ERIK BÄSKE · LATTICE FIELD THEORY GROUP · HUMBOLDT-UNIVERSITÄT ZU BERLIN

Spectral reconstruction of the inclusive Tau decay-rate with C^* boundary conditions

Including QED effects and quark mass splittings into the calculation of the inclusive decay-rate of the τ into hadrons

43rd International Symposium on Lattice Field Theory
University of Maryland, July 26th - August 1st 2026

§1 · SECTION ONE

Motivation.

Why care about $\Gamma(\tau \to X\nu_\tau)$?

High precision tests of the Standard Model

THE CKM-MATRIX

We want to calculate Cabibbo-Kobayashi-Maskawa matrix elements V_{ud}, V_{us}.

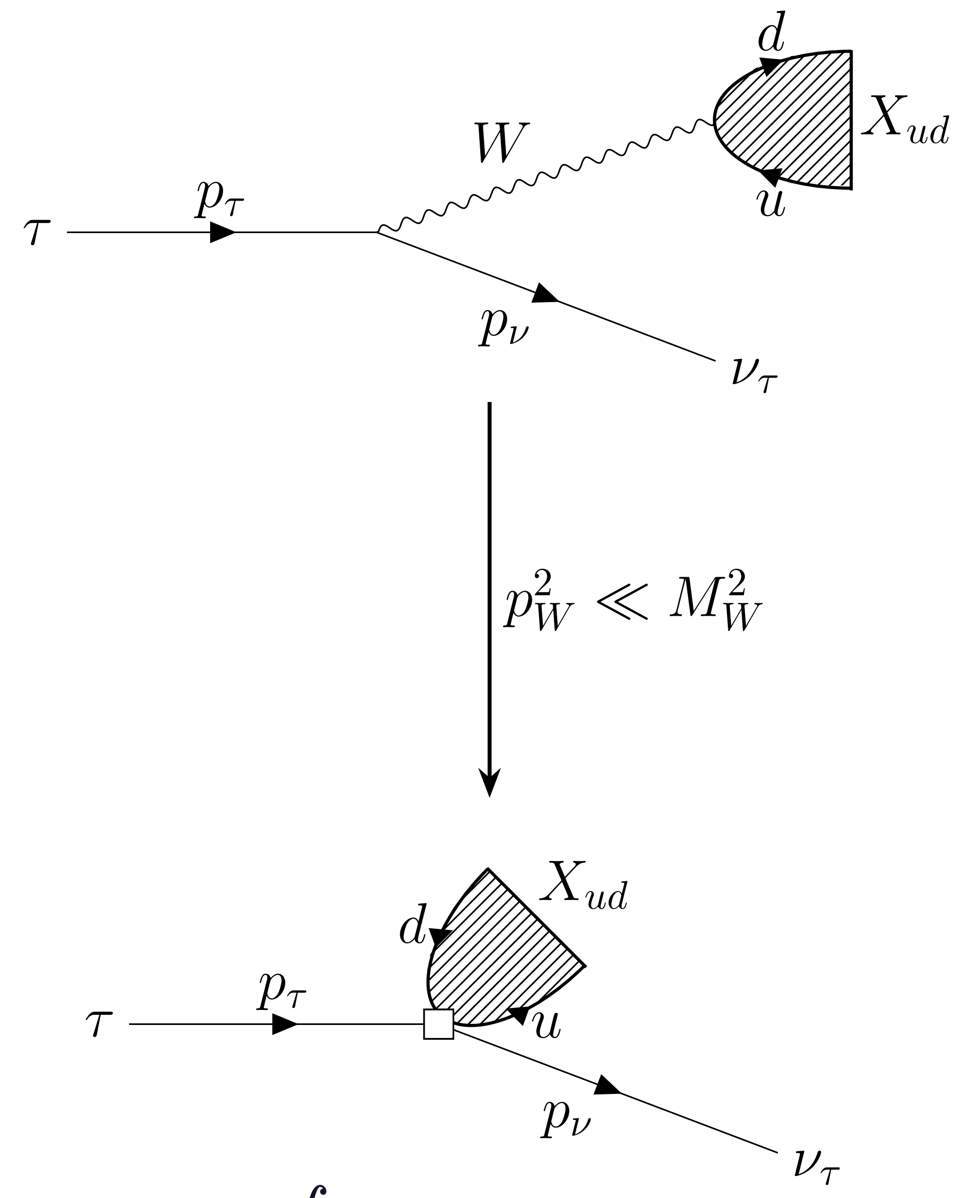
LAGRANGIAN

$$\mathcal{L}_{\mathrm{SM}} \subset -\frac{g}{2}\left(\bar{u}_L\ \bar{c}_L\ \bar{t}_L\right)\gamma_\mu W_\mu^+ V_{\mathrm{CKM}}\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \qquad V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Processes that depend on the CKM-matrix

- Tau decay $\tau \to X\nu$
- Semi-leptonic decays $K_L^0 \to \pi e \nu$ or $D \to \pi \ell \nu$ etc.
- β-decay
- ...

FERMI-THEORY



$$H_{\mathrm{eff}} \propto \int \mathrm{d}^3x\ \bar{u}_L\gamma_\mu d_L\ \bar{\tau}_L\gamma_\mu\nu_L$$

[PDG, Phys. Rev. D 110 (2024)], [FLAG, Phys.Rev.D 113 (2026)], [M. Davier et al., Rev. Mod. Phys. 78, 1043]

High precision tests of the Standard Model

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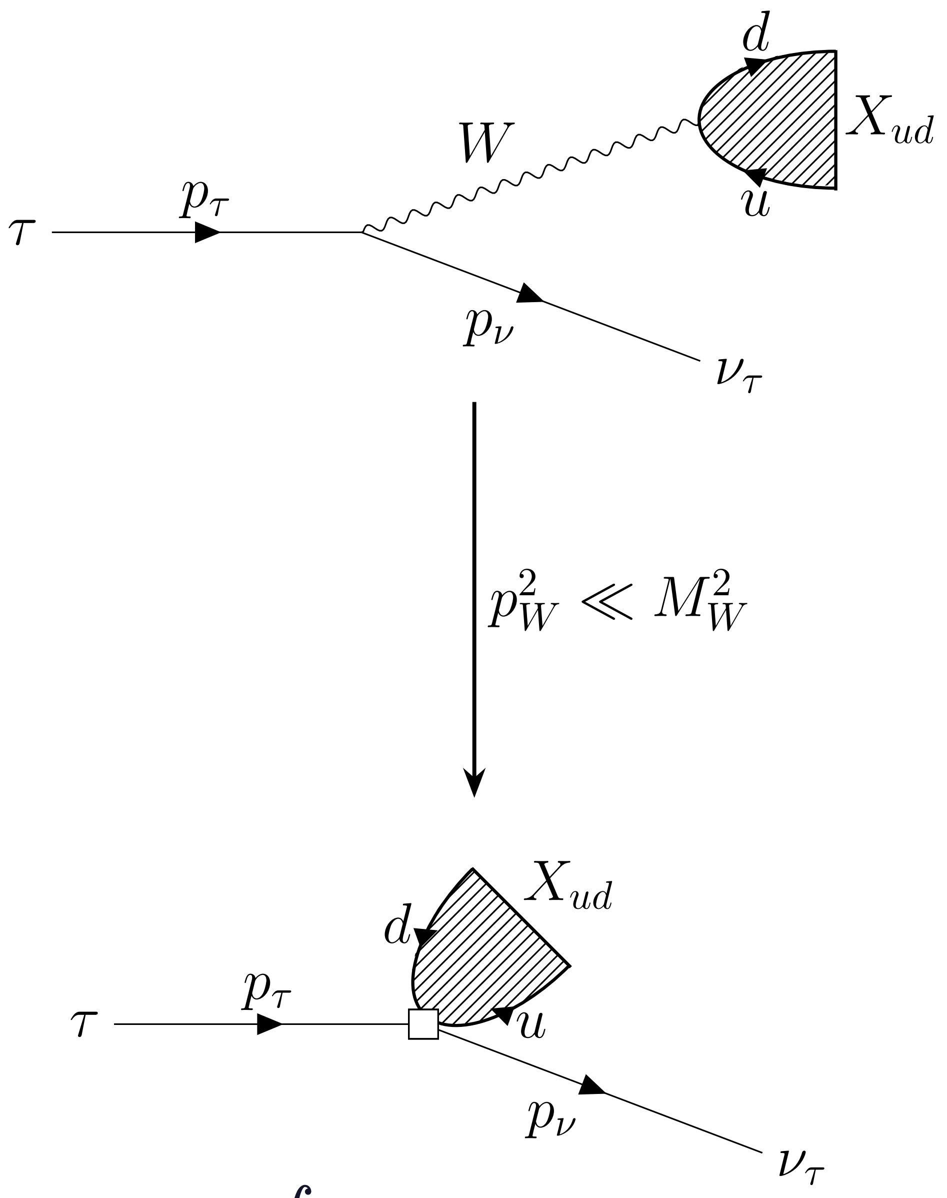
Processes that depend on the CKM-matrix

- **Tau decay** $\tau \to X\nu$
- Semi-leptonic decays
- β-decay
- ...

- simple initial state
- related to $(g-2)_\mu$
- sensitive to NP

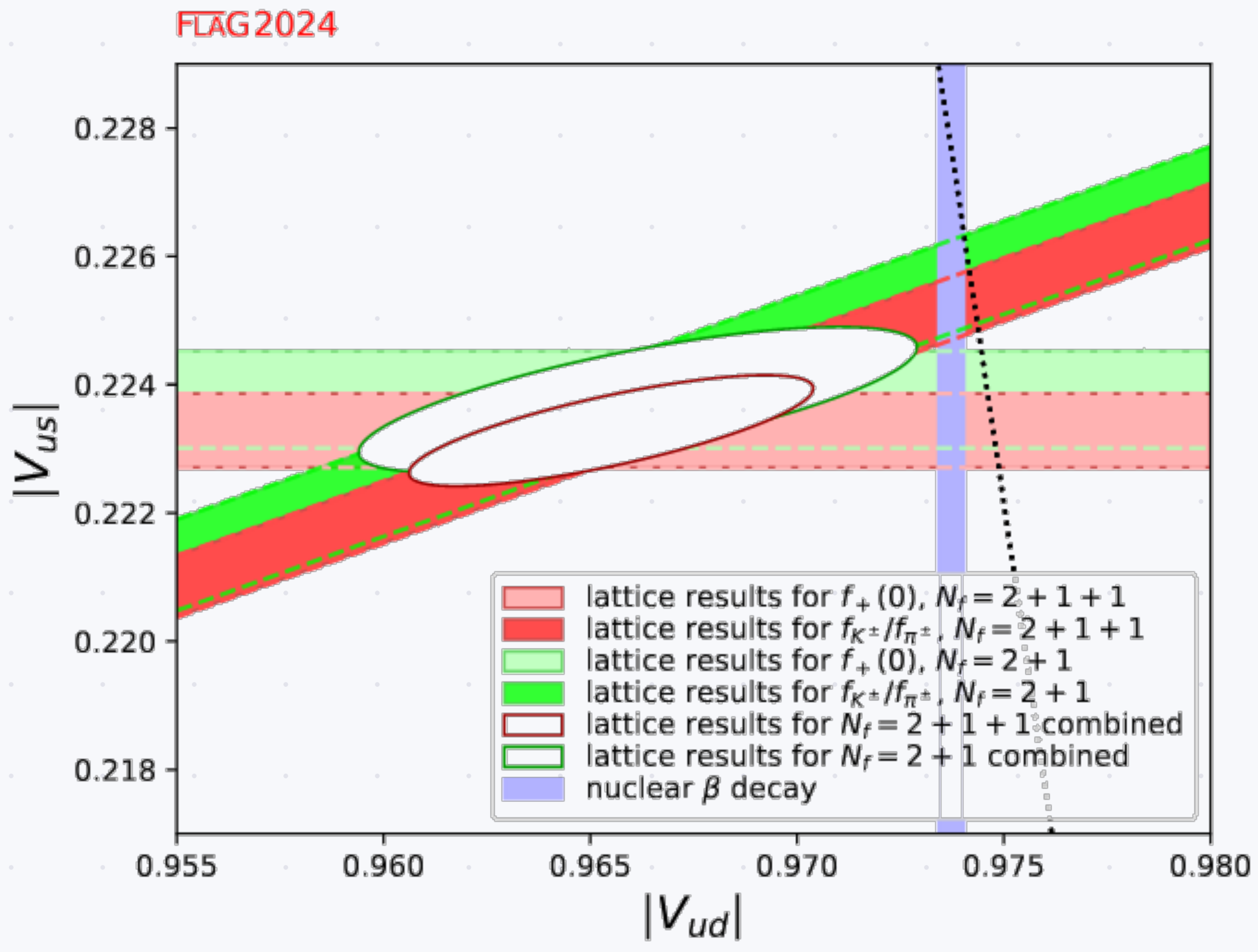
FERMI-THEORY

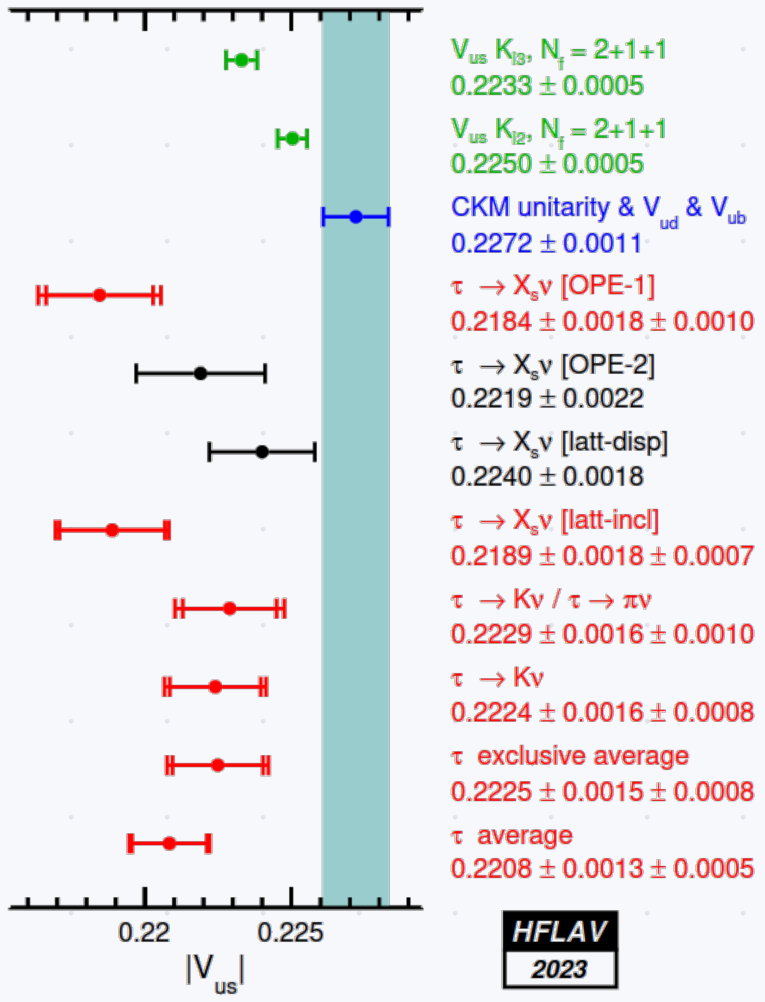


$$H_{\mathrm{eff}} \propto \int \mathrm{d}^3x\ \bar{u}_L\gamma_\mu d_L\ \bar{\tau}_L\gamma_\mu\nu_L$$

Current status of lattice determinations

Neglecting isospin-breaking effects:





DISCREPANCIES

Lattice results disagree ($2.8\,\sigma$) with experimentally measured β-decays and CKM-unitarity.

DISCREPANCIES

Lattice results for semi-leptonic Kaon decays and inclusive τ decays disagree ($2-3\sigma$)

Outline

§ 1 **Motivation**

Why care about $\Gamma(\tau \to X\nu_\tau)$?

§ 2 **The decay-rate**

Real time quantities from Euclidean data

§ 3 **Simulation set-up**

Simulations with dynamical τ and γ

§ 4 **Matrix Element from 4-point function**

Isolating the external τ states using the interpolating fields

§ 5 **Matrix Element from 2-point function**

Isolating the external τ states using the thermal trace

§ 6 **Outlook**

What we still have to do

§2 · SECTION TWO

The decay-rate

Real time quantities from Euclidean data

[ETMC, Phys.Rev.D 108 (2023)], [M. Di Carlo et al., arXiv:2603.29016], [B. Lucini et al., JHEP 02 (2016) 076], [RC* Collaboration, Eur.Phys.J.C 80 (2020)]

Old fashioned perturbation theory

DECAY-RATE

$$\Gamma(\tau \to X\nu) = \lim_{T\to\infty} \frac{1}{T} \sum_X \frac{|\langle X\nu|\tau\rangle|^2}{\langle X\nu|X\nu\rangle\langle\tau|\tau\rangle} = \frac{1}{2m_\tau}\langle\tau|\mathcal{H}(0)\,(2\pi)^4\delta^4(\hat{P}-p_i)\,\mathcal{H}(0)|\tau\rangle\left(1+\mathcal{O}(G_F^2)\right)$$

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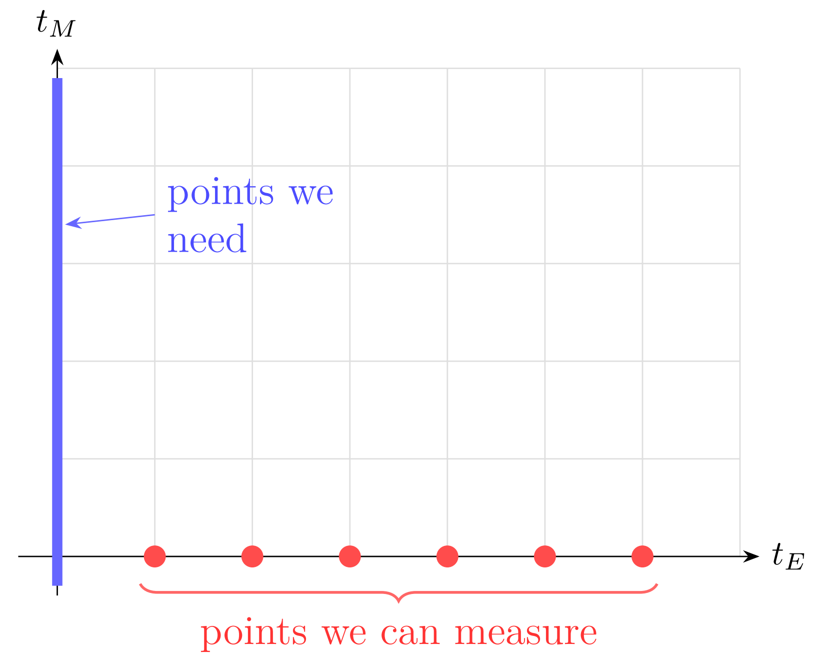
CHALLENGES:

- Analytic continuation/spectral reconstruction

$$\langle\tau|\mathcal{H}(0)\ \delta(H-m_\tau)\ \mathcal{H}(0)|\tau\rangle = \int \mathrm{d}t\ \langle\tau|\mathcal{H}(0)e^{-i(H-m_\tau)t}\ \mathcal{H}(0)|\tau\rangle$$
$$= \langle\tau|\mathcal{H}(it,\underline{0})\mathcal{H}(0)|\tau\rangle$$

ANALYTIC CONTINUATION



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CHALLENGES:

- Analytic continuation/spectral reconstruction
- Isolating the matrix element

$$\langle O_1 \ldots O_n\rangle_T \xrightarrow{?} \langle\tau|\ldots|\tau\rangle$$

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CHALLENGES:

- Analytic continuation/spectral reconstruction
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- QED effects: inclusion of dynamical τ and γ

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CHALLENGES:

- Analytic continuation/spectral reconstruction
- Isolating the matrix element
- QED effects: inclusion of dynamical τ and γ
- Renormalization of the effective electroweak Hamiltonian

[M. Hansen et al., Phys. Rev. D 99 (2019)], [ETMC, Phys.Rev.D 108 (2023)], [RM123 Collaboration, Phys.Rev.D 87 (2013)], [M. Bruno et al., arXiv:2607.00564]

Spectral reconstruction à la HLT

We can turn the *ill-posed* problem into a merely *ill-conditioned* one by first smearing the δ-function

SMEARING

$$\delta(x) = \lim_{\sigma\to 0} \delta_\sigma(x) = \lim_{\sigma\to 0}\lim_{\varepsilon\to 0} \sum_{n=0}^{N(\varepsilon)} c_n(\sigma) e^{-nx}$$

Expanding $\delta(\hat{H} - m_\tau)$ in a basis of $e^{-an(\hat{H}-m_\tau)}$ gives the decay-rate as

$$\Gamma(\tau) = \lim_{\sigma\to 0}\lim_{\varepsilon\to 0} \sum_{n=0}^{N(\varepsilon)} c_n(\sigma)\, \frac{1}{2} \sum_r \int \mathrm{d}^3x \, \langle \tau^r(\underline{0}) | \mathcal{H}(an, \underline{x}) \mathcal{H}(0) | \tau^r(\underline{0}) \rangle$$

determined by the HLT-method: optimize c_n s.t. kernel is reproduced accurately

matrix element at discrete Euclidean times

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In iso-QCD (neglecting isospin breaking or in the RM123 method) a very different kernel is used

$$\Gamma \propto \int \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3 2E_\nu} \ell_{\mu\nu}(p_\tau, p_\nu)\, \langle \Omega | J^\mu_{ud}(0)\, (2\pi)^4 \delta^4(P - (p_\tau - p_\nu)) J^{\nu\dagger}_{ud}(0) | \Omega \rangle$$

kernel from analytic lepton piece

hadronic spectral density

§4 · SECTION FOUR

Simulation set-up

Simulations with dynamical τ and γ

[K. Osterwalder and E. Seiler, Annals Phys. 110 (1978)], [Alpha Collaboration, JHEP 08 (2001)], [M. Di Carlo et al., arXiv:2603.29016], [B. Lucini et al., JHEP 02 (2016) 076], [RC* Collaboration, Eur.Phys.J.C 80 (2020)]

Parameters and action

ACTIONS

Fermion action

$$S_F[u,d,s,c,U,A] = S_{\text{Wilson}}[u,U,A] + S_{\text{Wilson}}[d,U,A] + S_{\text{Wilson}}[s,U,A] + S_{\text{Wilson}}[c,U,A]$$

Valence τ action

$$S_\tau^{\text{val}}[\tau, A] = \bar{\tau}\left(\tilde{\gamma}\cdot\nabla^S - \frac{ar}{2}\nabla_\mu^F\nabla_\mu^B + m_0 + \frac{i}{4}c_{\text{SW}}\sigma_{\mu\nu}F_{\mu\nu}^{\text{U(1)}} + i\gamma_5\mu\right)\tau$$

Gauge action

$$S_G[U,A] = S_{\text{LW}}^{\text{SU(3)}}[U] + S_{\text{Wilson}}^{\text{U(1)}}[A]$$

PARAMETERS

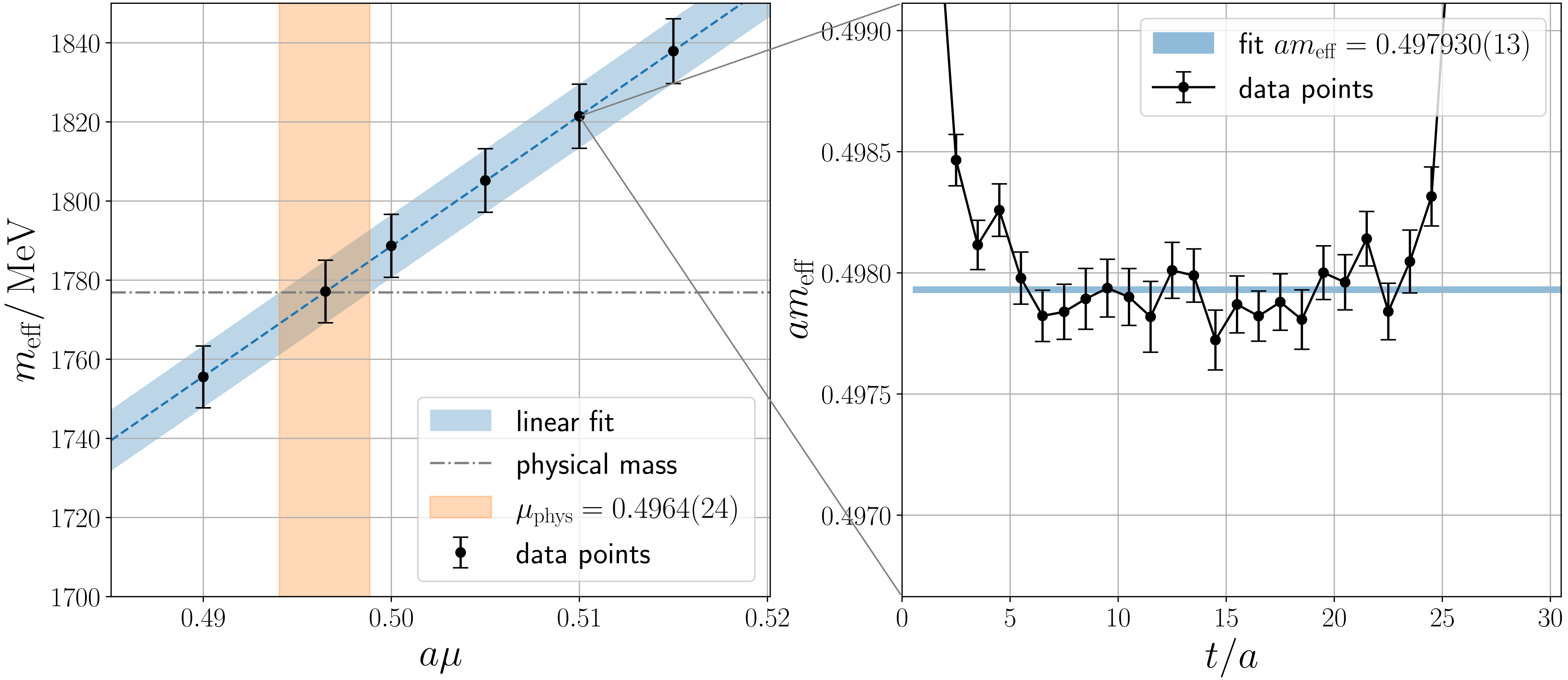
Lattice size	64×32^3
Boundary conditions	periodic in time C^* in $k=3$ spatial directions
Lattice spacing	$\beta = 3.24$ $a = 0.0539(2)\,\text{fm}$
Quark flavours	$N_f = 1+2+1$
Pion mass	$m_\pi \approx 380\,\text{MeV}$ $m_\pi L \approx 3$
Electromagnetic coupling	$\alpha_{\text{EM}} = 1/137$
Mass of the τ	$m_\tau = 1777(8)\,\text{MeV}$
Number of configurations	$N_{\text{cnfg}} = \mathcal{O}(100)$

C^* BOUNDARY CONDITIONS

$$\psi(x+\hat{L}_1/2) = \psi(x)^{\mathcal{C}} = C^{-1}\bar{\psi}(x)^T$$
$$\bar{\psi}(x+\hat{L}_1/2) = \bar{\psi}(x)^{\mathcal{C}} = -\psi^T(x)C$$
$$U_\mu(x+\hat{L}_1/2) = U_\mu(x)^{\mathcal{C}} = U_\mu(x)^*$$

Tuning of the τ

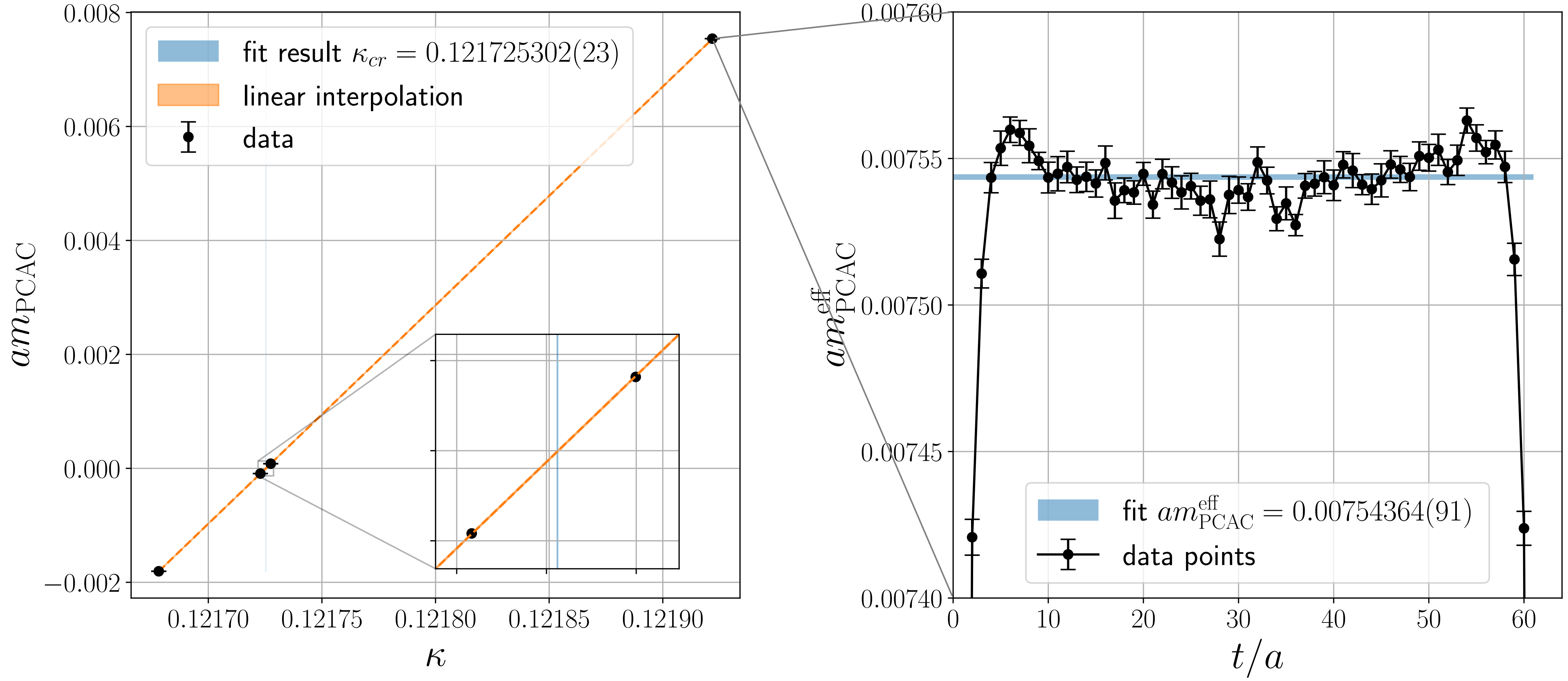
PHYSICAL MASS INTERPOLATION



$$C_2(t) = \langle \tau(t)\bar{\tau}(0)\rangle = \frac{Z_\tau}{2m_\tau}\sum_r \left(e^{-tm_\tau}u^r(\underline{0})\bar{u}^r(\underline{0}) + e^{-(T-t)m_\tau}v^r(\underline{0})\bar{v}^r(\underline{0})\right) + \ldots$$

Tuning of the τ

MAXIMAL TWIST INTERPOLATION



$$m_{\mathrm{PCAC}}(t - t_0) = \frac{\langle \sum_{x,y} \partial_0^S A_0^1(t,\underline{x}) P^1(t_0,\underline{y}) \rangle}{2\langle \sum_{x,y} P^1(t,\underline{x}) P^1(t_0,\underline{y}) \rangle}$$

with mass term $m_0 + i\mu\gamma_5\sigma^3$, maximal twist at $m_R \sim m_{\mathrm{PCAC}} = 0$

§3 · SECTION THREE

The matrix element from a 4-point function

Isolating the external τ states using the interpolating fields

The matrix element using interpolating fields

Our task is now to calculate the matrix element

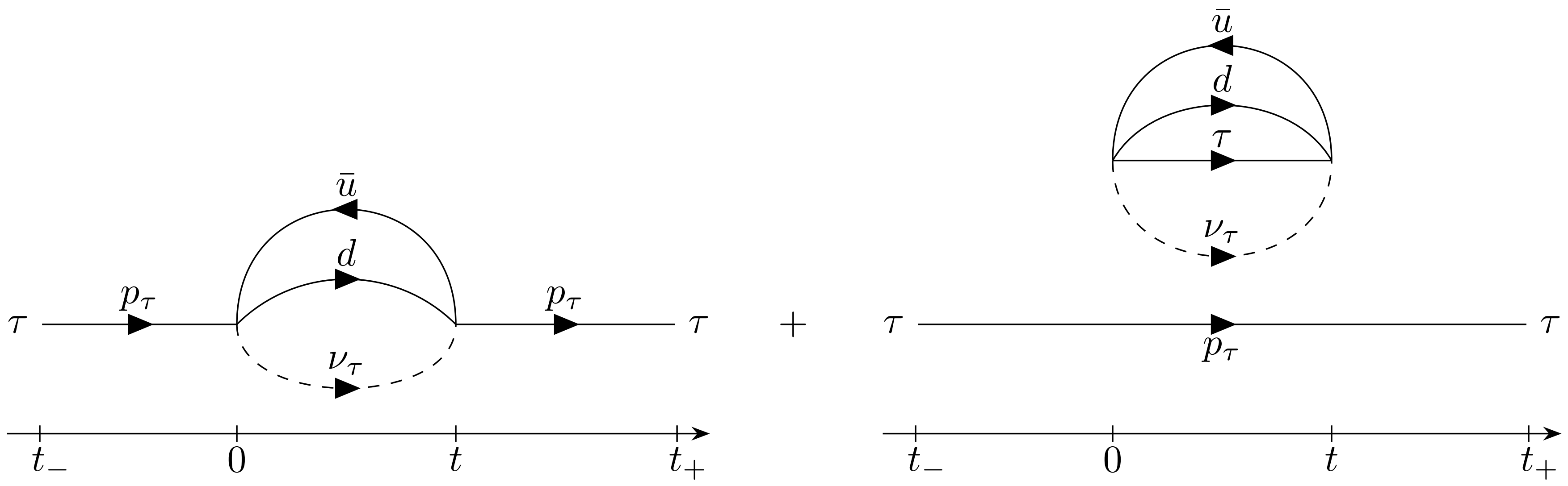
$$\frac{1}{2}\sum_r \langle \tau^r(\underline{0})|\mathcal{H}(t,\underline{x})\mathcal{H}(0)|\tau^r(\underline{0})\rangle$$

Using the interpolating field of the τ

$$\langle\Omega|\tau(t_+)\ldots\bar{\tau}(t_-)|\Omega\rangle = \langle\tau|\ldots|\tau\rangle\Big(\langle\Omega|\tau|\tau\rangle\langle\tau|\bar{\tau}|\Omega\rangle e^{-(t_+-t_-)m_\tau}\Big) + \ldots$$

TARGET 4-POINT FUNCTION

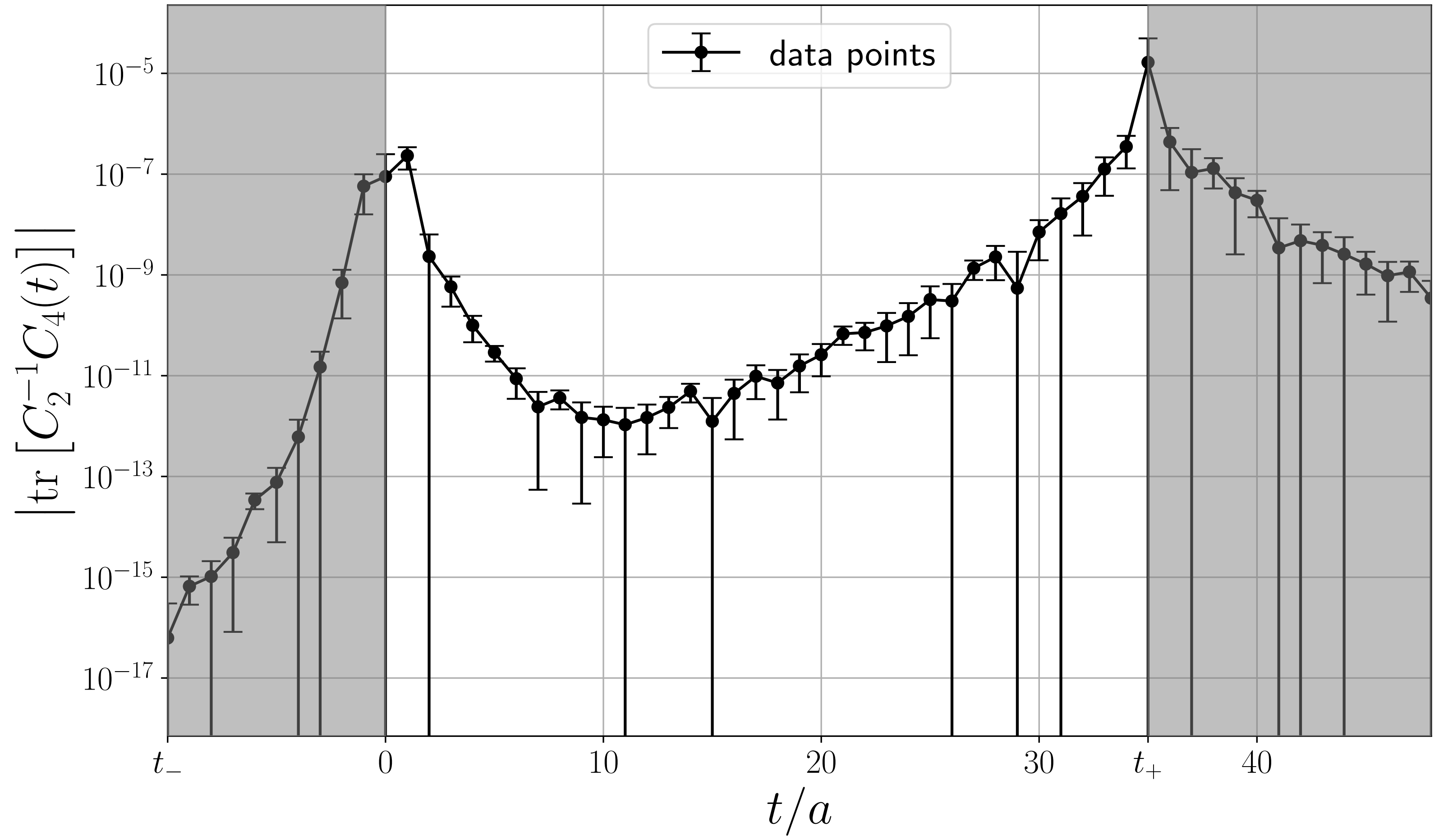
$$C_4(t_+,t,0,t_-) = \langle 0|\tau(t_+,\underline{p}=\underline{0})\;\mathcal{H}(t,\underline{x})\;\mathcal{H}(0)\;\bar{\tau}(t_-,\underline{p}=\underline{0})|0\rangle$$



Extracting the matrix element

Need to project the 4×4 spinor matrix to the trace over physical polarizations

MATRIX ELEMENT DATA



$$\mathrm{tr}\left[C_2^{-1}(t_+ - t_-)C_4(t_+, t, 0, t_-)\right] = \sum_r \langle \tau^r(\underline{0})|\mathcal{H}(t)\mathcal{H}(0)|\tau^r(\underline{0})\rangle + \ldots$$

§5 · SECTION FIVE

The matrix element from a 2-point function

Isolating the external τ states using the thermal trace

Computing the matrix element using a trace

Recall that with periodic boundary conditions we compute

$$\langle O \rangle_T = \frac{1}{Z(T)} \mathrm{tr}\left[e^{-TH} O\right] = \frac{1}{Z(T)} \left(e^{-TE_0} \langle \Omega | O | \Omega \rangle + \cdots + e^{-Tm_\tau} \sum_r \langle \tau^r(\underline{0}) | O | \tau^r(\underline{0}) \rangle + \dots \right)$$

How to isolate this part?

Computing the matrix element using a trace

Recall that with periodic boundary conditions we compute

$$\langle O \rangle_T = \frac{1}{Z(T)} \mathrm{tr}\left[e^{-TH} O\right] = \frac{1}{Z(T)} \left(e^{-TE_0} \langle \Omega | O | \Omega \rangle + \dots + e^{-Tm_\tau} \sum_r \langle \tau^r(\underline{0}) | O | \tau^r(\underline{0}) \rangle + \dots \right)$$

How to isolate this part?

Idea: τ-number is conserved (QCD+QED)

$$\tau \to e^{i\alpha} \tau \qquad \bar{\tau} \to \bar{\tau} e^{-i\alpha} \qquad \Leftrightarrow \qquad \hat{U}(\alpha) = e^{i\alpha \hat{N}_\tau} \qquad \hat{U}(\alpha) | \psi \rangle = e^{i\alpha N_\tau(\psi)} | \psi \rangle$$

$\Rightarrow$ projector to odd τ-number

$$\langle \mathbb{P}_{\mathrm{odd}} O \rangle_T = \left\langle \frac{1 - (-1)^{\hat{N}_\tau}}{2} O \right\rangle_T = \frac{1}{Z(T)} e^{-Tm_\tau} \sum_r \langle \tau^r(\underline{0}) | O | \tau^r(\underline{0}) \rangle + \begin{cases} \tau(\underline{p}) \\ \tau + \pi \\ \tau + \gamma \\ \tau\tau\tau \\ \dots \end{cases}$$

Flavour projection in practice

How to calculate $\langle \mathbb{P}_{\mathrm{odd}} O \rangle_T$ in practice?
Recall: thermal expectation value

$$\begin{aligned}\langle O \rangle_T &= \frac{1}{Z(T)} \sum_n \langle n | e^{-TH} O | n \rangle \\ &= \frac{1}{Z(T)} \int \mathcal{D}[U, A, \psi] e^{-S_{[0,T)}} O \Big|_{\tau(T) = -\tau(0)}\end{aligned}$$

Flavour projection in practice

How to calculate $\langle \mathbb{P}_{\text{odd}} O \rangle_T$ in practice?
With $U(\alpha) = e^{i\alpha N_\tau}$

$$\begin{aligned}\langle U(\alpha) O \rangle_T &= \frac{1}{Z(T)} \sum_n \langle n | U(\alpha) e^{-TH} O | n \rangle \\ &= \frac{1}{Z(T)} \int \mathcal{D}[U, A, \psi] e^{-S_{[0,T)}} O \Big|_{\tau(T) = -e^{i\alpha}\tau(0)}\end{aligned}$$

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$$= \frac{1}{Z(T)} \int \mathcal{D}[U, A, \psi] e^{-S_{[0,T)}} O \Big|_{\tau(T) = -e^{i\alpha}\tau(0)}$$

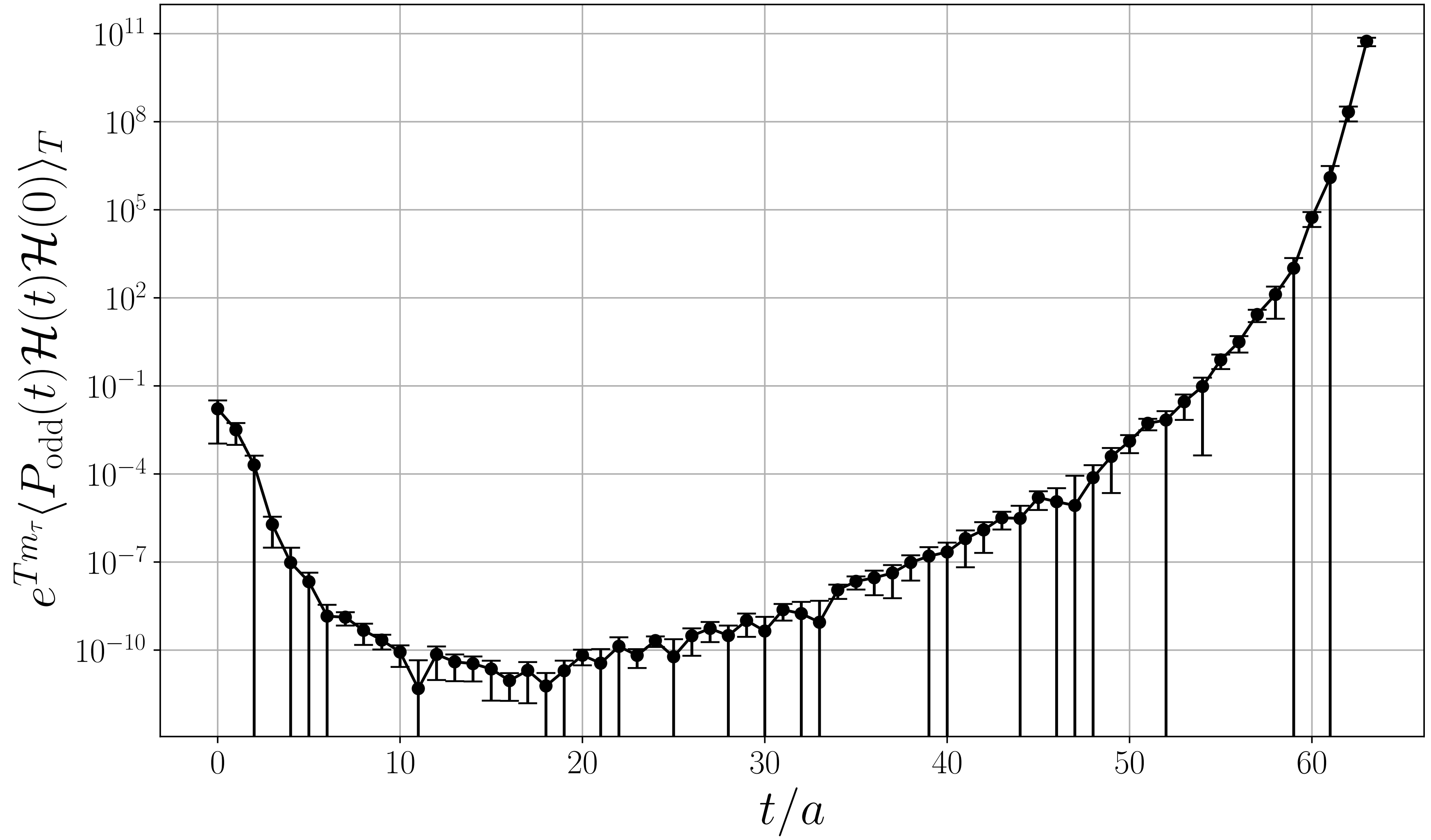
FLAVOUR PROJECTION

$$\langle \mathbb{P}_{\text{odd}} O \rangle_T = \frac{1}{2} \langle O \rangle_T - \frac{1}{2} \langle (-1)^{N_\tau} O \rangle_T = \frac{1}{2} \left(\langle O \rangle \Big|_{\tau \text{ anti-periodic}} - \langle O \rangle \Big|_{\tau \text{ periodic}} \right)$$

Extracting the matrix element

MATRIX ELEMENT DATA

$$\langle \mathbb{P}_{\mathrm{odd}} \mathcal{H}(t)\mathcal{H}(0)\rangle_T = \frac{e^{-Tm_\tau}}{Z(T)} \sum_r \langle \tau^r(\underline{0})|\mathcal{H}(t)\mathcal{H}(0)|\tau^r(\underline{0})\rangle + \dots$$



§6 · SECTION FIVE OF FIVE

Outlook & to-do list

What we still have to do

Outlook & to-do list

NOISE REDUCTION
Analyse source/gauge variance
try different sources
noise reduction techniques

↓

SPECTRAL RECONSTRUCTION
Reconstruct Γ using the HLT

↓

COMPARISON
Compare with RM123 results from collaborators at ETMC

RENORMALIZATION
Define suitable renormalization conditions and measure renormalization constants

↓

MATCHING TO THE SM
Run to high energies to match to perturbation theory

INFINITE-VOLUME LIMIT
Analyse finite volume effects (more severe for QED)

↓

CONTINUUM LIMIT
Implement $\mathcal{O}(a)$-improvement and take $a \to 0$ limit

↓

GOING TO THE PHYSICAL POINT
Current ensembles are far away from the physical point
$m_\pi \approx 380\,\mathrm{MeV} \gg 135\,\mathrm{MeV}$

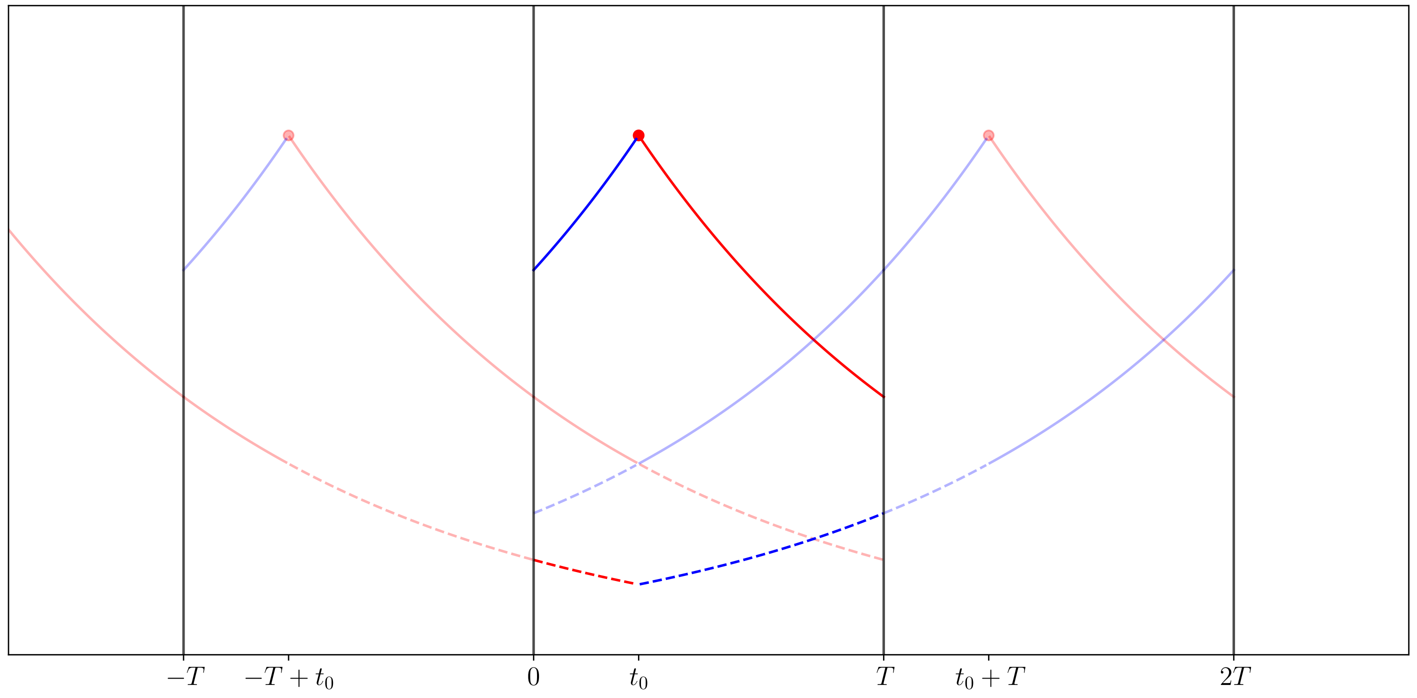
Appendix

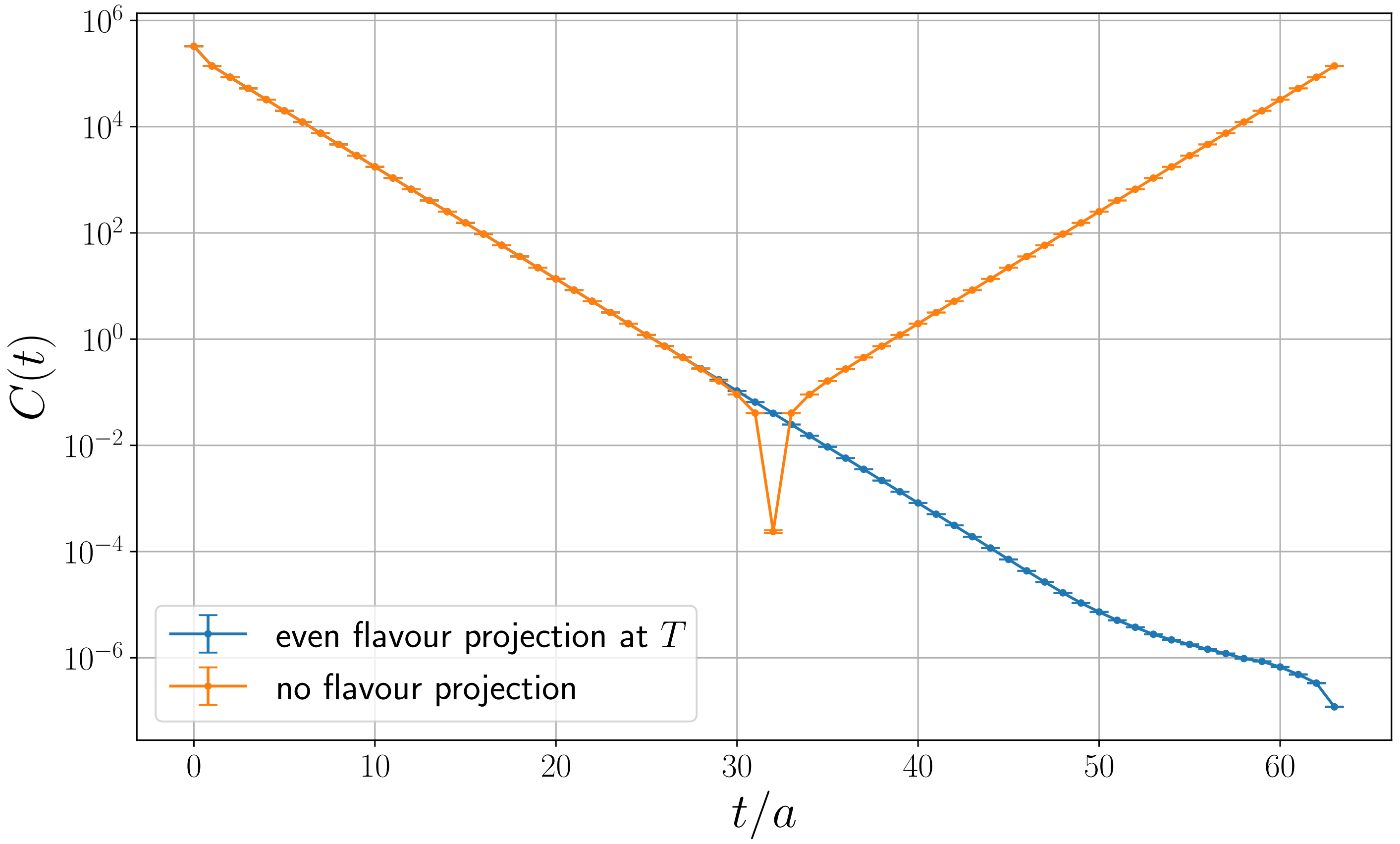
Flavour projection in action

2-POINT FUNCTION OF THE TAU

$$C_2(t,t_0) = \langle \tau(t)\bar{\tau}(t_0)\rangle_T \sim \langle\Omega|\tau|\tau\rangle\langle\tau|\bar{\tau}|\Omega\rangle e^{-tm_\tau} + \langle\bar{\tau}|\tau|\Omega\rangle\langle\Omega|\bar{\tau}|\bar{\tau}\rangle e^{-(T-t)m_\tau} + \ldots$$

$$C_2^{\text{even}}(t,t_0) = \langle \mathbb{P}_{\text{even}}\tau(t)\bar{\tau}(t_0)\rangle_T \sim \langle\Omega|\tau|\tau\rangle\langle\tau|\bar{\tau}|\Omega\rangle e^{-tm_\tau} + \ldots$$



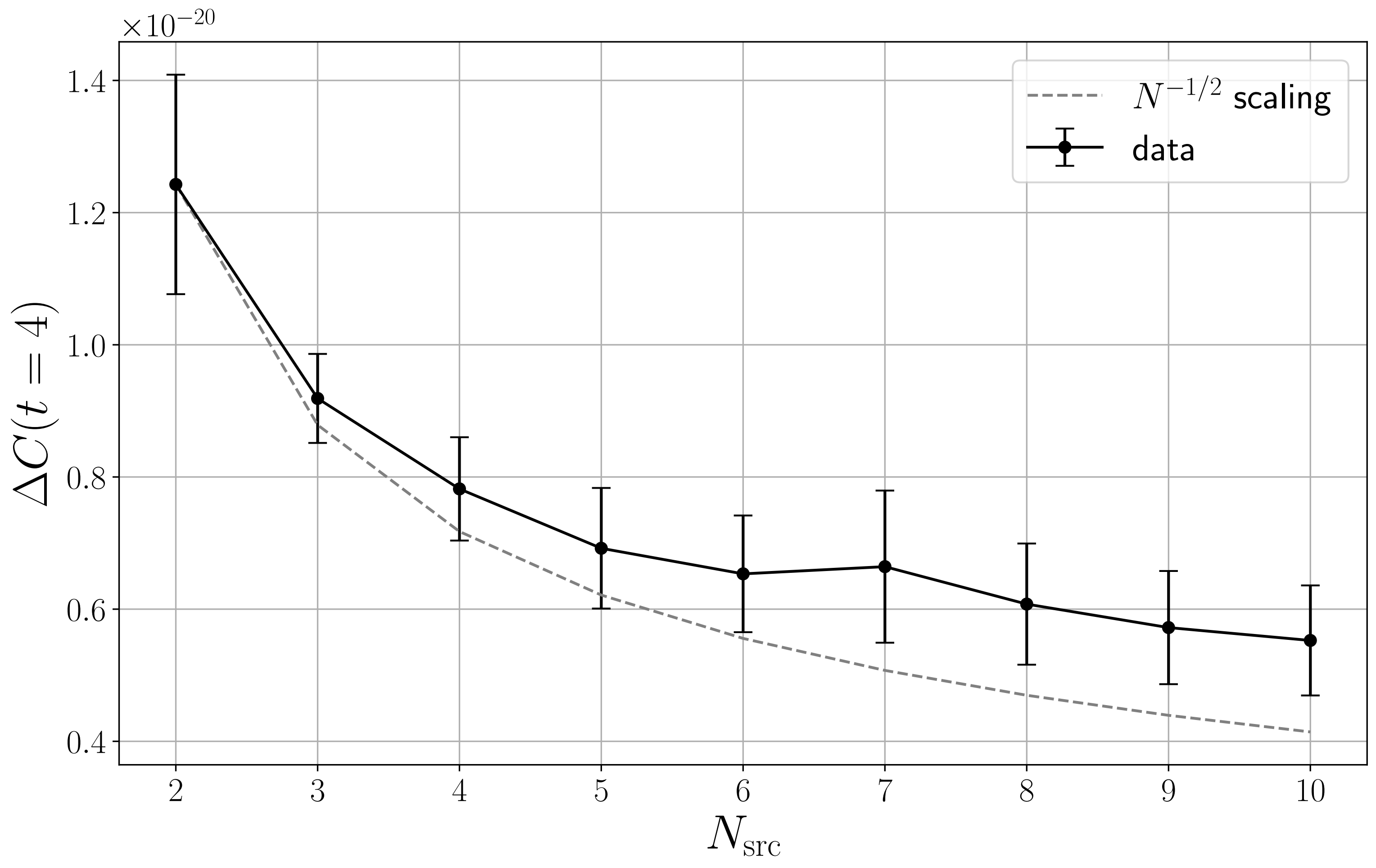


Random point sources

2-POINT FUNCTION NOISE

Varying the number of randomized point sources $\underline{x}_0^{(n)}$

$$C_2(t) = \frac{1}{N_{\mathrm{src}}} \sum_n^{N_{\mathrm{src}}} \langle \mathbb{P}_{\mathrm{odd}} \mathcal{H}(t, \underline{p} = 0) \mathcal{H}(0, \underline{x}_0^{(n)}) \rangle_T$$



Mixing pattern

The effective Hamiltonian $\mathcal{H} \propto v^L V^L$ mixes with...

Operator \ Action	Wilson u, d, τ	Wilson u, d Overlap τ	Overlap u, d, τ
$s^L S^L = \bar{u}_R d_L \; \bar{\tau}_R \nu_L$	◆		
$s^R S^L = \bar{u}_L d_R \; \bar{\tau}_R \nu_L$	◆		
$v^L V^L = \bar{u}_L \gamma_\mu d_L \; \bar{\tau}_L \gamma_\mu \nu_L$	◆	◆	◆
$v^R V^L = \bar{u}_R \gamma_\mu d_R \; \bar{\tau}_L \gamma_\mu \nu_L$	◆	◆	
$t^L T^L = \bar{u}_R \sigma_{\mu\nu} d_L \; \bar{\tau}_R \sigma_{\mu\nu} \nu_L$	◆		

Wilson averages

New goal: renormalize only *certain* correlators instead of the full operator

Notice that the $\mathbb{Z}_2$ subgroup

$$\psi \to \gamma_5 \psi \qquad \bar{\psi} \to -\bar{\psi}\gamma_5$$

of $\mathrm{U}(1)_A$ is a spurionic symmetry

$$\bar{\psi} D(r, m, \mu)\psi = \bar{\psi}' D(-r, -m, \mu)\psi' \qquad D = \gamma \cdot \tilde{\nabla} - \frac{ar}{2}\nabla^F_\mu \nabla^B_\mu + m + i\mu\gamma_5$$

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of $\mathrm{U}(1)_A$ is a spurionic symmetry

$$\bar{\psi} D(r, m, \mu) \psi = \bar{\psi}' D(-r, -m, \mu) \psi' \qquad D = \gamma \cdot \tilde{\nabla} - \frac{ar}{2} \nabla_\mu^F \nabla_\mu^B + m + i\mu\gamma_5$$

SYMMETRY	$s^L S^L$	$s^R S^L$	$v^L V^L$	$v^R V^L$	$t^L T^L$
$\mathbb{Z}_2^A(\tau)$	$+$	$+$	$-$	$-$	$+$
$\mathbb{Z}_2^A(u)$	$+$	$-$	$-$	$+$	$+$
$\mathbb{Z}_2^A(d)$	$-$	$+$	$-$	$+$	$-$

[R. Frezzotti and G. C. Rossi, JHEP 08 (2004)]

Wilson averages

New goal: renormalize only *certain* correlators instead of the full operator

WILSON AVERAGE

$$\langle [v^L V^L]_R \rangle^{\mathrm{WA}} = \frac{1}{2}\Big(\langle [v^L V^L]_R \rangle_{r_\tau} + \langle [v^L V^L]_R \rangle_{-r_\tau} \Big)$$

$$= \left\langle \frac{Z_1(r_\tau)+Z_1(-r_\tau)}{2} s^L S^L + \frac{Z_2(r_\tau)+Z_2(-r_\tau)}{2} s^R S^L + \frac{Z_3(r_\tau)+Z_3(-r_\tau)}{2} v^L V^L + \frac{Z_4(r_\tau)+Z_4(-r_\tau)}{2} v^R V^L + \frac{Z_5(r_\tau)+Z_5(-r_\tau)}{2} t^L T^L \right\rangle_{\mathrm{cont}} + \mathcal{O}(a)$$

$$= \left\langle Z_3(r_\tau) v^L V^L + Z_4(r_\tau) v^R V^L \right\rangle_{\mathrm{cont}} + \mathcal{O}(a)$$

SYMMETRY	$s^L S^L$	$s^R S^L$	$v^L V^L$	$v^R V^L$	$t^L T^L$
$\mathbb{Z}_2^A(\tau)$	$+$	$+$	$-$	$-$	$+$
$\mathbb{Z}_2^A(u)$	$+$	$-$	$-$	$+$	$+$
$\mathbb{Z}_2^A(d)$	$-$	$+$	$-$	$+$	$-$

Constraints from the anomalous chiral symmetry

ANOMALY IN THE MASSLESS THEORY

In the massless theory the contributions from $Q \neq 0$ sectors automatically vanish by the index theorem

$$Z_\theta = \int \mathcal{D}[A,\psi,\bar{\psi}]\; e^{-S_G[A]-\bar{\psi}D\psi-i\theta Q} = \sum_{n=0} \int_{Q[A]=n} \mathcal{D}A\; e^{-S_G[A]-i\theta n} \det D = \int_{Q[A]=0} \mathcal{D}A\; e^{-S_G[A]} \det D = Z_0$$

Since $\mathcal{H}$ does not mix with lower dimensional operators, the renormalization constants can be chosen to be mass independent. So we can calculate them in the massless theory and here

$$\begin{aligned} Z[J] &= \int \mathcal{D}[A,\psi',\bar{\psi}']\; e^{-S_G[A]-\bar{\psi}'D\psi'+JO[\psi',\bar{\psi}']} \qquad (\psi' = e^{i\alpha\gamma_5}\psi \quad \bar{\psi}' = \bar{\psi}e^{i\alpha\gamma_5}) \\ &= \int \mathcal{D}[A,\psi,\bar{\psi}] e^{-i\theta Q[A]}\; e^{-S_G[A]-\bar{\psi}D\psi+J\left(\rho(\alpha)O[\psi,\bar{\psi}]\right)} \\ &= \int \mathcal{D}[A,\psi,\bar{\psi}]\; e^{-S_G[A]-\bar{\psi}D\psi+(J\rho(\alpha))O[\psi,\bar{\psi}]} \\ &= Z[J\rho(\alpha)] \end{aligned}$$

The same symmetry constraint then holds for the (renormalized) vertex functional $\Gamma[J] = \Gamma[J\rho(\alpha)]$. Suppose now that O transforms in some (1-dimensional) irrep ρ_q of $\mathrm{U}(1)_A$ and O' in a different inequivalent irrep $\rho_{q'}$ then

$$JZO' \stackrel{!}{=} J\rho_q(\alpha)Z\rho_{q'}(-\alpha)O' = e^{i(q-q')\alpha}JZO'$$

which cannot be true for all α when $q \neq q'$. This implies that JO' cannot appear in Γ_R.