appendix: Different Smearing Kernels