Structure functions for lepton-nucleon scattering from four-point functions on the lattice

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for the χQCD collaboration

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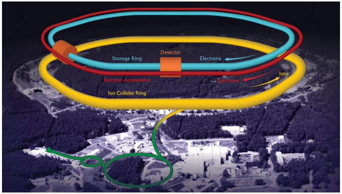
Introduction: Lepton-Nucleon scattering

- One of the most fundamental processes in high-energy physics
- Essential for understanding the structure of nucleons $\Rightarrow$ purpose of the Electron-Ion-Collider (EIC)
- Proxy for discovering physics beyond the standard model $\Rightarrow$ Neutrino experiments (DUNE, Hyper-K)



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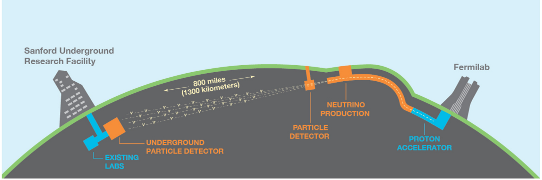

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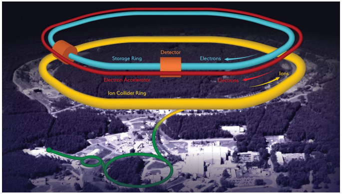

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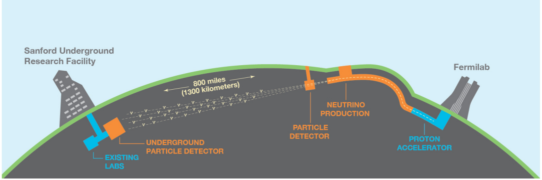

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- The cross section for lepton-nucleon scattering (e.g. deeply inelastic scattering, DIS) can be factorized in a leptonic part (leptonic tensor $L_{\mu\nu}$) and a hadronic part (hadronic tensor $W_{\mu\nu}$): $\sigma^2 \propto L^{\mu\nu} W_{\mu\nu}$
- In general, $W_{\mu\nu}$ is a non-perturbative object. It is parameterized in terms of Lorentz invariant structure functions $F_i(x, Q^2)$
- **This work**: Calculate the hadronic tensor directly from two-current matrix elements on the lattice. **Goal**: Determine x and Q^2 dependence of all structure functions for all electroweak currents

The hadronic tensor and structure functions

Definition of the hadronic tensor

$$W_{\mu\nu}(p,q,s) := \frac{1}{4\pi}\int \mathrm{d}^4 z\; e^{iqz}\,\langle p,s|\left[J_\mu^\dagger(z),J_\nu(0)\right]|p,s\rangle$$

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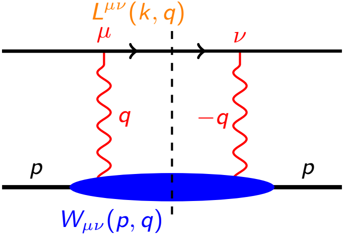

Cross section

$$\frac{\sigma^2}{\mathrm{d}x\mathrm{d}y} = \frac{2\pi y \alpha^2}{Q^4}\sum_j \eta_j L_j^{\mu\nu} W^j_{\mu\nu}$$

Electroweak currents

$$J_\mu^{\mathrm{em}} \qquad J_\mu^{W^+} = J_\mu^{W^-,\dagger} \qquad J_\mu^Z$$

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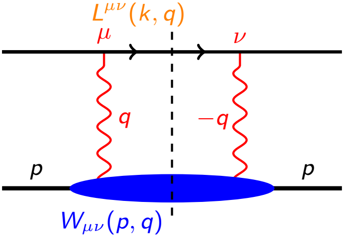

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$W_{\mu\nu}$ can be decomposed in terms of **structure functions** by Lorentz symmetry:

Structure functions

- F_1, F_2
- F_3 (only $W^\pm$, Z)
- Spin: g_1, g_2, g_3 ,g_4, g_5

Kinematic variables:

- Momentum transfer: $Q^2 = \vec{q}^2 - \nu^2$, $\nu = q^0$
- Bjorken-x: $x = \frac{Q^2}{2p\cdot q}$

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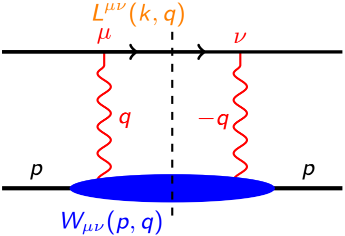

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The hadronic tensor on the lattice

Problem: **Euclidean time**:

$$W_{\mu\nu}(p,q,s) := \frac{1}{4\pi}\int \mathrm{d}z^0 e^{i\nu z^0} \int \mathrm{d}^3 z\ e^{-i\vec{q}\vec{z}} \langle p,s| \left[J^\dagger_\mu(z), J_\nu(0)\right] |p,s\rangle$$

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There is no lattice calculation that can address this "directly". Instead, identify the time integral as **inverse Laplace transform w.r.t. Euclidean time difference τ:**

$$W_{\mu\nu}(p,q,s) := \frac{2E_{\vec{p}}}{2\pi i} \int_{-i\infty}^{i\infty} \mathrm{d}\tau e^{\nu\tau} W^{\mathrm{E}}_{\mu\nu}(\vec{p},\vec{q},s,\tau)$$

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Hence, consider: *[K.F. Liu and S.J. Dong, Phys.Rev.Lett. 72 (1994) 1790-1793; K.F. Liu, Phys.Rev.D 62 (2000) 074501]*

Hadronic tensor in Euclidean spacetime

$$W^{\mathrm{E}}_{\mu\nu}(\vec{p},\vec{q},s,\tau) = \frac{1}{4E_{\vec{p}}}\int \mathrm{d}^3 z\ e^{-i\vec{q}\vec{z}} \langle p,s| J^\dagger_\mu(\vec{z},\tau) J_\nu(\vec{0},0) |p,s\rangle \to \mathrm{Lattice}\left(\frac{4\mathrm{pt}}{2\mathrm{pt}}\right)$$
$$= \int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}}\ e^{-\nu\tau} W_{\mu\nu}(p,q,s)$$

Minimal energy transfer $\nu_{\min}$ defined by on-shell threshold $(p+q)^2 \geq m^2$

Lattice calculation strategy

(1) Four-point functions (Two-current correlator)

$$W^{\mathrm{E}}_{\mu\nu}(\vec{p},\vec{q},\tau) = \frac{e^{-E_{\vec{p}}\tau}}{2} \frac{\langle \Gamma \chi_N(\vec{p},\frac{t}{2})\ J_\mu(\vec{q},\bar{\tau}+\frac{\tau}{2})\ J_\nu(-\vec{q},\bar{\tau}-\frac{\tau}{2})\ \overline{\chi}_N(\vec{p},-\frac{t}{2})\rangle}{\langle \chi_N(\vec{p},t)\ \overline{\chi}_N(\vec{p},0)\rangle}\Bigg|_{t\gg\bar{\tau}\gg 0}$$

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$$W^{\mathrm{E}}_{\mu\nu}(\vec{p},\vec{q},\tau) \to A(\vec{q}^{\,2},\vec{p}\cdot\vec{q},\vec{p}^{\,2},\tau), B(\vec{q}^{\,2},\vec{p}\cdot\vec{q},\vec{p}^{\,2},\tau),\ldots$$

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(3) Inverse problem

$$\int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}} e^{-\nu\tau} F_1(x,Q^2) = -m^2 A(\vec{q}^{\,2},\vec{p}\cdot\vec{q},\vec{p}^{\,2},\tau)$$

$$\int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}} e^{-\nu\tau} \frac{2x}{Q^2} F_2(x,Q^2) = B(\vec{q}^{\,2},\vec{p}\cdot\vec{q},\vec{p}^{\,2},\tau)$$

$$\ldots$$

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Lattice setup and calculations

Four-point functions

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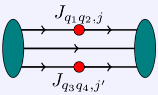
- $\overline{\chi}_N$, χ_N interpolating fields for the nucleon; Γ: spin/parity projector
  $\rightarrow$ consider spin average for now
- $J_\mu J_\nu$: Currents of interest
  $\rightarrow$ consider $V_\mu V_\nu$ or $A_\mu A_\nu$ current combinations $\Rightarrow$ F_1 and F_2
- t: source-sink separation, $\bar{\tau}$: averaged current time; avoid excited state contaminations: $t \gg \bar{\tau} \gg 0$ $\rightarrow$ consider $t/a = 8, 10, 12$
- τ: time difference between currents; Laplace transform $\tau \rightarrow \nu$
- $\vec{p}$: Nucleon momentum $\rightarrow$ consider $\vec{p} \in \{(0,0,0),(1,1,1)\}2\pi/(La)$
- $\vec{q}$: 3-Momentum transfer
- Calculate for all $\vec{q}$, τ, $\bar{\tau}$ $\rightarrow$ stochastic sources, use approach of *[G. Bali et al. JHEP 09 (2021) 106]*
- Start with CLS ensemble S100 ($n_f = 2+1$ Wilson-clover fermions)

id	β	a[fm]	$L^3 \times T$	m_π[MeV]	m_K[MeV]	$m_\pi L$	L[fm]	# conf
S100	3.4	0.085	$32^3 \times 128$	223	476	3.1	2.7	984

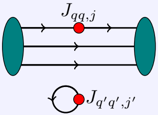
Wick contractions

Five types of diagrams

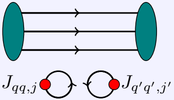
$$C^{jj'}_{1,q_1\ldots q_4} =$$



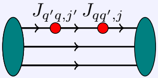
$$S^{jj'}_{1,qq'} =$$



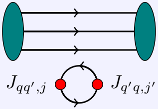
$$D^{jj'}_{qq'} =$$



$$C^{jj'}_{2,qq'} =$$



$$S^{jj'}_{2,qq'} =$$



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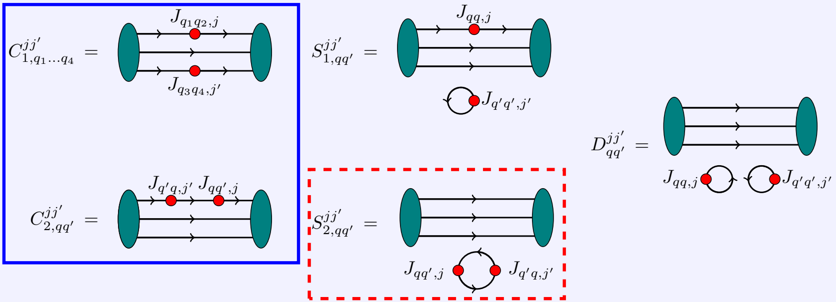

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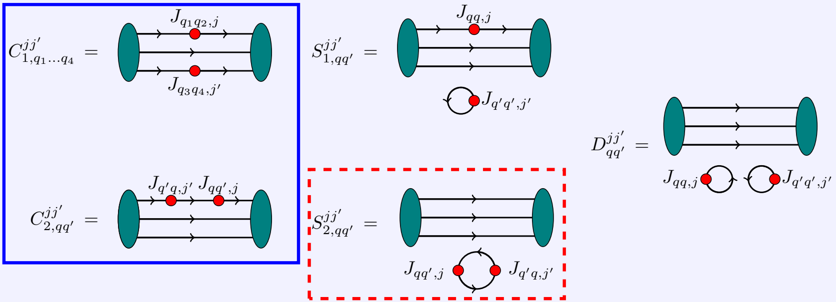
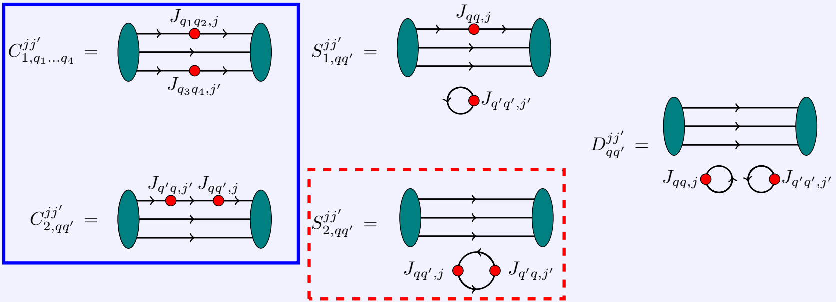


Contributions to $\langle p|\, J^{\rm em}_\mu(-\vec{q},\tau)\; J^{\rm em}_\nu(\vec{q},0)\,|p\rangle$ (proton)

$$\frac{4}{9} C^{VV,\mu\nu}_{1,uuuu}(\vec{q},\tau) - \frac{2}{9} C^{VV,\mu\nu}_{1,uudd}(\vec{q},\tau) - \frac{2}{9} C^{VV,\nu\mu}_{1,uudd}(-\vec{q},-\tau) + \frac{4}{9} C^{VV,\mu\nu}_{2,u}(\vec{q},\tau) + \frac{4}{9} C^{VV,\nu\mu}_{2,u}(-\vec{q},-\tau)$$
$$+\frac{1}{9} C^{VV,\mu\nu}_{2,d}(\vec{q},\tau) + \frac{1}{9} C^{VV,\nu\mu}_{2,d}(-\vec{q},-\tau) + \frac{2}{9} S^{VV,\mu\nu}_{1,u}(\vec{q},\tau) + \frac{2}{9} S^{VV,\nu\mu}_{1,u}(-\vec{q},-\tau) - \frac{1}{9} S^{VV,\mu\nu}_{1,d}(\vec{q},\tau)$$
$$- \frac{1}{9} S^{VV,\nu\mu}_{1,d}(-\vec{q},-\tau) + \frac{5}{9} S^{VV,\mu\nu}_{2}(\vec{q},\tau) + \frac{1}{9} D^{VV,\mu\nu}(\vec{q},\tau)$$

Wick contractions

Five types of diagrams



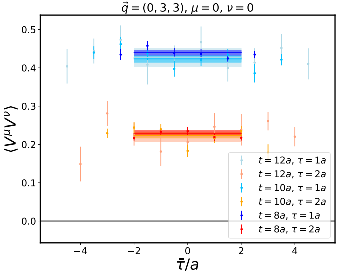
Contributions to $\langle p|\, J^{W^+}_\mu(-\vec{q},\tau) J^{W^-}_\nu(\vec{q},0)\,|p\rangle$ (proton)

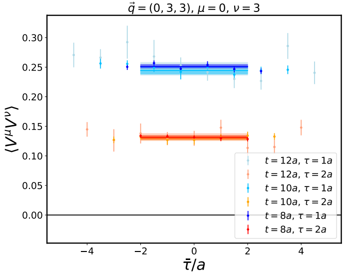
$$\frac{1}{4} C^{\nu\mu}_{1,duud}(-\vec{q},-\tau) + \frac{1}{4} C^{\mu\nu}_{2,u}(\vec{q},\tau) + \frac{1}{4} C^{\nu\mu}_{2,d}(-\vec{q},-\tau) + \frac{1}{4} S^{\mu\nu}_{2}(\vec{q},\tau)$$

for $\quad V^\mu V^\nu - V^\mu A^\nu - A^\mu V^\nu + A^\mu A^\nu$

Matrix element data

Results for $\langle p|\, V^\mu(\vec{q}, \bar{\tau} + \tau/2)\ V^\nu(-\vec{q}, \bar{\tau} - \tau/2)\,|p\rangle$; the point $\bar{\tau} = 0$ corresponds to the average of the source and sink timeslice





- No significant dependence on $t = t_{\rm snk} - t_{\rm src}$ or $\bar{\tau}$ (both VV, AA)
- Excited state contamination (of the initial state) is small
- Average over certain $\bar{\tau}$ region ($\to$ bands)

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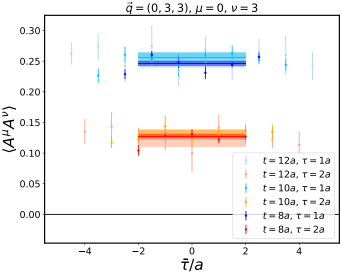

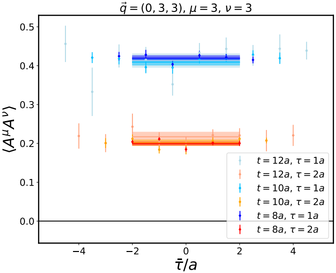

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$$\ldots$$

Euclidean structure functions

Hadronic tensor (spin average) in Euclidean space:

$$W^{\mathrm{E}}_{\mu\nu}(\vec{p},\vec{q},\tau) = \int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}}\, e^{-\nu\tau}\, W_{\mu\nu}(\vec{p},\vec{q},\nu)$$

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Start from Lorentz covariant decomposition in Minkowski space (for VV/AA):

$$\begin{aligned} W_{\mu\nu}(p, q) = {} & m^2 g_{\mu\nu} A_{\mathrm{M}}(p \cdot q, q^2) + p_\mu p_\nu B_{\mathrm{M}}(p \cdot q, q^2) + q_\mu q_\nu C_{\mathrm{M}}(p \cdot q, q^2) \\ & + (p_\mu q_\nu + q_\mu p_\nu)\, D_{\mathrm{M}}(p \cdot q, q^2) \end{aligned}$$

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Apply Laplace transform

Euclidean structure functions

$$W^{\mathrm{E}}_{00} = m^2 A + E^2 B + \frac{\partial^2}{\partial\tau^2} C - 2E \frac{\partial}{\partial\tau} D$$

$$W^{\mathrm{E}}_{0j} = W^{\mathrm{E}}_{j0} = p_j E B - q_j \frac{\partial}{\partial\tau} C + \left(q_j E - p_j \frac{\partial}{\partial\tau} \right) D$$

$$W^{\mathrm{E}}_{jk} = -m^2 \delta_{jk} A + p_j p_k B + q_j q_k C + (p_j q_k + q_j p_k) D$$

Euclidean structure functions

Hadronic tensor (spin average) in Euclidean space:

$$W^{\mathrm{E}}_{\mu\nu}(\vec{p},\vec{q},\tau) = \int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}} e^{-\nu\tau} W_{\mu\nu}(\vec{p},\vec{q},\nu)$$

Apply Laplace transform

Euclidean structure functions

$$W^{\mathrm{E}}_{00} = m^2 A + E^2 B + \frac{\partial^2}{\partial\tau^2} C - 2E \frac{\partial}{\partial\tau} D$$

$$W^{\mathrm{E}}_{0j} = W^{\mathrm{E}}_{j0} = p_j E B - q_j \frac{\partial}{\partial\tau} C + \left(q_j E - p_j \frac{\partial}{\partial\tau} \right) D$$

$$W^{\mathrm{E}}_{jk} = -m^2 \delta_{jk} A + p_j p_k B + q_j q_k C + (p_j q_k + q_j p_k) D$$

Relation to (Minkowski) structure functions:

$$\int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}} e^{-\nu\tau} F_1(x,Q^2) = -m^2 A(\vec{q}^{\,2}, \vec{p}\cdot\vec{q}, E_{\vec{p}}, \tau)$$

$$\int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}} e^{-\nu\tau} \frac{2xm^2}{Q^2} F_2(x,Q^2) = m^2 B(\vec{q}^{\,2}, \vec{p}\cdot\vec{q}, E_{\vec{p}}, \tau)$$

$$Q^2 = \vec{q}^{\,2} - \nu^2 \qquad x = \frac{\vec{q}^{\,2} - \nu^2}{E_{\vec{p}}\nu - \vec{p}\cdot\vec{q}}$$

x and Q^2 are functions of ν **AND** $\vec{q}^{\,2}$, $E_{\vec{p}}$ and $\vec{p}\cdot\vec{q}$
$\Rightarrow$ Can exploit this when reconstructing F_i

Euclidean structure functions: Results

- Correct for lattice artifacts by multiplying by $\exp\{\tau(E^{\mathrm{Wilson}}_{\vec{p}+\vec{q}} - E_{\vec{p}+\vec{q}})\}$ (dispersion relation for Wilson fermions), only keep data for $\vec{q}$ close to lattice diagonal.
- Perform Laplace transform on experimental results (use data from *[SLAC '92]*) to compare with our Euclidean structure functions.
- **Notice:** Cover deep inelastic region only at large $|\vec{q}|$. At small $|\vec{q}|$, resonances and elastic scattering dominate.
- Lattice result for A matches perfectly the experimental data for $|\vec{q}| > 4.5$ GeV.
- Discrepancies for small $\vec{q}$ (elastic/resonance contributions the exp. data is **not** aware of).
- Non-zero $\vec{p}$: have same trends, discrepancies larger for $\frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|} = \cos\phi < 0$.
- Also have (preliminary) results for other electroweak currents.

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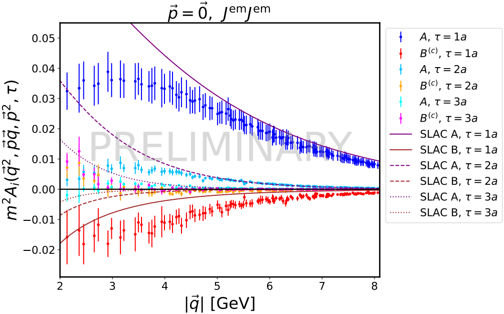

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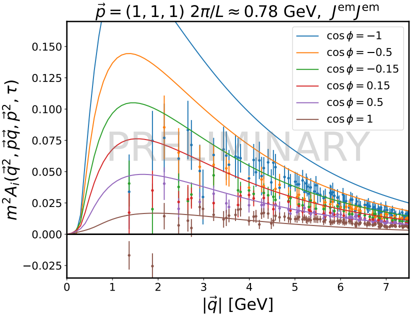

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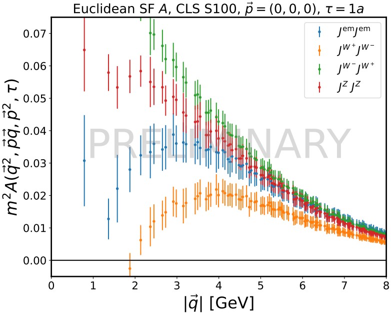

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Lattice calculation strategy

(1) Four-point functions

$$W^{\mathrm{E}}_{\mu\nu}(\vec{p},\vec{q},\tau) = \frac{e^{-E_{\vec{p}}\tau}}{2} \frac{\langle \Gamma \chi_N(\vec{p},\frac{t}{2})\; J_\mu(\vec{q},\bar{\tau}+\frac{\tau}{2})\; J_\nu(-\vec{q},\bar{\tau}-\frac{\tau}{2})\; \overline{\chi}_N(\vec{p},-\frac{t}{2})\rangle}{\langle \chi_N(\vec{p},t)\; \overline{\chi}_N(\vec{p},0)\rangle}\Bigg|_{t\gg\bar{\tau}\gg 0}$$

(2) Euclidean structure functions

$$W^{\mathrm{E}}_{\mu\nu}(\vec{p},\vec{q},\tau) \to A(\vec{q}^{\,2},\vec{p}\cdot\vec{q},\vec{p}^{\,2},\tau), B(\vec{q}^{\,2},\vec{p}\cdot\vec{q},\vec{p}^{\,2},\tau),\ldots$$

(3) Inverse problem

$$\int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}} e^{-\nu\tau} F_1(x,Q^2) = -m^2 A(\vec{q}^{\,2},\vec{p}\cdot\vec{q},\vec{p}^{\,2},\tau)$$

$$\int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}} e^{-\nu\tau} \frac{2x}{Q^2} F_2(x,Q^2) = B(\vec{q}^{\,2},\vec{p}\cdot\vec{q},\vec{p}^{\,2},\tau)$$

$$\ldots$$

Inverse problem for structure functions

- For now consider only $A \leftrightarrow F_1$
- Employ Callan-Gross relation $F_2 = 2xF_1$ (approximately valid as long as x is not small)

Defining relation: Laplace transform

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Recast as integral over (x, Q^2):

Inverse problem

$$m^2 A(\vec{q}^{\,2}, \vec{p}\cdot\vec{q}, E_{\vec{p}}, \tau) = \int \mathrm{d}Q^2 \int_0^1 \mathrm{d}x \; K(\vec{q}^{\,2}, \vec{p}\cdot\vec{q}, E_{\vec{p}}, \tau, x, Q^2) F_1(x, Q^2)$$

$$K(\vec{q}^{\,2}, \vec{p}\cdot\vec{q}, E_{\vec{p}}, \tau, x, Q^2) := e^{-\tau(R - E_{\vec{p}}x)} \left(1 - \frac{E_{\vec{p}}x}{R} - \frac{\vec{p}\cdot\vec{q}}{E_{\vec{p}}R}\right) \Theta(R - R_1 + E_{\vec{p}}(1-x)) \times$$
$$\times\, \delta(Q^2 - (R - E_{\vec{p}}x)^2 - \vec{q}^{\,2})$$

with $R := \sqrt{E_{\vec{p}}^2 x^2 + 2\vec{p}\cdot\vec{q}x + \vec{q}^{\,2}}$, $R_1 := R|_{x=1}$.

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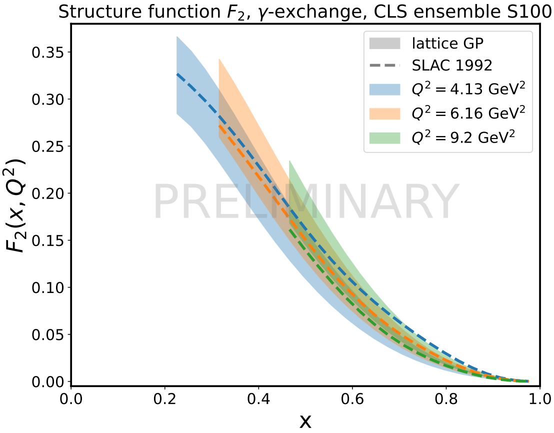
$\Rightarrow$ **Use Gaussian process regression** *[H. Dutrieux et al. Phys.Rev.D 111 (2025) 3]*

A first reconstruction of F_2

Use data for Euclidean structure function data for $|\vec{q}| > 4\ \mathrm{GeV}$ and $\cos\phi > 0$.



- **Reasonably good agreement between our result and experimental data**
- **Caveat: This is only a first calculation; still have to study systematics of GPR in our case**

Conclusion

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First principle calculation of the hadronic tensor (structure functions) of lepton-nucleon scattering using four-point functions.

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- Calculated four-point functions for VV and AA combinations for all $\vec{q}$ and two nucleon momenta $\vec{p}$. We use the CLS ensemble S100 (32×128, $m_\pi = 223\ \mathrm{MeV}$, $a = 0.085\ \mathrm{fm}$).
- Determined Euclidean structure functions for all electroweak currents. Good agreement with the Laplace transform of the exp. data for F_2^{em} for large $|\vec{q}|$.
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Open issues

- S_2 diagram missing.
- Study systematics of Gaussian process regression in our case.
- Increase statistics, add more momenta.
- Reconstruct $F_i^{W^\pm}$, F_i^Z; consider nucleon spin ($\Rightarrow$ spin structure functions g_i).
- Physical limit: Consider finite volume and discretization artifacts, physical masses.

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Thanks for your attention!

Backup Slides

Electroweak currents

Electroweak currents ($n_f = 2$)

$$J_\mu^{\mathrm{em}}(x) := \frac{2}{3} V_\mu^{uu}(x) - \frac{1}{3} V_\mu^{dd}(x)$$

$$J_\mu^{W^+}(x) := \frac{1}{2}\left[V_\mu^{ud}(x) - A_\mu^{ud}(x)\right] = J_\mu^{W^-,\dagger}(x)$$

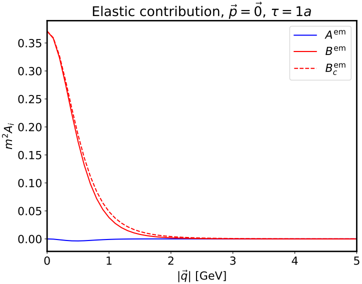
$$J_\mu^{Z}(x) := \frac{1}{4}\left[V_\mu^{uu}(x) - A_\mu^{uu}(x) - V_\mu^{dd}(x) + A_\mu^{dd}(x)\right] - \sin^2\theta_{\mathrm{W}} J_\mu^{\mathrm{em}}(x)$$

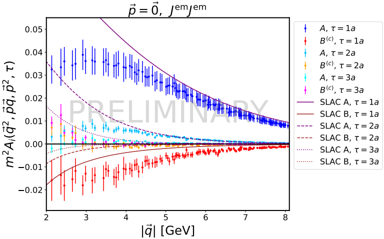
$$V_\mu^{qq'}(x) := \bar{q}(x)\gamma_\mu q'(x) \qquad A_\mu^{qq'}(x) := \bar{q}(x)\gamma_\mu\gamma_5 q'(x)$$

Elastic contributions

Consider electromagnetic currents and calculate:

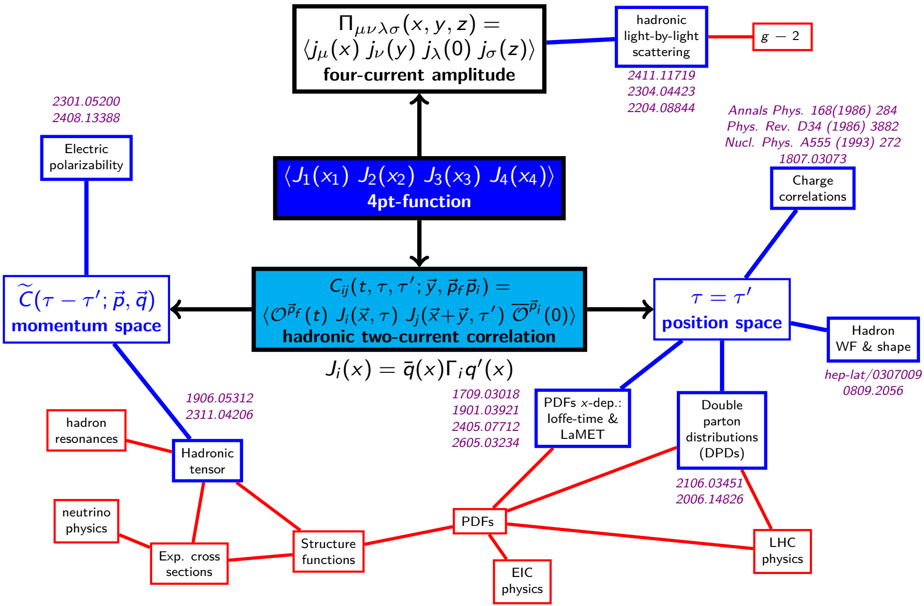
$$W^{\mathrm{E}}_{\mu\nu,\mathrm{elastic}} = \frac{1}{8E_{\vec{p}}E_{\vec{p}+\vec{q}}} e^{-\tau E_{\vec{p}+\vec{q}}} \frac{1}{2} \sum_{\lambda\lambda'} \underbrace{\langle p, \lambda | \, J^{\mathrm{em}}_{\mu} \, | \vec{p}+\vec{q}, \lambda' \rangle}_{\to \text{form factors}} \langle \vec{p}+\vec{q}, \lambda' | \, J^{\mathrm{em}}_{\nu} \, | p, \lambda \rangle$$





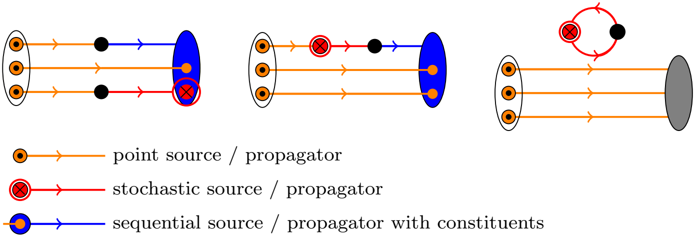
- Elastic contribution only relevant for $|\vec{q}| < 2\ \mathrm{GeV}$

The scope of four-point functions



Techniques

Following previous work *[Bali et al, JHEP 09 (2021) 106]*:



- Point source at fixed position, sequential and/or stochastic (C_1) sources at the sink; corresponding propagators have to be re-calculated for each momentum / source-sink separation
- Gaussian smearing at source and sink; if $\vec{p} \neq \vec{0} \rightarrow$ Momentum smearing *[Bali et al, Phys.Rev.D 93 (2016) 9 094515]*
- Stochastic propagators (C_2, S_2); can be re-used along different momenta, etc...
- Do contractions in momentum space (current momentum $\vec{q}$) for desired current/insertion timeslice combination (low computational cost); FFT for position space ($\vec{y}$)

Structure functions

Use Lorentz symmetry to decompose *[PDG]*:

$$
\begin{aligned}
W_{\mu\nu}(p,q,s) = {} & \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) F_1(x,Q^2) + \frac{\hat{p}_\mu \hat{p}_\nu}{p\cdot q} F_2(x,Q^2) - i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho p^\sigma}{2p\cdot q} F_3(x,Q^2) \\
& + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho}{p\cdot q} \left[s^\sigma g_1(x,Q^2) + \left(s^\sigma - \frac{s\cdot q}{p\cdot q} p^\sigma \right) g_2(x,Q^2) \right] \\
& + \frac{1}{2p\cdot q} \left[(\hat{p}_\mu \hat{s}_\nu + \hat{s}_\mu \hat{p}_\nu) - \frac{2s\cdot q}{p\cdot q} \hat{p}_\mu \hat{p}_\nu \right] g_3(x,Q^2) \\
& + \frac{s\cdot q}{p\cdot q} \left[\frac{\hat{p}_\mu \hat{p}_\nu}{p\cdot q} g_4(x,Q^2) + \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) g_5(x,Q^2) \right]
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with $\hat{p}_\mu = p_\mu - \frac{p\cdot q}{q^2} q_\mu$, $\hat{s}_\mu = s_\mu - \frac{s\cdot q}{q^2} q_\mu$.

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$$+ i\epsilon_{\mu\nu\rho\sigma}\frac{q^\rho}{p\cdot q}\left[s^\sigma g_1(x,Q^2) + \left(s^\sigma - \frac{s\cdot q}{p\cdot q}p^\sigma\right) g_2(x,Q^2)\right]$$
$$+ \frac{1}{2p\cdot q}\left[\left(\hat{p}_\mu \hat{s}_\nu + \hat{s}_\mu \hat{p}_\nu\right) - \frac{2s\cdot q}{p\cdot q}\hat{p}_\mu \hat{p}_\nu\right] g_3(x,Q^2)$$
$$+ \frac{s\cdot q}{p\cdot q}\left[\frac{\hat{p}_\mu \hat{p}_\nu}{p\cdot q} g_4(x,Q^2) + \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) g_5(x,Q^2)\right]$$

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Structure functions

- F_1, F_2
- F_3 (only $W^\pm$, Z)
- Spin: g_1, g_2, g_3 ,g_4, g_5

Kinematic variables:

- Momentum transfer: $Q^2 = \vec{q}^2 - \nu^2$, $\nu = q^0$
- Bjorken-x: $x = \dfrac{Q^2}{2p\cdot q}$

Kinematic regions

- Integration over ν includes several kinematic regions.
- Use experimental data (SLAC 1992) *[Phys.Lett.B 828 (1992) 475-482]* in order to see how the integrand looks like
- Can also reconstruct Euclidean SF data

$$\int_{\nu_{\min}}^{\infty} \frac{\mathrm{d}\nu}{2E_{\vec{p}}} e^{-\nu\tau} F_1(x, Q^2) = -m^2 A(\vec{q}^2, \vec{p}\cdot\vec{q}, E_{\vec{p}}, \tau)\,,$$

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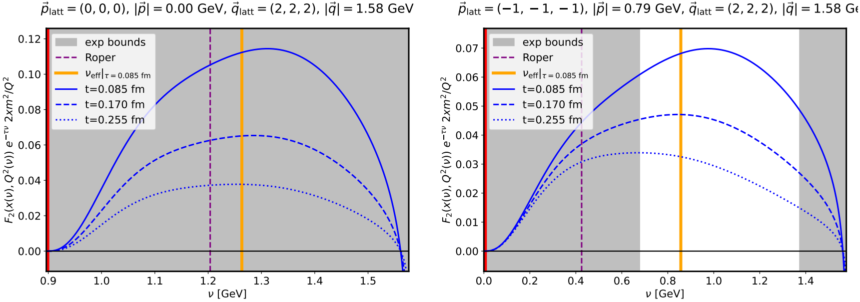
- For non-vanishing hadronic tensor, need on-shell state: $W^2 = (p+q)^2 \geq m^2$
- If $\nu > |\vec{q}|$ have light-like q (this is not DIS anymore).
- DIS region ($W^2_{\min} = 6~\mathrm{GeV}^2$, $Q^2_{\min} > 2~\mathrm{GeV}^2$):
$\sqrt{W^2_{\min} + (\vec{p}+\vec{q})^2} - E_{\vec{p}} < \nu < \sqrt{\vec{q}^2 - Q^2_{\min}}$
- **Notice: These are not integration limits!** The only non-trivial limit is imposed by $W^2 \geq m^2$ ($\Rightarrow \nu_{\min}$).
- Reconstructed effective energy is almost constant $\Rightarrow$ **Almost pure exponential**

Kinematic regions

Experimental data (SLAC 1992) Phys.Lett.B 828 (1992) 475-482



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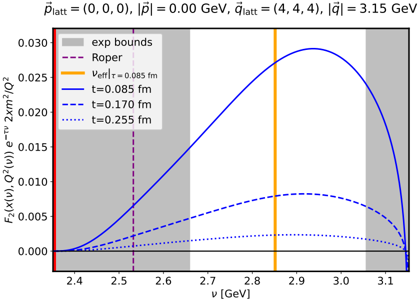

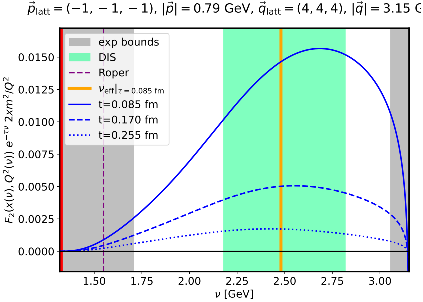

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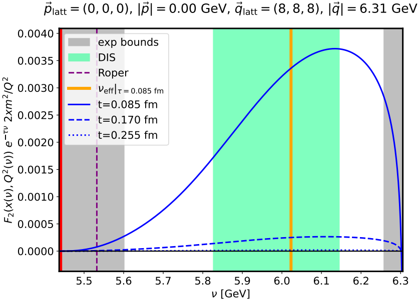

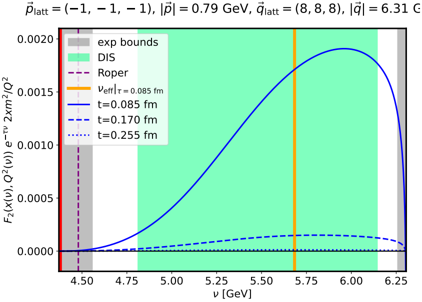

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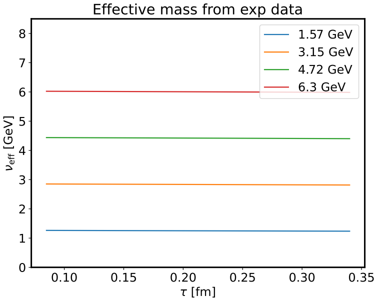
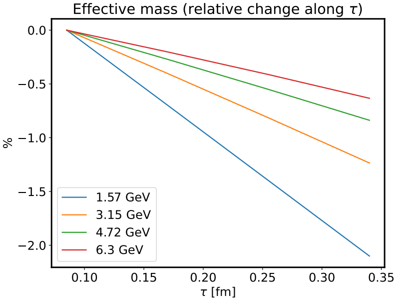
Kinematic regions

Experimental data (SLAC 1992) Phys.Lett.B 828 (1992) 475-482





- For non-vanishing hadronic tensor, need on-shell state: $W^2 = (p+q)^2 \geq m^2$
- If $\nu > |\vec{q}|$ have light-like q (this is not DIS anymore).
- DIS region ($W^2_{\mathrm{min}} = 6\ \mathrm{GeV}^2$, $Q^2_{\mathrm{min}} > 2\ \mathrm{GeV}^2$):
  $\sqrt{W^2_{\mathrm{min}} + (\vec{p}+\vec{q})^2} - E_{\vec{p}} < \nu < \sqrt{\vec{q}^2 - Q^2_{\mathrm{min}}}$
- **Notice: These are not integration limits!** The only non-trivial limit is imposed by $W^2 \geq m^2$ ($\Rightarrow \nu_{\mathrm{min}}$).
- Reconstructed effective energy is almost constant $\Rightarrow$ **Almost pure exponential**

Gaussian process regression

Follow *[Dutrieux et al. Phys.Rev.D 111 (2025) 3, 034515, arXiv:2604.21476]*:

Bayes' theorem

$$\mathbb{P}(F|A,I) = \frac{\mathbb{P}(A|F)\mathbb{P}(F|I)}{\mathbb{P}(A|I)}$$

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- Prior distribution: $\mathbb{P}(F|I) \propto \exp\left\{-\frac{1}{2}(F-g)^T G^{-1}(F-g)\right\}$
- Likelihood (χ^2): $\mathbb{P}(A|F) \propto \exp\left\{-\frac{1}{2}(A-K\cdot F)C^{-1}(A-K\cdot F)\right\}$
- F: Function to be determined
- g: initial guess (set $g=0$)
- A: Lattice data for Euclidean structure functions; C: Covariance matrix
- G: Gaussian kernel (ensure smoothness, prior assumption on functional form $F^{(0)}(x)$):
  $G(x,Q^2;x',Q^{2\prime}) = \sigma^2 F^{(0)}(x)F^{(0)}(x') \exp\left\{-\frac{(\log(x)-\log(x'))^2}{2\ell_x^2} - \frac{(Q^2-Q^{2\prime})^2}{2\ell_{Q^2}^2}\right\}$
- K: Kernel of our inverse problem (see previous slide)

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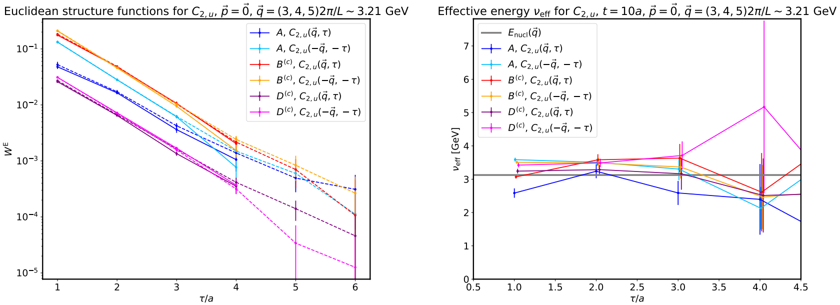
Posterior mean for F

$$\langle F\rangle = g + GK^T\left(C+KGK^T\right)^{-1}(A-K\cdot g)$$

Euclidean structure functions

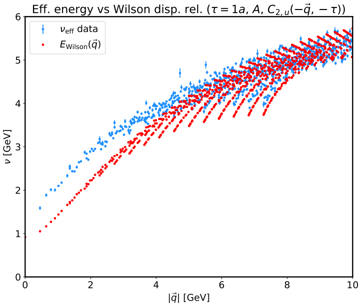
Contributions from diagram C_2, $\vec{p} = \vec{0}$:

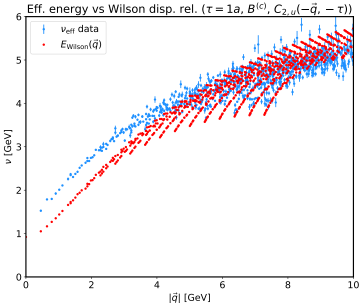


- Solid line: $t/a = 8$, dashed line $t/a = 10$
- Have good signal for $\tau/a \leq 4$
- As expected, observe exponential decay along τ; effective energy

$$a\nu_{\rm eff} = \log\left\{\frac{A(\vec{q}^2, 0, 0, \tau)}{A(\vec{q}^2, 0, 0, \tau + a)}\right\}$$

Wilson fermion dispersion relation





- Lower bound of the Laplace integral ($\nu_{\min}$) is given by elastic scattering
- One-fermion state $\Rightarrow$ Obeys dispersion relation for Wilson fermions:

$$\sinh^2(aE^{\mathrm{Wilson}}) = \sum_j \sin^2(ap_j) + \left[am + \sum_j (1 - \cos(ap_j))\right]^2$$

- The corresponding anisotropy effects are clearly visible in the lattice data (effective energy)