Probing Criticality in the (1+1)D Real Scalar phi4 Theory with Entanglement Entropy and HOTRG

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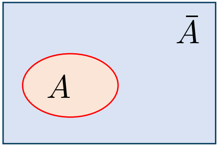
Motivation

- **Phase structure analysis in field theory**
  - Ultimate goal is to make clear the QCD phase diagram.
  - Precise numerical analysis of critical phenomena is one of the important research topics.
- **Entanglement Entropy (EE)**
  - Measure of quantum entanglement in a many-body system.
  - EE can be used as a probe of the quantum critical point.
- **Effective Central Charge (CC)**
  - CC can be extracted from EE.
  - CC also can be used as a probe of the quantum critical point

Entanglement Entropy

- **Defined by the von Neumann entropy, using the density matrix ρ**

$$S_A = -\mathrm{Tr}_A\left[\rho_A \log \rho_A\right], \quad \rho_A = \mathrm{Tr}_{\bar{A}}\, \rho$$



- **Difficulty of the naive calculation in QFT**
  - ρ has infinitely many degrees of freedom.
  - To calculate it efficiently,
    we take advantage of the tensor renormalization group (TRG)

Entanglement Entropy

- $(1+1)D$ **quantum system on a cylinder (PBC)**

Calabrese & Cardy, arXiv:hep-th/0405152v3

$$S_A(\ell, L) \begin{cases} = \dfrac{c}{3} \log\left[\dfrac{L}{\pi a} \sin\left(\dfrac{\pi \ell}{L}\right)\right] + \text{const.} & \text{(on critical)} \\ \approx \dfrac{c}{3} \log \xi + \text{const.} & \text{(off-critical)} \end{cases}$$

ℓ : Sub-system size, L : Total system size,
ξ : Correlation length, a : Lattice spacing,
c : Central charge.

Effective Central Charge

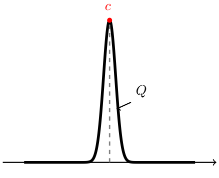
- **Defined as the difference in EE for the same L but different ℓ**

M. Campostrini et al., PRB 89, 094516 (2014)

$$Q(X,Y,L) \equiv 3 \times \left[\frac{S(XL,L) - S(YL,L)}{\log(\sin \pi X) - \log(\sin \pi Y)} \right], \quad (0 < Y < X < 1)$$

- **According to the previous relation,**

$$\lim_{L\to\infty} Q(X,Y,L) = \begin{cases} c & \text{(on critical)} \\ 0 & \text{(off-critical)} \end{cases}$$



- **We expect to detect the critical point from the behavior of Q.**

Entanglement Entropy with TRG

- **Previous studies in $(1+1)D$**
  - $O(2)$ planar Ising model: Yang et al., PRE 93 (2016) 012138
  - $O(3)$ NLσ model: X. Luo & Y. Kuramashi, JHEP03(2024)020
  - Ising model:
    EE for arbitrary sub-system size ℓ
    D. Kadoh, G. Tanaka, S. Takeda & TH, JHEP03(2026)002

- **Our target: $(1+1)D$ real scalar ϕ^4 theory**
  - One of the simplest interacting QFT models
  - Global $\mathbb{Z}_2$ symmetry: $\phi \to -\phi$
  - Ising universality class at criticality
  - **We use it as a benchmark for numerical calculations in QFT**

Model

- **Lattice action**

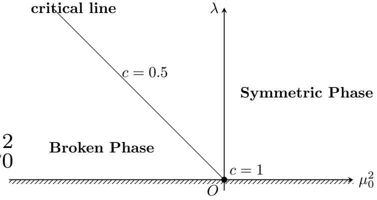
$$S_{\text{latt.}} = \sum_{n \in \text{ all lattice sites}} \left\{ \frac{1}{2} \sum_{\nu = x, \tau} (\phi_{n+\hat{\nu}} - \phi_n)^2 + \frac{\mu_0^2}{2}\phi_n^2 + \frac{\lambda}{4}\phi_n^4 \right\}$$

μ_0 : bare mass, $\lambda > 0$: coupling constant,

$\mu_0^2 = a^2 \mu_{0,\text{phys}}^2$, $\lambda = a^2 \lambda_{\text{phys}}$,

PBC: $\phi_{n+L_x\hat{x}} = \phi_n$.

- **If λ is fixed, we can detect $\mu_{0,c}^2$ by varying μ_0^2**



TN Representation for Partition Function

- The scalar fields are discretized using the Gauss-Hermite quadrature, and the path integral is represented as a network of finite-dimensional tensors.

- TN representation of partition function:

$$Z = \sum_{\{x,\tau\}} \prod_n T_{x_n \tau_n x_{n-\hat{1}} \tau_{n-\hat{2}}},$$

where $x_n, \tau_n, x_{n-\hat{1}}, \tau_{n-\hat{2}} = 1, 2, \ldots, D$

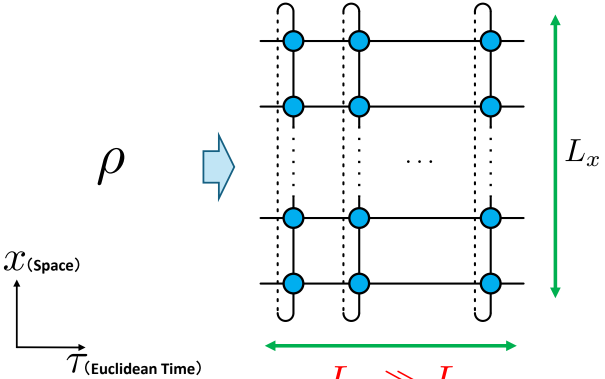
D: **Bond dimension**

TN Representation for Density Matrix

- $(1+1)D$ case at zero temperature

Yang et al., PRE 93 (2016) 012138

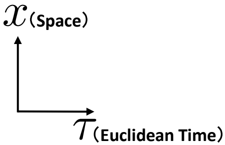
X. Luo & Y. Kuramashi, JHEP03(2024)020

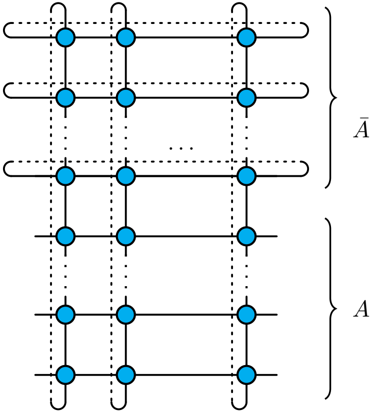


$L_\tau \gg L_x$

TN Representation for Density Matrix

$$\rho_A = \mathrm{Tr}_{\bar{A}}\ \rho$$

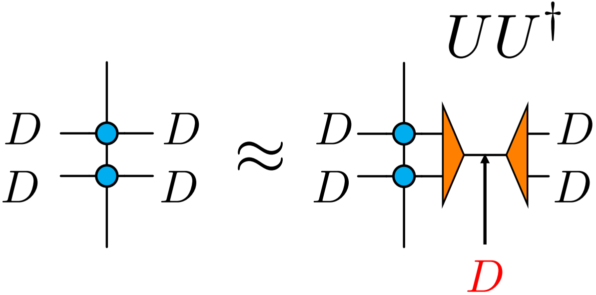


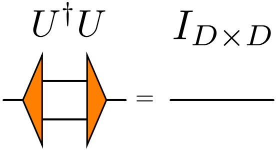


Coarse-graining

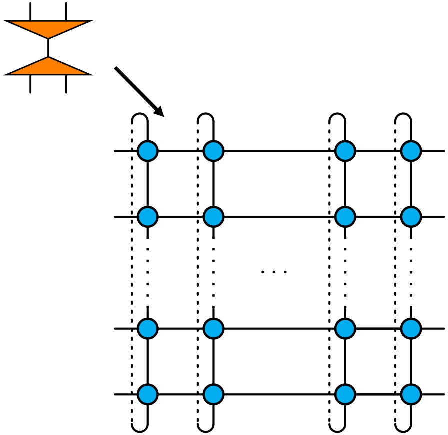
- **Higher-Order Tensor Renormalization Group (HOTRG)**

Z.Y. Xie et al., PRB 86, 045139 (2012)



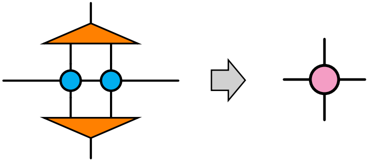


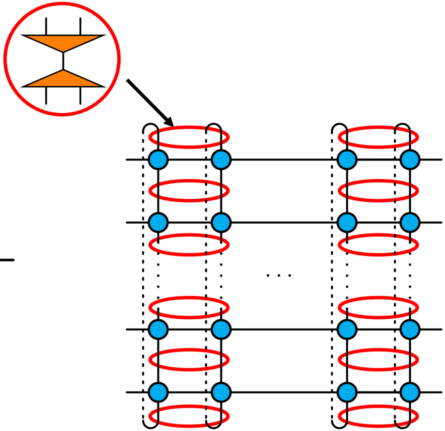
Coarse-graining



Coarse-graining

$O(D^7)$





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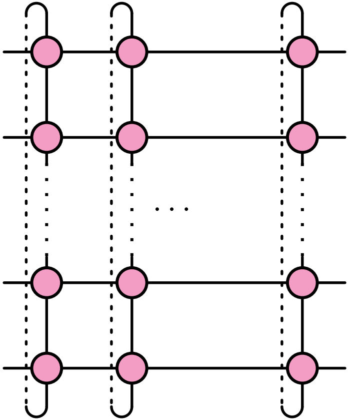

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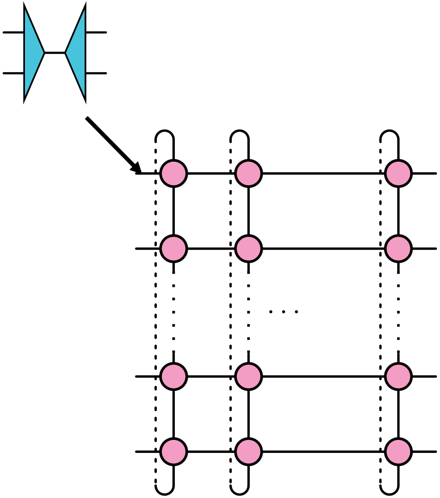

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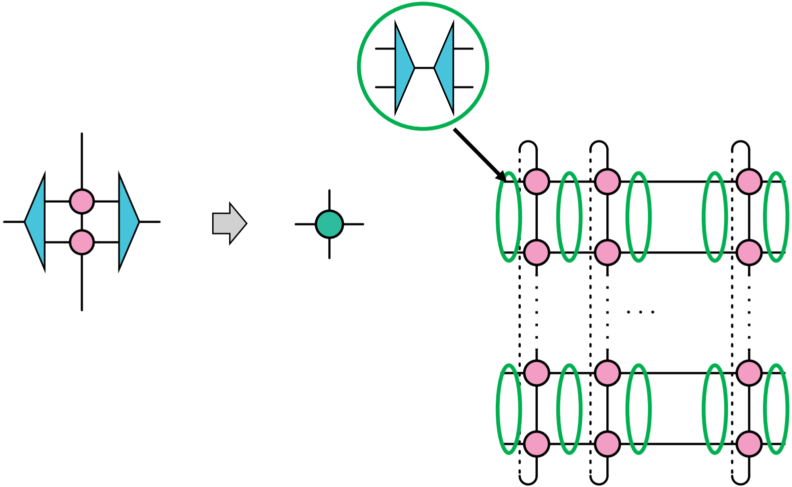

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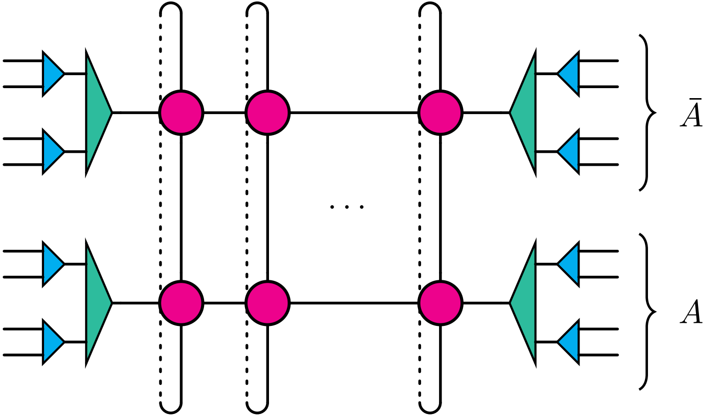
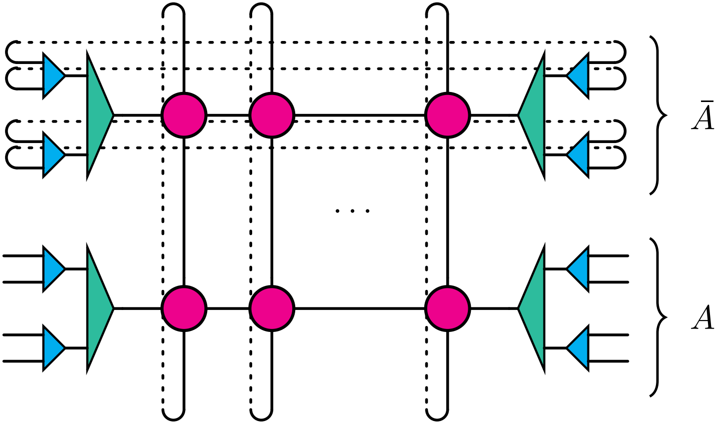


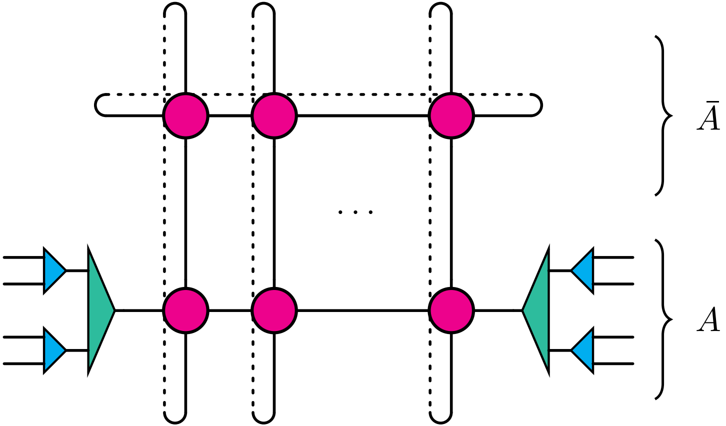
$\ell = L/2$

- $\rho_A = \mathrm{Tr}_{\bar{A}}\, \rho$
- **Outside isometries for $\bar{A}$ are cancelled out.**



$\ell = L/2$

- $S_A = -\mathrm{Tr}_A \left[\rho_A \log \rho_A\right]$
- **Unitary matrices do not contribute to EE.**
- **We can remove these isometries.**

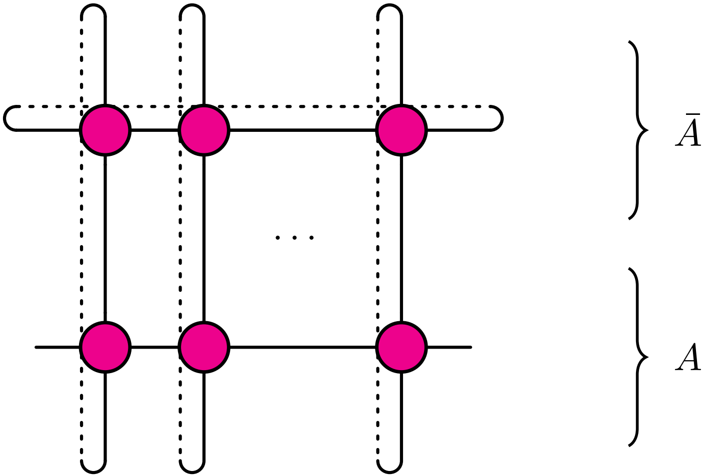


$\ell = L/2$

- **EVD**

$$\rho_A \to \{\sigma_i\}$$

$$S_A = -\mathrm{Tr}_A\left[\rho_A \log \rho_A\right]$$
$$\approx -\sum_{i=1}^{D} \sigma_i \log \sigma_i$$



Results: Entanglement Entropy S_A

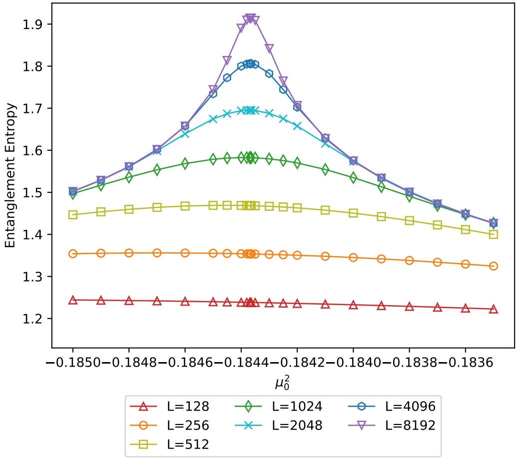
- **Preliminary survey**

  Coupling constant: $\lambda = 0.1$

  Aspect ratio: $L_\tau / L_x = 16$

  Sub-system size: $\ell = L_x/2$

  Bond dimension: $D = 96$



Results: Effective Central Charge Q

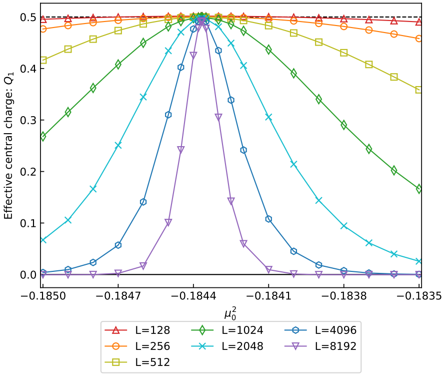
- **Preliminary survey**

  Coupling constant: $\lambda = 0.1$

  Aspect ratio: $L_\tau / L_x = 16$

  Bond dimension: $D = 96$

- **Peak height is nearly 0.5**
- **Peak position is almost L-independent!**



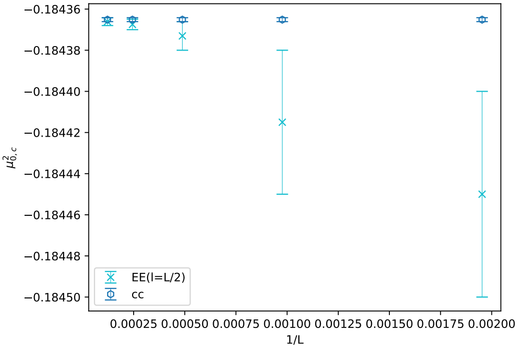
$$Q(X = 1/2, Y = 1/4, L_x) \equiv 3 \times \left[\frac{S(L_x/2, L_x) - S(L_x/4, L_x)}{\log(\sin \pi/2) - \log(\sin \pi/4)} \right]$$

Estimating peak position

- **Preliminary result**
- We estimate the peak position of the effective central charge Q using **Golden-section search.**

Method	$\mu_{0,c}^2$
Monte Carlo (2025)	$-0.18436329(97)$
This work	$-0.18436513(95)$

S. Dürr & T.S.H. Kiel PRD 113, 114520 (2026)



Conclusion

- **EE is calculated by using HOTRG as a benchmark in QFT.**
- **The effective central charge can also be extracted from EE.**
- $\mu_{0,c}^2$ **is estimated from the effective central charge for one** λ**.**
  - Bond dimension dependence is small.
  - L-dependence is small.

Future work

- Analyzing the critical point for smaller values of λ to take the continuum limit.
- Extension to 3D and 4D models

Gauss-Hermite Quadrature

$$\int_{-\infty}^{\infty} dy\, e^{-y^2}\, g(y) \approx \sum_{\alpha=1}^{K} w_\alpha g(y_\alpha)$$

- g: Given function
- e^{-y^2}: Weight function
- y_α: Root of the K-th Hermite polynomial
- $w_\alpha = 2^{K-1} K! \sqrt{\pi} / (K^2 H_{K-1}(y_\alpha)^2)$: Gauss-Hermite Weight

Partition function w/ TN

$$
\begin{aligned}
Z &= \int [d\phi] \exp\left(-S_{\mathrm{latt.}}\right) \\
&= \int [d\phi] \exp\left(-\sum_{n}\left[\frac{1}{2}\sum_{\nu}\left(\phi_{n+\hat{\nu}}-\phi_{n}\right)^{2}+V(\phi_{n})\right]\right) \\
&= \int [d\phi] \prod_{n,\nu} f(\phi_{n},\phi_{n+\hat{\nu}})
\end{aligned}
$$

$$
f(\phi,\varphi) = \exp\left[-\frac{1}{2}(\phi-\varphi)^{2}-\frac{1}{4}V(\phi)-\frac{1}{4}V(\varphi)\right], \quad V(\phi) = \frac{\mu_0^2}{2}\phi^2+\frac{\lambda}{4}\phi^4
$$

Partition function w/ TN

- **Using Gauss-Hermite quadrature, we can approximate this integral as following:**

$$Z = \int [d\phi] \prod_{n,\nu} f(\phi_n, \phi_{n+\hat{\nu}})$$
$$\approx \sum_{\{\alpha\}} \left[\prod_n w_{\alpha_n} e^{\phi^2_{\alpha_n}} \prod_\nu f(\phi_{\alpha_n}, \phi_{\alpha_{n+\hat{\nu}}}) \right]$$

Partition function w/ TN

- **Performing singular value decomposition(SVD),**

$$f(\phi,\varphi) \underset{\text{SVD}}{=} \sum_{k=1}^{\infty} \tilde{U}_k(\phi)\tilde{s}_k\tilde{V}_k^{\dagger}(\varphi) \approx \sum_{k=1}^{K} U_k(\phi)s_k V_k^{\dagger}(\varphi)$$

- **We obtain the TN representation of partition function:**

$$Z(K) = \sum_{\{x,\tau\}} = \sum_{\{x,\tau\}} \prod_n T(K)_{x_n \tau_n x_{n-\hat{1}} \tau_{n-\hat{2}}},$$

$$T(K)_{ijkl} = \sqrt{s_i s_j s_k s_l} \sum_{\alpha=1}^{K} w_\alpha e^{\phi_\alpha^2} U_{\alpha i} U_{\alpha j} V_{k\alpha}^{\dagger} V_{l\alpha}^{\dagger}$$

where $i,j,k,l = 1,\ldots,D$

Density matrix w/ TN

$$\rho_{\{\phi_f\},\{\phi_i\}} = \frac{1}{Z}\int_{\substack{\phi(\tau=0)=\phi_i \\ \phi(\tau=\beta)=\phi_f}} [d\phi] \exp\left(-S_{\mathrm{latt.}}\right)$$
$$\approx \frac{1}{Z(K)} \mathrm{tTr}_{\mathrm{bulk}} \left[\prod_n T(K)\right]_{\{\phi_f\},\{\phi_i\}}$$

$$\mathrm{Tr}\rho = 1, \qquad T(K)_{ijkl} = \sqrt{s_i s_j s_k s_l} \sum_{\alpha=1}^{K} w_\alpha e^{\phi_\alpha^2} U_{\alpha i} U_{\alpha j} V^\dagger_{k\alpha} V^\dagger_{l\alpha}$$

where $i, j, k, l = 1, \ldots, D$

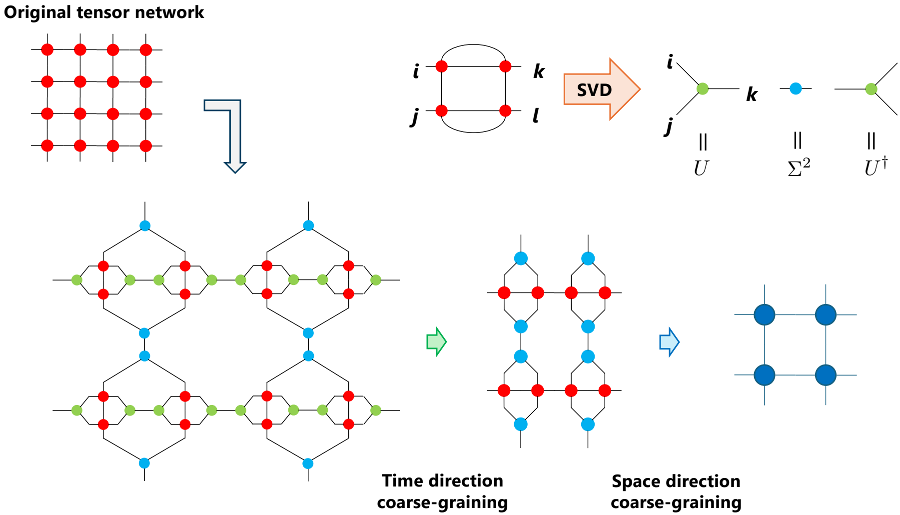
HOTRG

- **1 step coarse-graining of HOTRG**
  - Time direction
  - Space direction

- **Information compression via low-rank approximation**
  - **Singular value decomposition(SVD)**

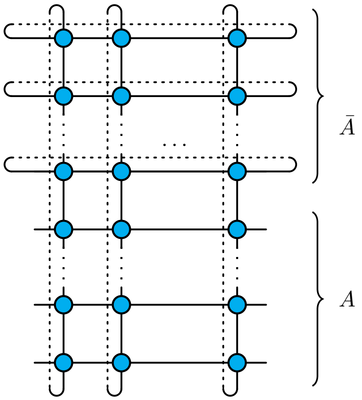
HOTRG

- U:
  Unitary matrix
- $UU^\dagger$:
  Isometry
- Σ^2:
  Singular value

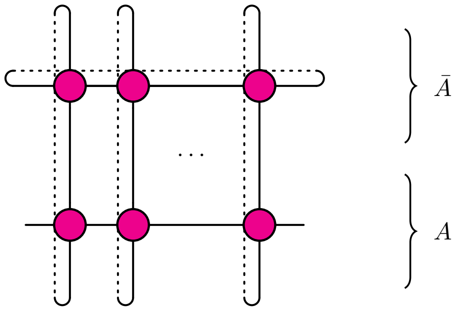


HOTRG Computational Cost

$$\mathcal{O}\left(D^{L_x L_\tau}\right)$$



$$\mathcal{O}\left(D^{7}\right)$$



HOTRG representation for Density Matrix for $\ell = L/4$

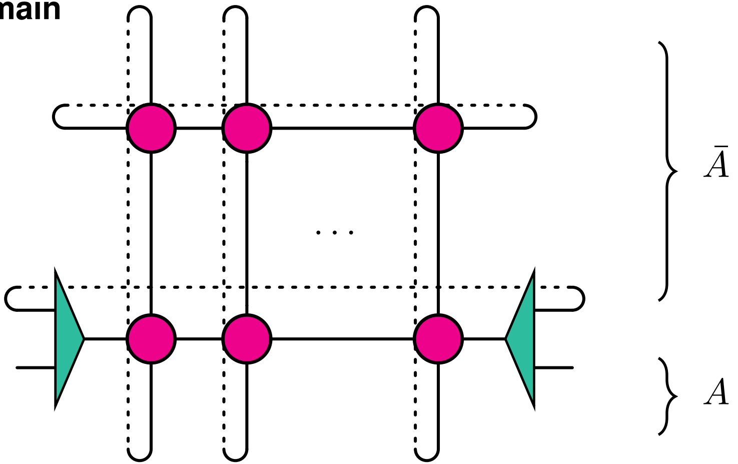
- **Two unitary matrices remain**

  $\cdots$ used in the previous coarse-graining step

- **EVD**

  $\tilde{\rho}_A \to \{\tilde{\sigma}_i\}$

$$
\begin{aligned}
S_A &= -\mathrm{Tr}_A\left[\tilde{\rho}_A \log \tilde{\rho}_A\right] \\
&\approx -\sum_{i=1}^{D} \tilde{\sigma}_i \log \tilde{\sigma}_i
\end{aligned}
$$



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- **Preliminary survey**

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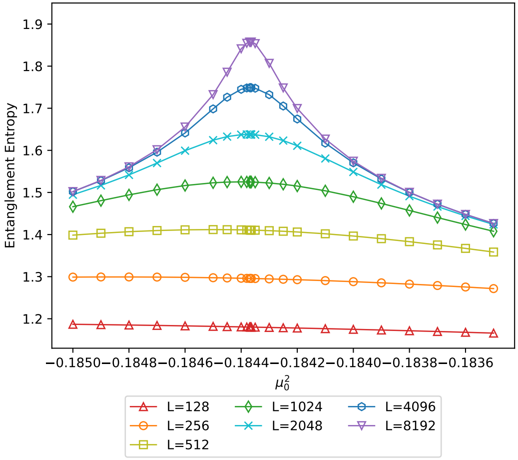
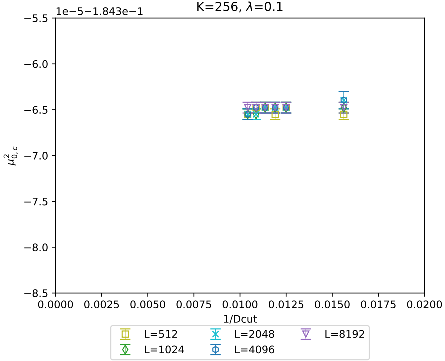


Results: D-dependence of peak position for Q

- **Preliminary results**
  The initial range is the same for all D.
  Updates until non-trivial D-dependence emerges.



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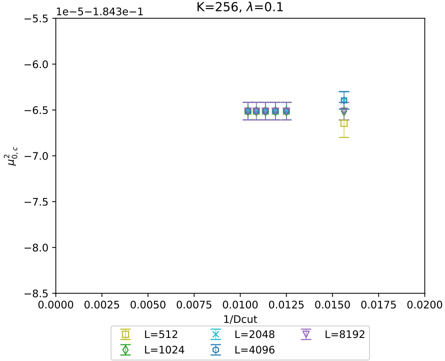

Go back one step and stop there.