Finite-volume effects to hadronic observables in lattice QCD+QED

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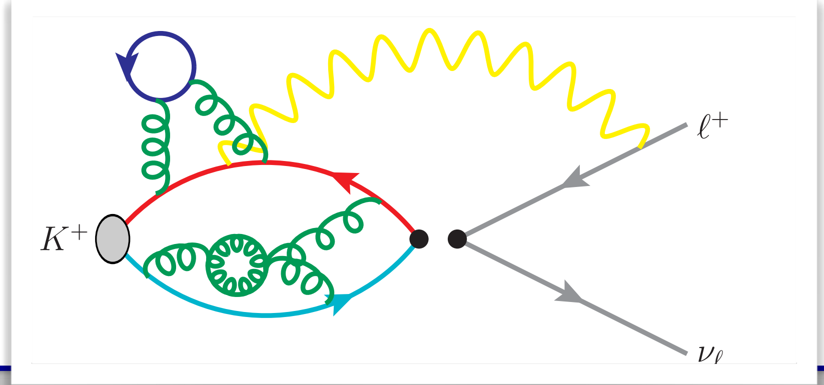
OUTLINE

- Motivations
- FVEs in light meson leptonic decays

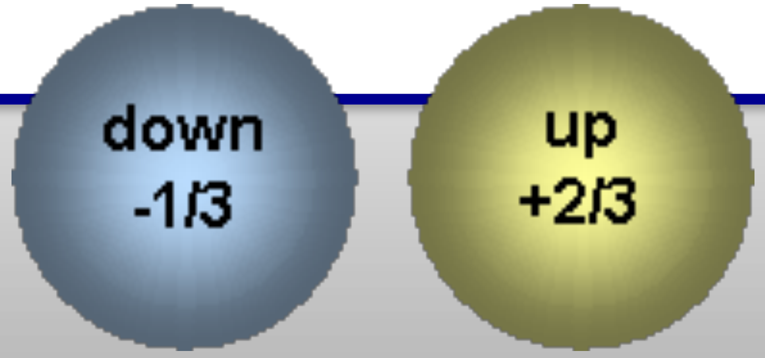


In collaboration with

Volodymyr Biloshytskyi and Laurent Lellouch, for BMWc

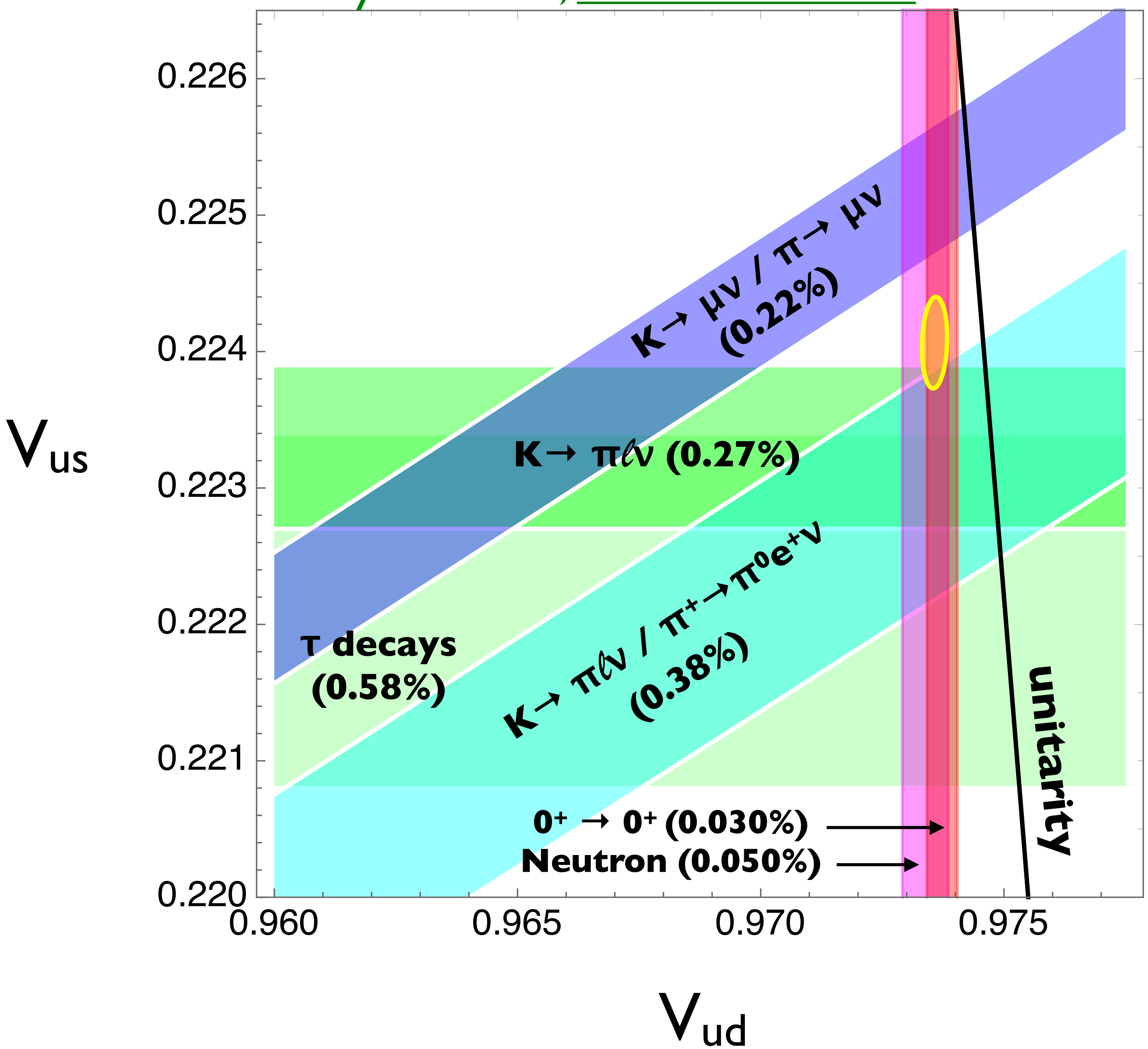
paper in preparation

Motivations



Cabibbo-angle anomaly

D. Bryman *et al.*, arXiv:2111.05338

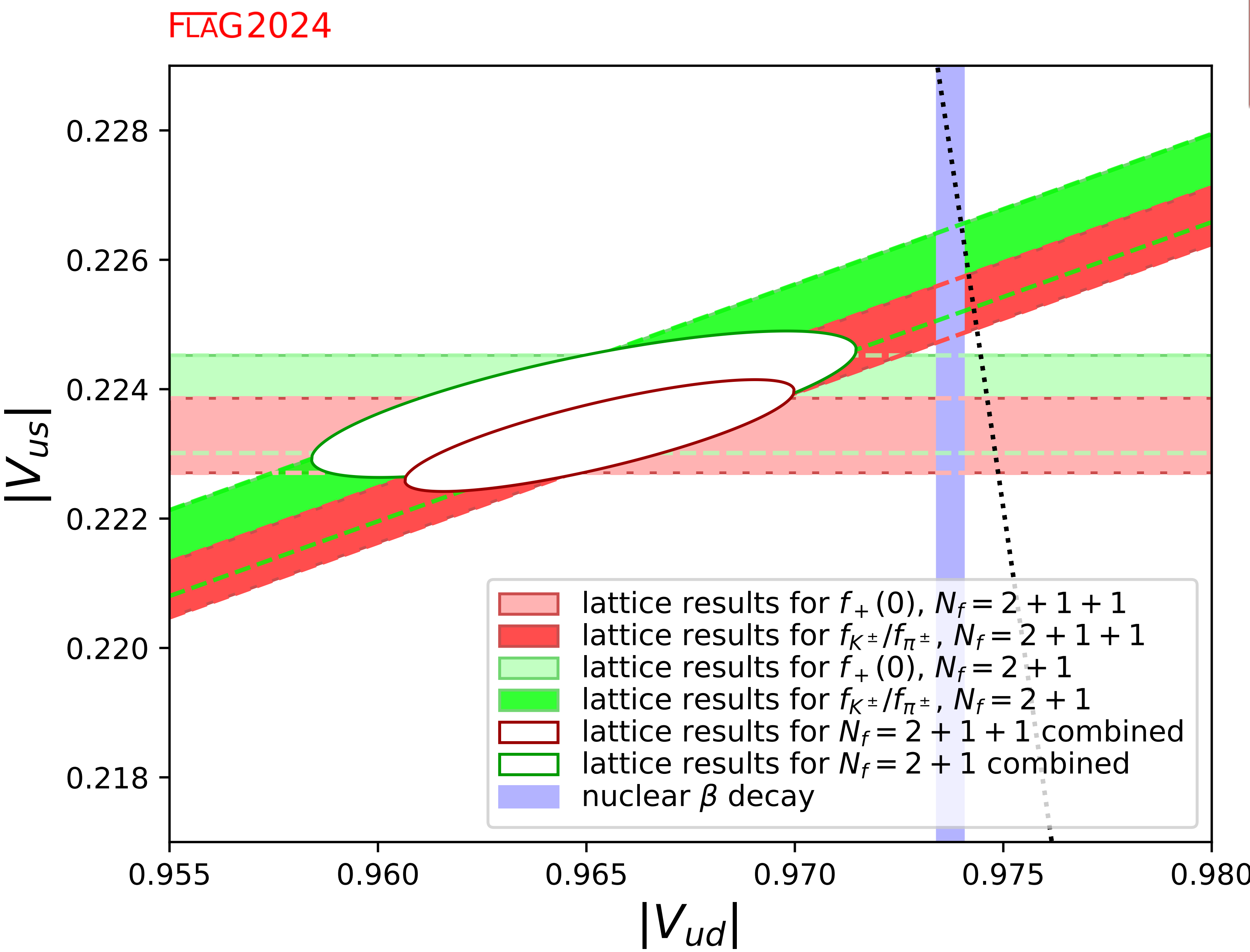


$$|V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2$$

$$|V_u|^2 \simeq 0.9983(7)$$ PDG

From superallowed nuclear β decays and kaon decays the test points to a small but persistent $\approx 2\sigma$ tension

Cabibbo-angle anomaly



FlaviAnet Kaon WG

$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27599(41)$$

$$|V_{us}|\, f_+(0) = 0.21654(41)$$

< 0.2%

FLAG Flavour Lattice Averaging Group

$$f_{K^\pm}/f_{\pi^\pm} = 1.1934(19)$$

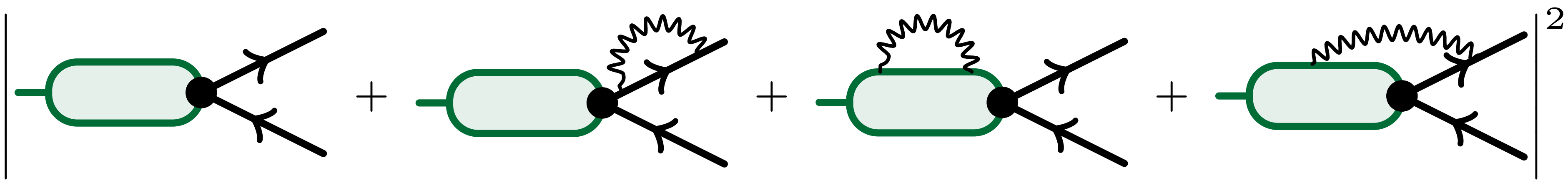
$$f_+(0) = 0.9698(17)$$

0.2%

Electromagnetic $O(\alpha_{em})$ and strong $O((m_d - m_u)/\Lambda_{QCD})$ isospin-breaking effects must be included

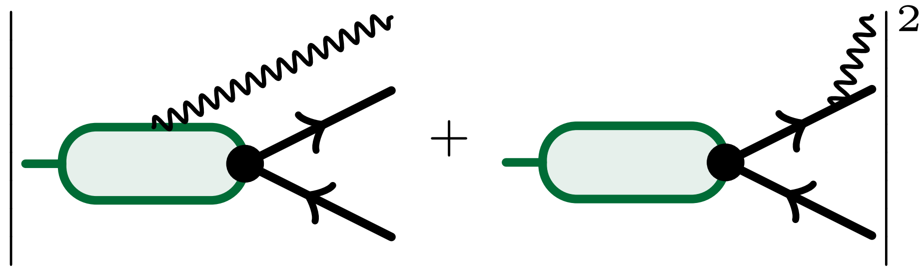
Leptonic decays at O(α)

$\Gamma(E) = \int_{2\text{ b.p.s.}}$



Γ_0

$+ \int_{3\text{ b.p.s.}}^{E_\gamma < E}$



$\Gamma_1(E)$

$\Gamma(E) = \Gamma_0 + \Gamma_1(E)$ with $0 \leq E_\gamma \leq E$ is infrared finite

F. Bloch and A. Nordsieck, 1937

- Both Γ_0 and $\Gamma_1(E)$ can be evaluated in a fully non-perturbative way in lattice simulations
- The first lattice calculations of $\Gamma[\pi, K \to \ell\nu(\gamma)]$ have been finalized and, within BMWc, we are working on a new one (see A. Cotellucci's contribution)

DG *et al.* arXiv:1711.06537 M. Di Carlo *et al.* arXiv:1904.08731 P. Boyle *et al.* arXiv:2211.12865

Finite-volume effects

- Available calculations performed using $\mathrm{QED_L}$ prescription (photon zero-momentum mode excluded)
- Infinite-volume reconstruction method and $\mathrm{QED_r}$ prescription have been recently proposed N.H. Christ *et al.*,arXiv:2304.08026 M. Di Carlo *et al.*,arXiv:2501.07936

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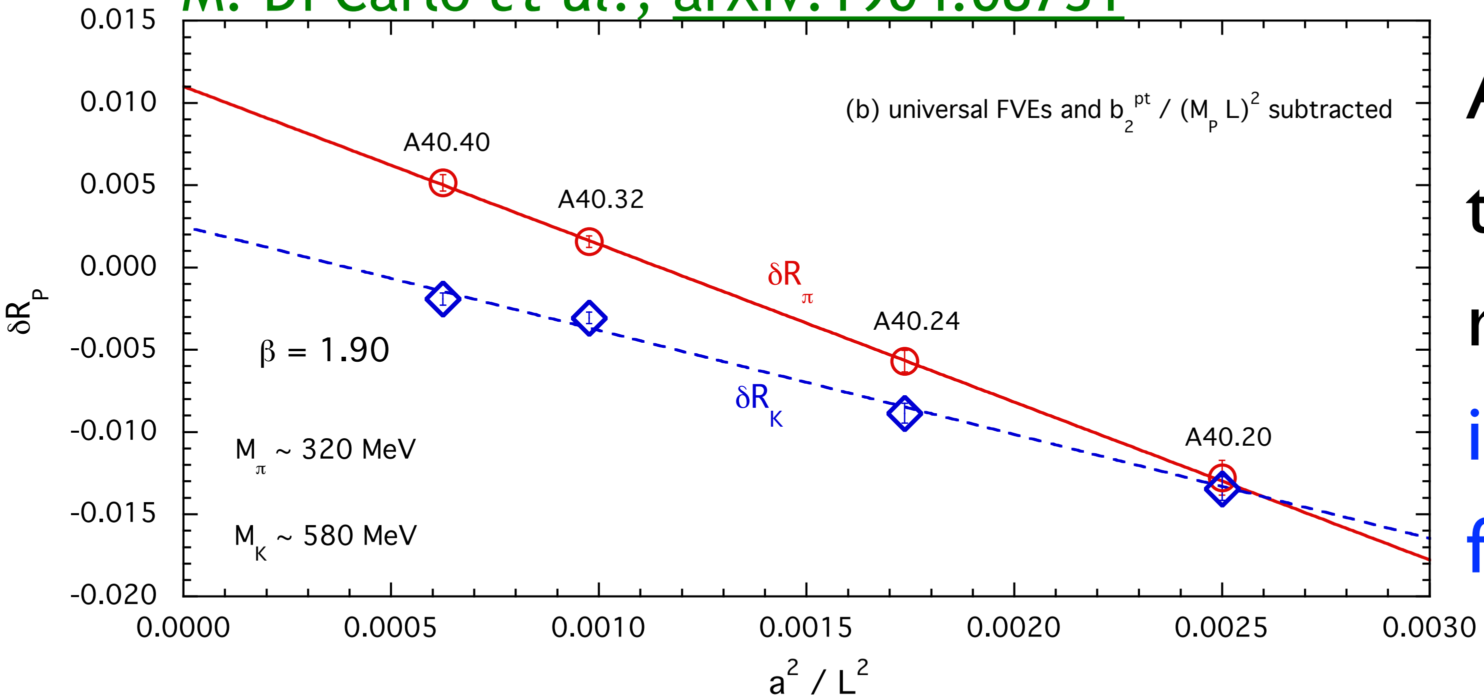

After subtracting universal FVEs up to $O(1/L)$, RM123S data displayed a residual volume dependence linear in $1/L^2$, interpreted as a sizable SD finite-volume effect

Finite-volume effects

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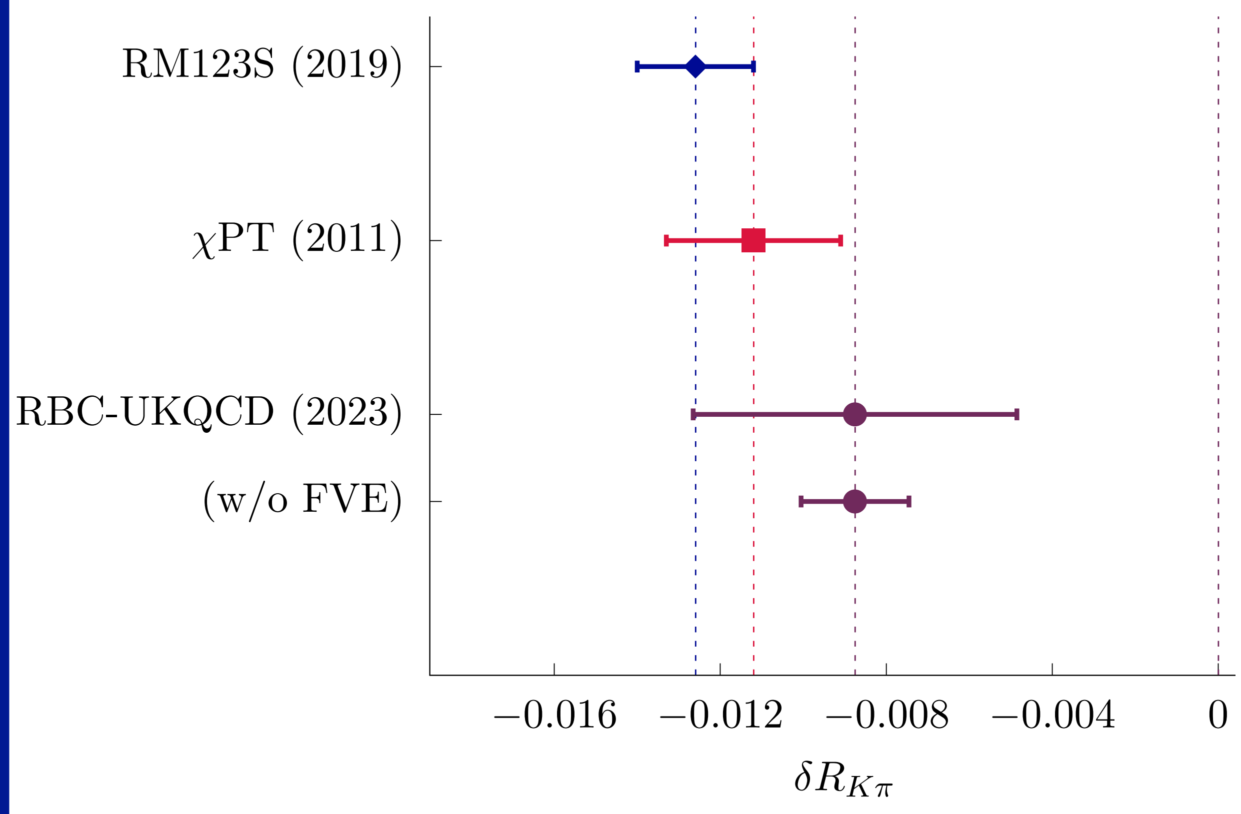
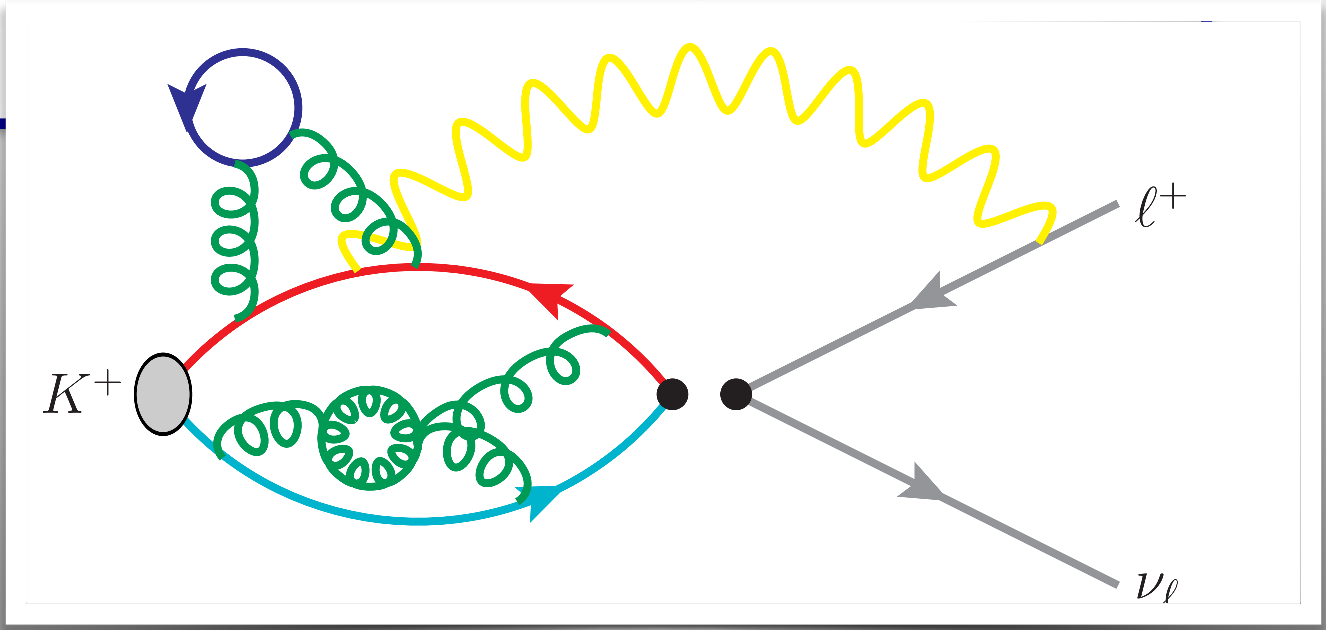


This interpretation became less compelling after the analysis of M. Di Carlo *et al.*, 2022 which determined the $O(1/L^2)$ SD terms and found them numerically small. The dominant uncertainty of RBC/UKQCD calculation is due to incomplete knowledge of the $O(1/L^3)$ FV corrections

An unusual hierarchy of FVEs emerged and here we revisit it in the light of exp-suppressed FV contributions in the pt approximation so far neglected

FVEs in light meson leptonic decays



The lattice strategy [N. Carrasco et al. 2015]

$$\Gamma\left(P_{\ell 2}^{\pm}\right) = \left(\Gamma_0 - \Gamma_0^{pt}\right) + \left(\Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E)\right)$$

- The contributions from soft virtual photon to Γ_0 and Γ_0^{pt} in the first term are exactly the same and the IR divergence cancels in the difference $\Gamma_0 - \Gamma_0^{\mathrm{pt}}$.
- The sum $\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)$ in the second term is also IR finite since it is a physically well defined quantity. This term can be thus calculated in perturbation theory with a different IR cutoff.
- The two terms are also separately gauge invariant.

$$\Delta\Gamma_0(L) = \Gamma_0(L) - \Gamma_0^{pt}(L)$$

$$\Gamma^{pt}(\Delta E) = \lim_{m_\gamma \to 0}\left[\Gamma_0^{pt}(m_\gamma) + \Gamma_1^{pt}(\Delta E, m_\gamma)\right]$$

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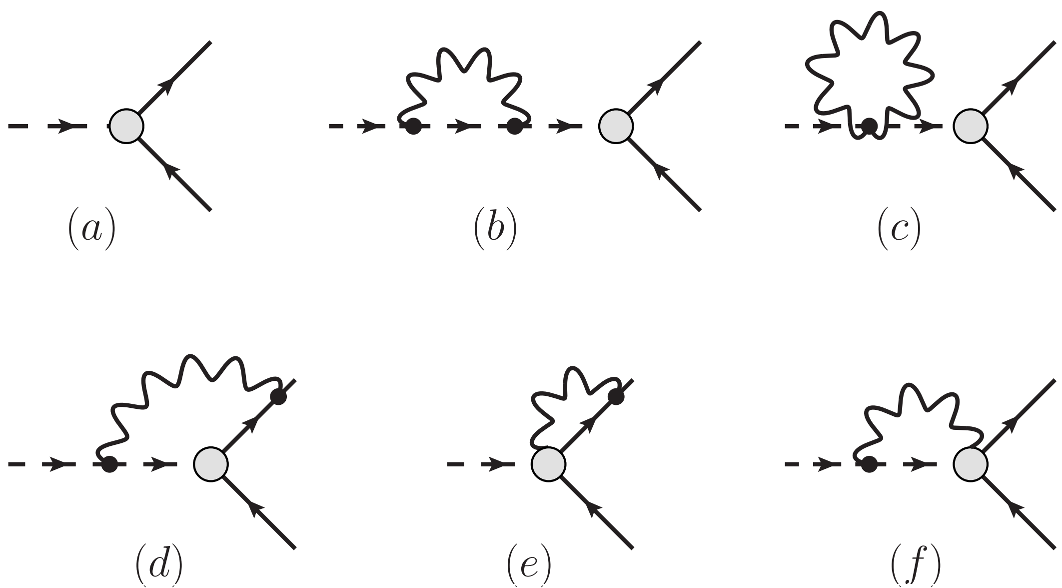
The point-like meson approximation



$$\Gamma_0^{\mathrm{pt}}(L) = \Gamma_0^{\mathrm{tree}} \left\{ 1 + 2\frac{\alpha_{\mathrm{em}}}{4\pi} Y(L) \right\}$$

$$\Gamma_0^{\mathrm{tree}} = \frac{G_F^2 |V_{q\,Q}|^2}{8\pi} \left[f_P^{(0)} \right]^2 m_P\, m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2} \right)^2$$

$$Y(L) = \tilde{C}_0(r_\ell) \log(m_P L) + C_0(r_\ell) + \frac{C_1(r_\ell)}{m_P L} + \frac{C_2(r_\ell)}{(m_P L)^2} + \frac{C_3(r_\ell)}{(m_P L)^3} + \ldots$$

$$r_\ell = m_\ell / m_P$$

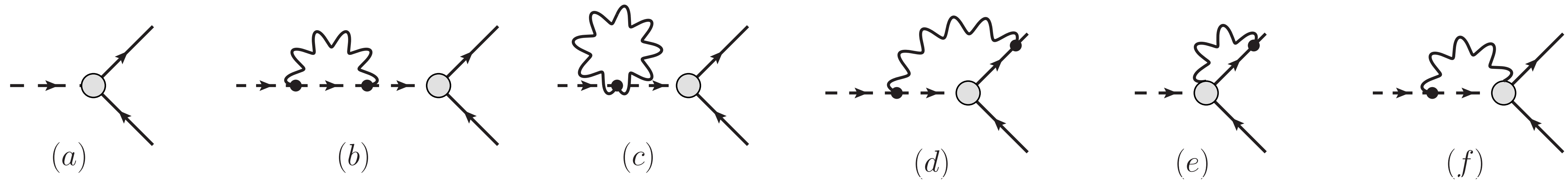
- $\tilde{C}_0$, C_0, C_1 are universal, although C_0 and C_1 depend on the IR regulator
- $\tilde{C}_0, C_{0,\ldots,3}$ are known both in $\mathrm{QED_L}$ and in $\mathrm{QED_r}$ (where $C_3 = 0$)

N. Tantalo *et al.*, arXiv:1612.00199 M. Di Carlo *et al.*, arXiv:2501.07936

$\mathrm{QED_L}$	$\mathrm{QED_r}$
$\ldots = O(e^{-\lambda L})$	$\ldots = O\left[\frac{1}{(m_P L)^5}, e^{-\lambda L} \right]$

New

The structure of the calculation



- The whole effect can be expressed in terms of four one-loop master integrals, plus the meson wave function renormalization

$$Y(L) = 16\pi^2 \left\{ i\frac{1+r_\ell^2}{1-r_\ell^2} S_1 - \frac{2ir_\ell^2}{1-r_\ell^2} S_2 - 2i(1+r_\ell^2) S_3 + \frac{2i}{1-r_\ell^2} S_5 + \frac{1}{2} S_P \right\}$$

V. Lubicz *et al.*,arXiv:1611.08479

- Each FV MI is decomposed in terms of the IV part, the finite-infinite volume difference ξ and a part specific of the QED formulation used

$$S_i^{\mathrm{FV}}(L) = S_i^{\mathrm{IV}} - \Delta S_i(L) + \xi_i(L)$$

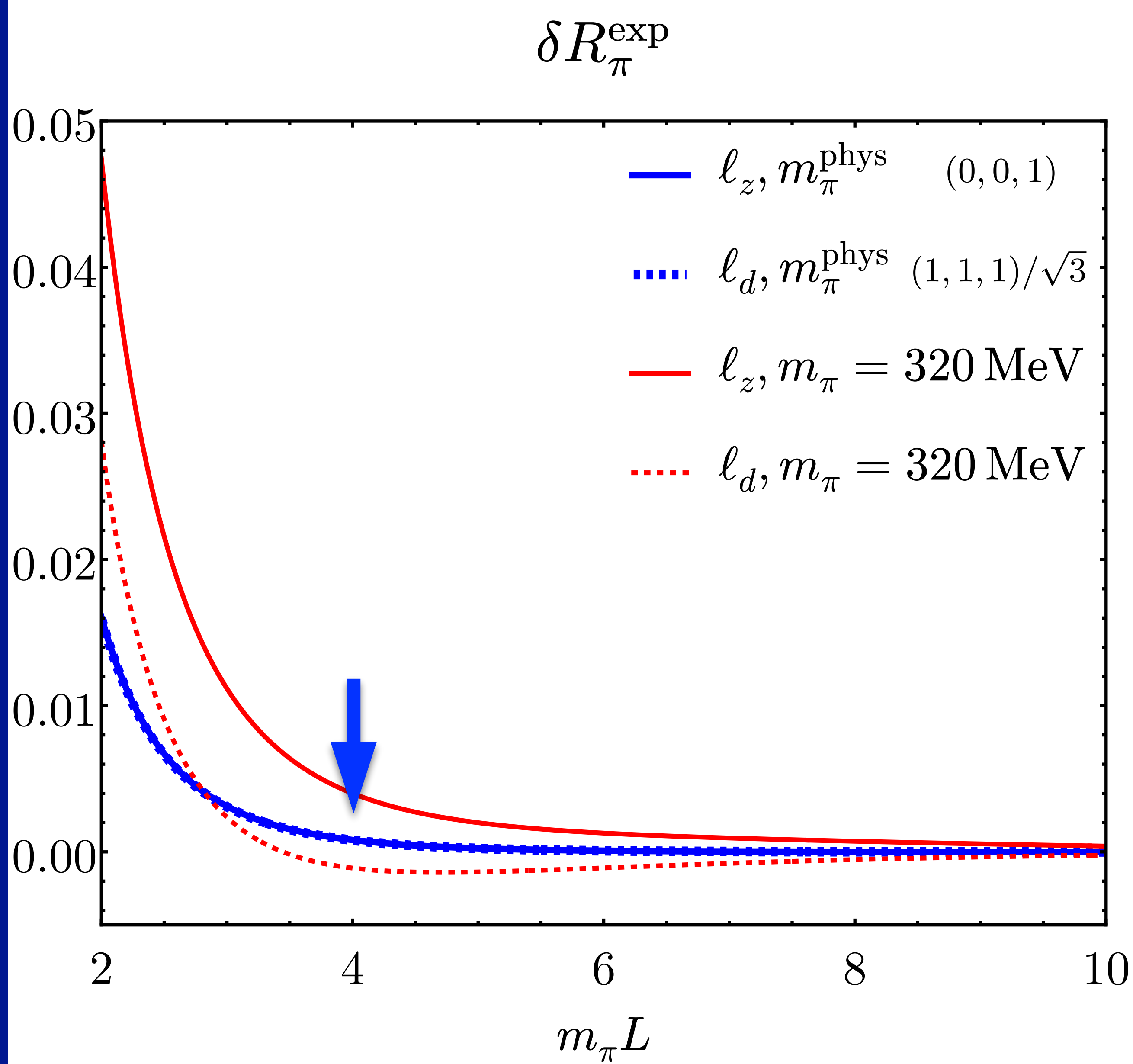
- minimal $\mathrm{QED_r}$:

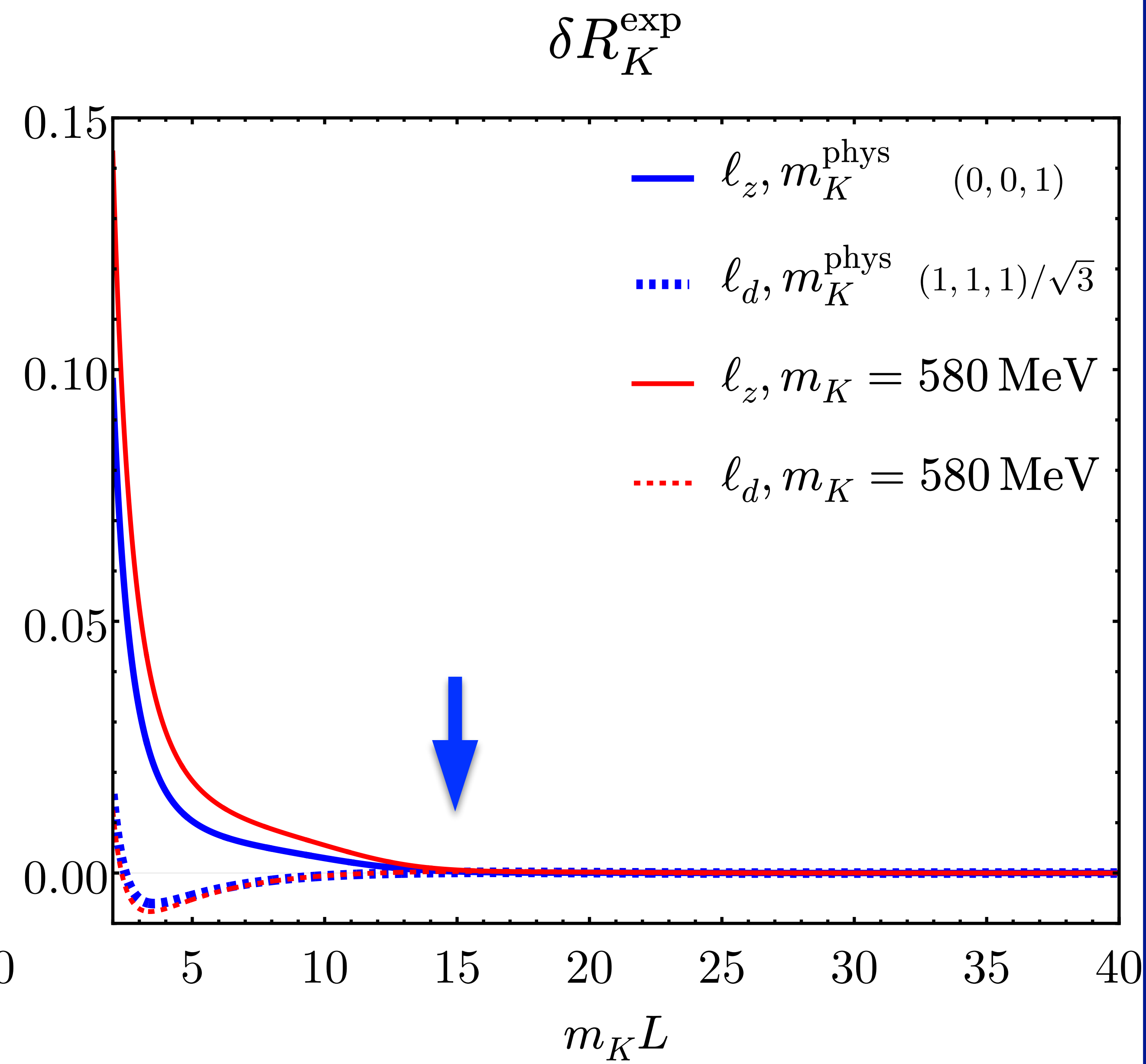
$$D^r_{\mu\nu}(x) = \frac{1}{L^3} \sum_{\vec{k}\neq 0} \int \frac{\mathrm{d}k_0}{2\pi} e^{ikx} \frac{\delta_{\mu\nu}}{k_0 + \vec{k}^2} \left[1 + h(\vec{n}, R)\right] \Big|_{\vec{n}=\vec{k}L/(2\pi)} \qquad h(\vec{n}) = \delta_{|\vec{n}|,1}/6$$

M. Di Carlo *et al.*,arXiv:2501.07936

- W-regularization to cure UV divergences: $\dfrac{1}{k^2 + i0^+} \to \dfrac{1}{k^2 + i0^+} - \dfrac{1}{k^2 - M_W^2 + i0^+}$

The pt exp-suppressed corrections

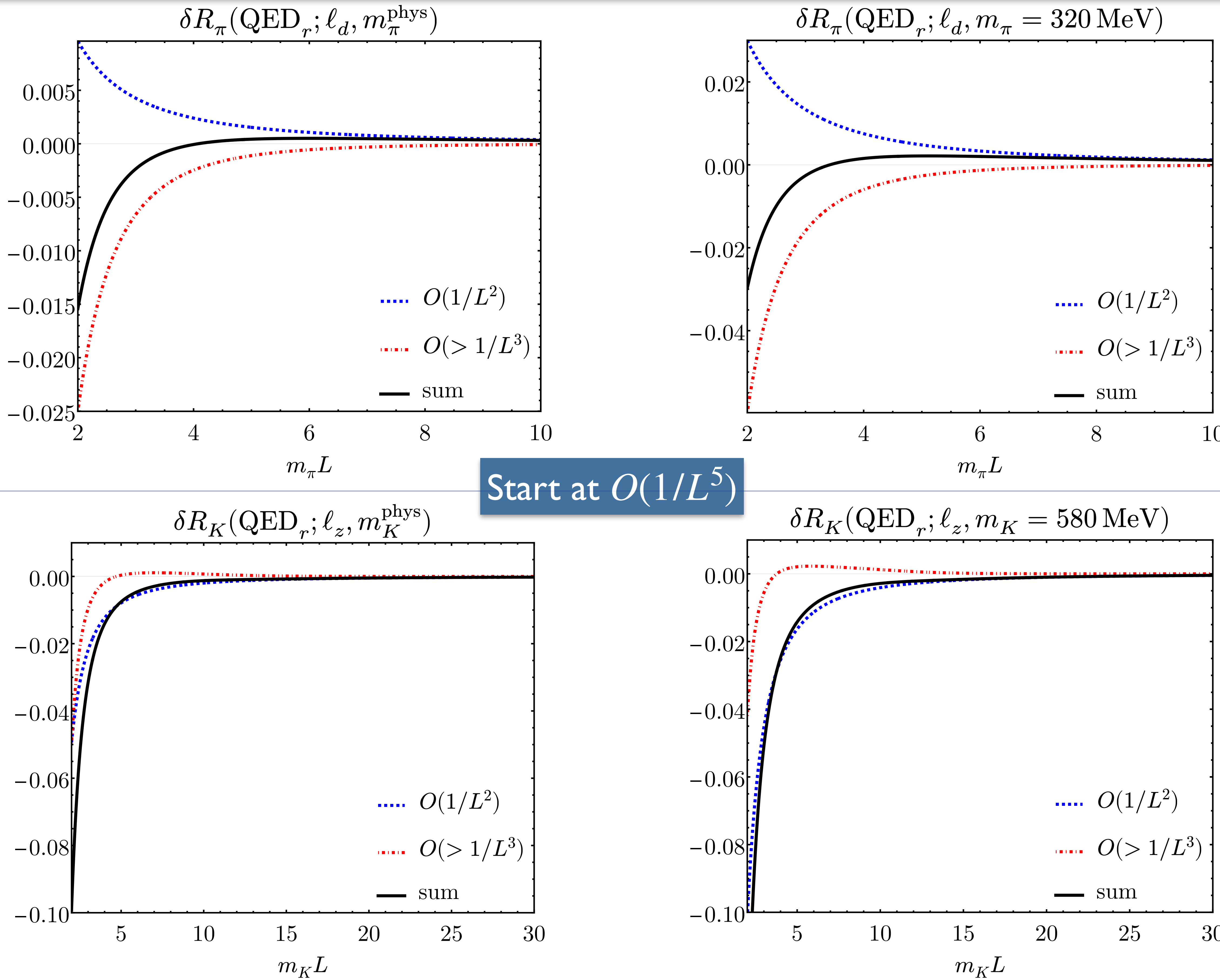




Non-trivial competition between $O(e^{-m_\ell L})$ and $O(e^{-m_P L})$ asymptotic behaviors

These corrections are independent of the FV QED prescription adopted

The pt HO power-law corrections for QED_r



π

K

Start at $O(1/L^5)$

Numerical impact

π

Finite-volume contributions to the IB correction δR_π, in units of 10^{-4}, for the physical charged-pion and muon masses at $L = 6$ fm.

$1/L$ expansion	$\hat{\ell} = (0,0,1)$		$\hat{\ell} = (1,1,1)/\sqrt{3}$	
	QED_L	QED_r	QED_L	QED_r
$c + \log L$	-150.4	-152.3	-150.4	-152.3
$1/L$	4.5	4.0	4.5	4.0
$1/L^2$	32.6	21.0	32.8	21.3
$1/L^3$	-31.7	$\equiv 0$	-31.7	$\equiv 0$
$> 1/L^3$	$\equiv 0$	-20.0	$\equiv 0$	-20.0
exp	6.0		5.9	
total	-139.0	-141.2	-138.9	-141.2

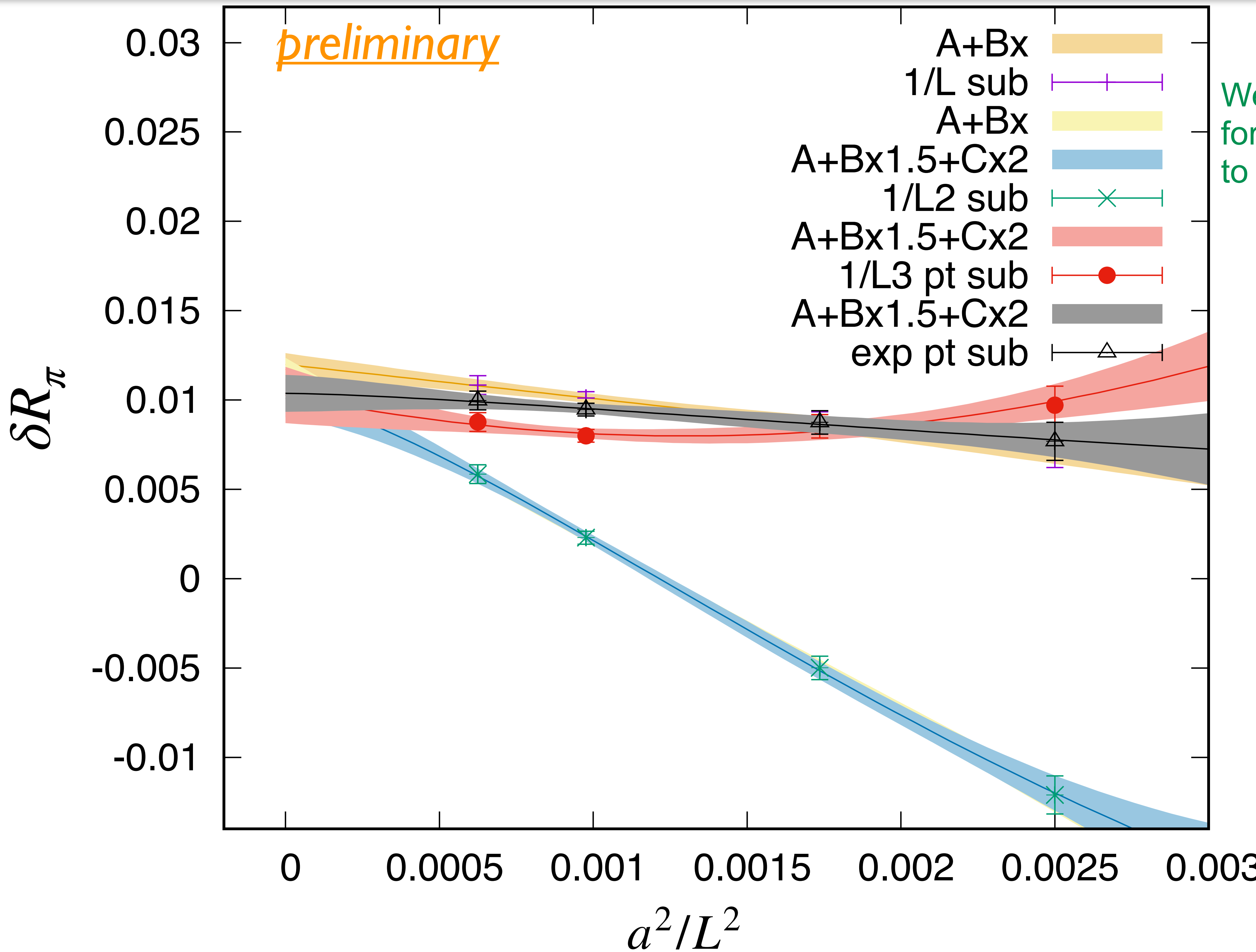
K

Finite-volume contributions to the IB correction δR_K, in units of 10^{-4}, for the physical charged-kaon and muon masses at $L = 6$ fm.

$1/L$ expansion	$\hat{\ell} = (0,0,1)$		$\hat{\ell} = (1,1,1)/\sqrt{3}$	
	QED_L	QED_r	QED_L	QED_r
$c + \log L$	-224.0	-232.0	-200.0	-203.3
$1/L$	6.1	5.6	5.2	4.6
$1/L^2$	-3.2	-8.8	11.0	8.8
$1/L^3$	-1.9	$\equiv 0$	-1.9	$\equiv 0$
$> 1/L^3$	$\equiv 0$	1.3	$\equiv 0$	-0.0
exp	3.1		1.2	
total	-220.0	-230.8	-184.5	-188.7

- The pt exp-suppressed FVEs for both channels are comparable in magnitude to the combined power-law corrections and to the size of the $1/L$ ones
- $\mathrm{QED_r}$ redistributes a significant part of the large $1/L^3$ correction (especially for pion channel) into the HO power-law tail. Like the $1/L^3$ term in $\mathrm{QED_L}$, this remainder is largely canceled by the $1/L^2$ contribution

First application to RM123S data



We thank G. Martinelli for giving us permission to use RM123S data

Remaining FV dep. is mild and compatible with an $O(1/L^3)$ term and HO SD effects

Conclusions and future perspectives

- We have computed the exp-suppressed FV corrections to the pt em contribution to the inclusive leptonic decay rates of charged pions and kaons. A paper on that will appear soon!

- Our exact expressions reveal a subtle interplay among terms governed by different exponential scales, which can be removed analytically from lattice data.

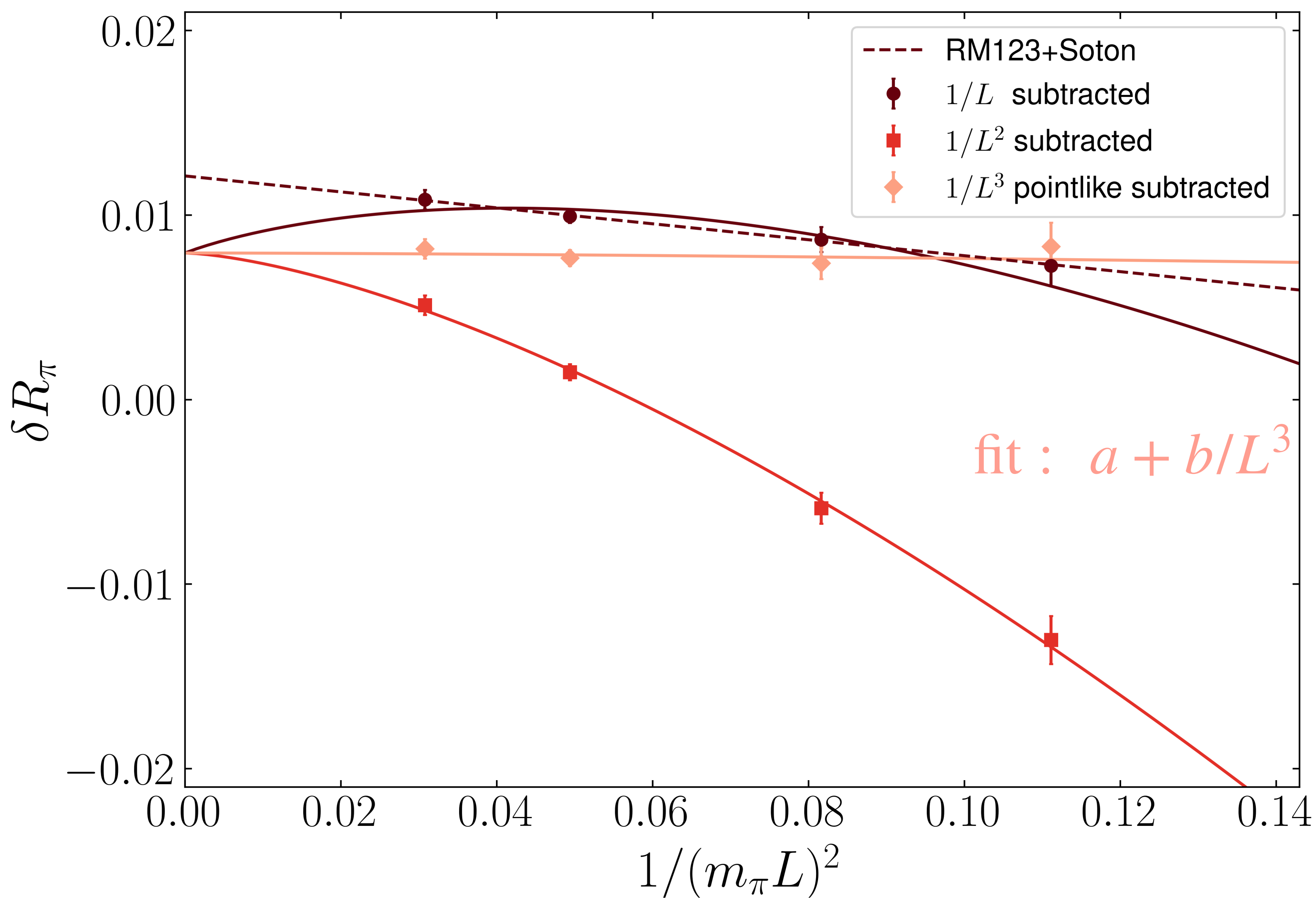
- $\mathrm{QED_r}$ redistributes a significant part of the large $1/L^3$ correction into the HO power-law remainder

- At the volumes used in present lattice simulations, these corrections are numerically comparable to the combined power-law contributions and subtracting them improves FV extrapolation

Backup slides

Observation raised by RBC/UKQCD

P. Boyle *et al.*, arXiv:2211.12865



RBC/UKQCD argues that the RM123S data can be described by $1/L^3$ behavior in order to get an estimate of the residual SD FVEs. This leads to a change in the FV extrapolated value, according to P. Boyle *et al.*

This puzzle is solved once the pt exp-suppressed FV terms are subtracted from the data, as we have shown here