Resolving vector-current spectral functions with singular value decomposition

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collaboration with:

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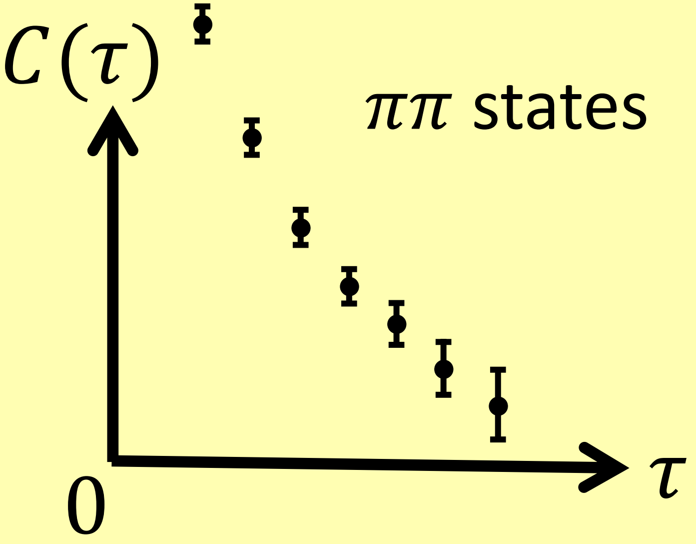
Jul. 29 @LATTICE2026

Analysis of vector-current correlator

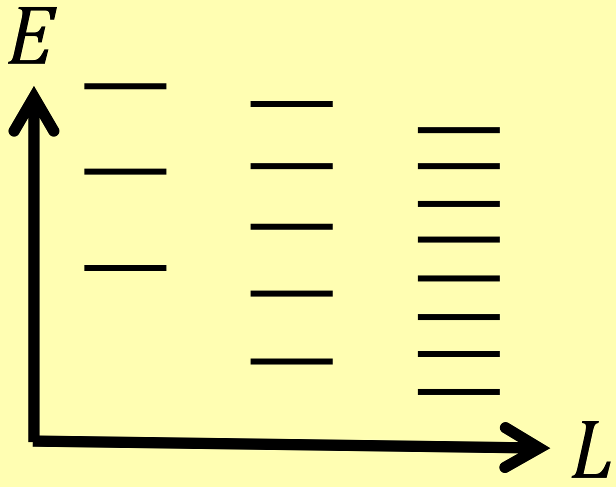
Standard method:

= study of ρ-resonance i.e. I=1 $\pi\pi$ states

Lattice data



Energy spectrum



Lüscher's method

M. Lüscher [Nucl. Phys. B. **354** (1991)]

$$\det[\, F^{-1}(E,L) + \mathcal{M}(E) \,] = 0$$

- Phase shift
- Scattering length
- Decay width

In larger volumes, the energy level becomes denser.
It will be difficult to distinguish those states.

Alternative methods to study hadron spectroscopy?

This work

Smeared spectral functions

Correlator: $$C(\tau) = \int dE \;\; \rho(E)\, e^{-E\tau}$$

1st-order moment: $$\langle E\rangle(\tau) = \int dE \;\; \rho(E)\, E\, e^{-E\tau}/C(\tau)$$

2nd-order moment: $$\Delta E(\tau) = \sqrt{\langle (E - \langle E\rangle)^2\rangle}$$

Written as a smeared spectrum with different smearing kernel.

- For lattice determination of $\langle E\rangle$ and ΔE, we use SVD of $e^{-E\tau}$.
- Study of sensitivity to determine m_ρ and Γ_ρ.

SVD of the Laplace kernel $e^{-E\tau}$

$$\boxed{e^{-E\tau} \xrightarrow{\text{SVD}} \sum_l U_l(\tau)\,\sigma_l\,V_l(E)}$$

$$\sum_E V_{l'}(E)\,V_l(E) = \delta_{ll'}$$
$$\sum_\tau U_{l'}(\tau)\,U_l(\tau) = \delta_{ll'}$$

R. Tsuji, S. Hashimoto [arXiv:2605.15674]

And talks by R. Tsuji (Jul. 28, 15:20-)
S. Hashimoto (Jul.27, plenary)

$$C(\tau) = \int dE\ \rho(E)\,e^{-E\tau}$$

$$= \sum_l U_l(\tau)\,\sigma_l \underbrace{\int dE\ \rho(E)\,V_l(E)}_{\equiv\ \rho_l}$$

$$\rho(E) = \sum_l \rho_l\,V_l(E)\,, \qquad \rho_l = \frac{1}{\sigma_l}\sum_\tau C^{\mathrm{lat}}(\tau)\,U_l(\tau)$$

Construction of smeared spectrum with SVD 4

$$\boxed{\rho(E) = \sum_l \rho_l V_l(E)}$$

$$\langle E^n \rangle(\tau) = \frac{1}{C(\tau)} \int dE\ E^n \rho(E)\, e^{-E\tau}$$

$$= \frac{1}{C(\tau)} \sum_l \rho_l \int dE\ E^n V_l(E)\, e^{-E\tau}$$

SVD of $e^{-E\tau}$

$+$

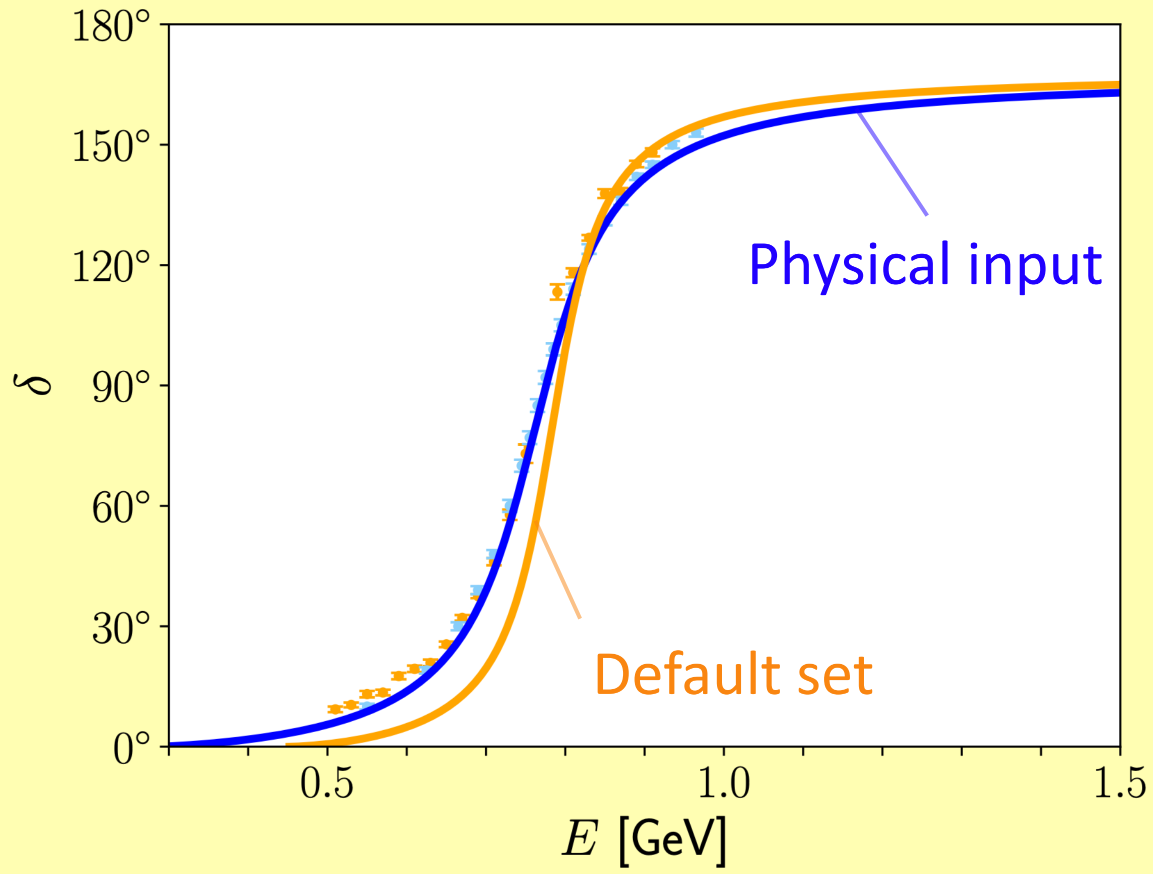
Lattice data $C^{\mathrm{lat}}(\tau)$

$\Rightarrow$ $\rho_l = \frac{1}{\sigma_l} \sum_\tau C^{\mathrm{lat}}(\tau)\, U_l(\tau)$ $\Rightarrow$ $\langle E^n \rangle(\tau)$

Breit-Wigner type

$$\delta^{\mathrm{BW}}(s) = \frac{m_\rho}{m_\rho^2 - s}\,\Gamma_\rho(s)$$

$$\Gamma_\rho(s) = \Gamma_\rho \left(\frac{s - 4m_\pi^2}{m_\rho^2 - 4m_\pi^2}\right)^{3/2} \frac{m_\rho}{\sqrt{s}}$$



To compare with our lattice:

JLQCD, domain-wall lattice,

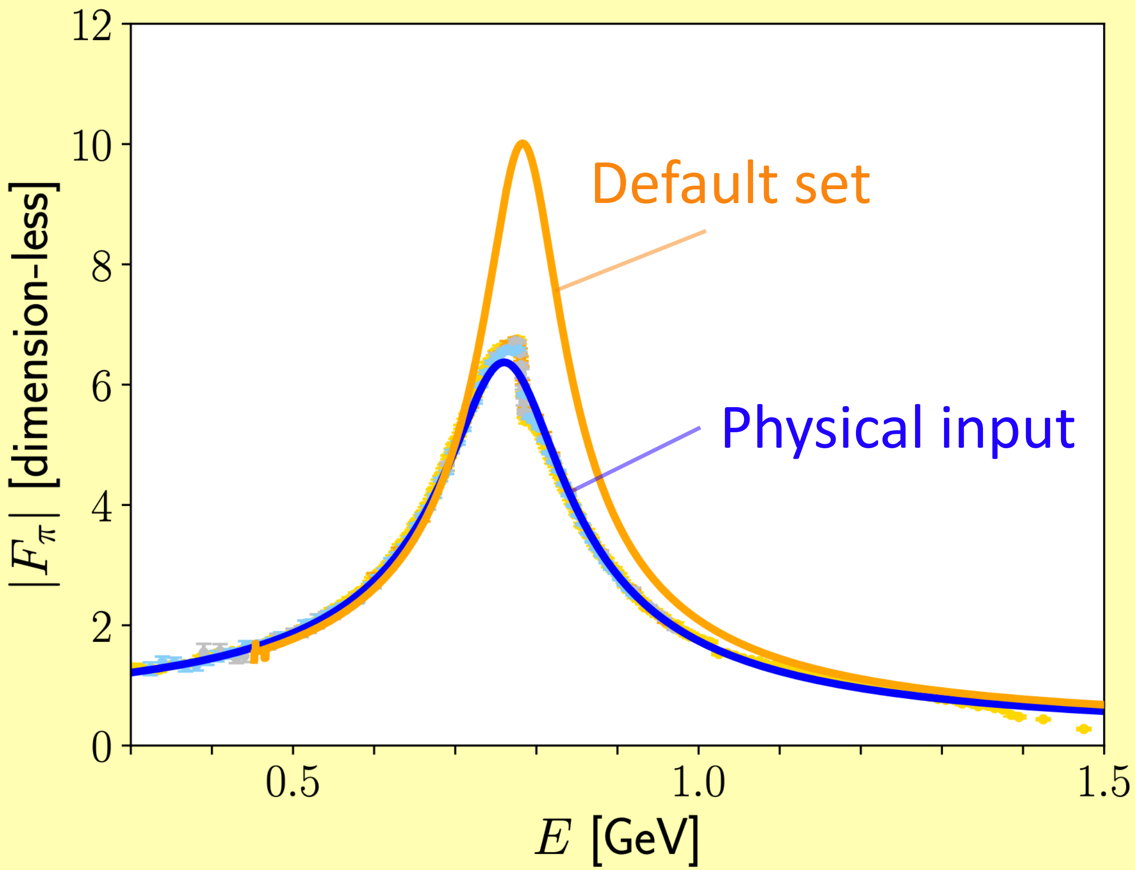
$32^3\times64$, $48^3\times96$,

$a \sim 0.08$ fm, $m_\pi = 226$MeV

Model params.	m_π [MeV]	m_ρ [MeV]	Γ_ρ [MeV]	$\sqrt{\langle r_\pi^2\rangle}$ [fm]
Physical	139.57	775.26	149.1	0.66
Our default set	226	792	98	0.635

Omnés representation

$$\ln|F_\pi|(s) = \frac{1}{6}\langle r_\pi^2\rangle s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} ds' \, \frac{\delta^{\mathrm{BW}}(s')}{s'(s'-s)}$$



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Construction of the model correlator

Finite volume:

SI, H. Fukaya, S. Hashimoto [PRD **111**, 114507]

$$C(\tau, L) = \sum_n \left|\langle 0| J_\mu |\pi\pi, n\rangle\right|^2 e^{-E_n \tau}$$

$$E_n = 2\sqrt{m_\pi^2 + k_n^2}$$

Lüscher's quantization condition

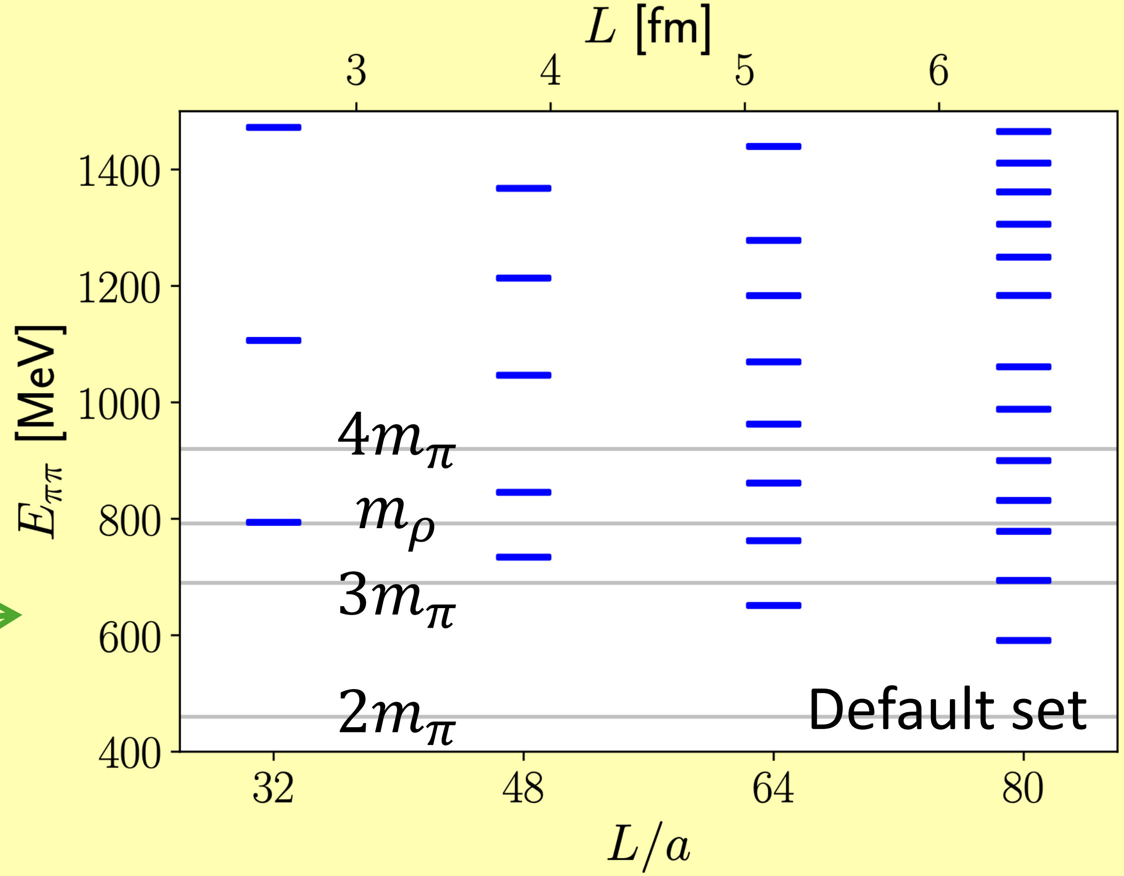
M. Lüscher [Nucl. Phys. B. **354** (1991)]

$$\det[\, \underbrace{F^{-1}(E, L)}_{\text{Finite V}} + \underbrace{\mathcal{M}(E)}_{\text{Infinite-V amplitude}} \,] = 0$$

P-wave only:

$$\underbrace{\phi\left(\frac{kL}{2\pi}\right)}_{\text{Sym. of the system}} + \underbrace{\delta^{\mathrm{BW}}(k)}_{\text{Scattering phase shift}} = 0$$

solve



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$$C(\tau, L) = \sum_n \left|\langle 0| J_\mu |\pi\pi, n\rangle\right|^2 e^{-E_n\tau}$$

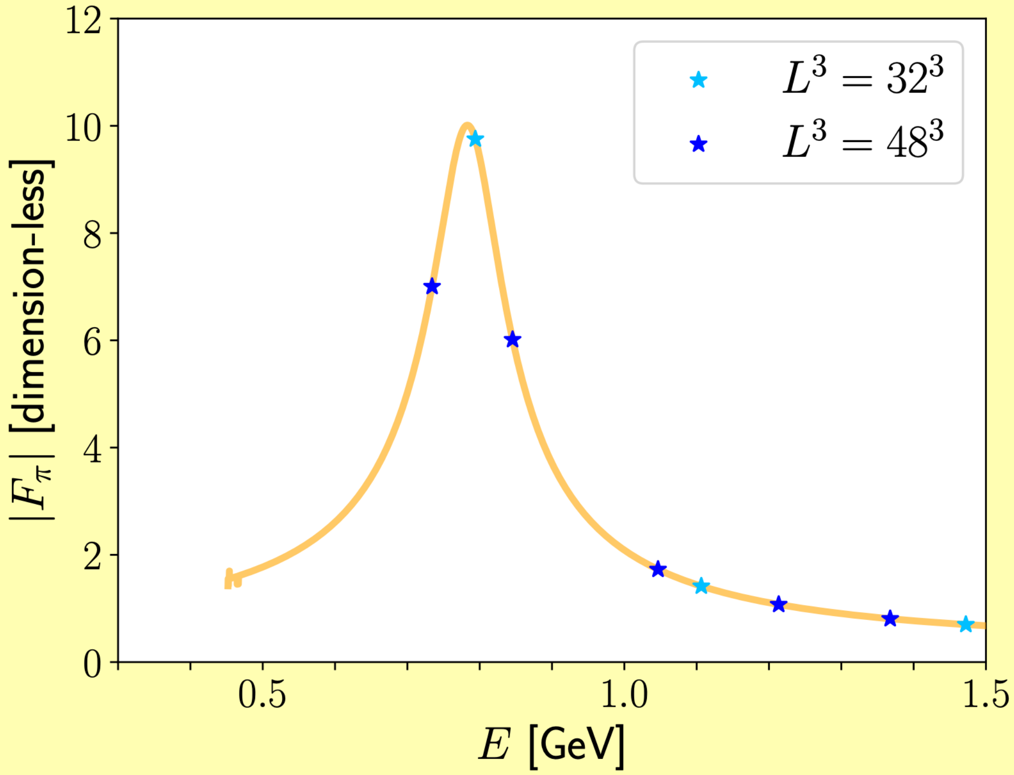
Lellouch-Lüscher-Meyer formula

H. B. Meyer [Phys. Rev. L. **107** (2022)]

$$|F_\pi|^2 = (\text{factor}) \cdot \left|\langle 0| J_\mu |\pi\pi, n\rangle\right|^2$$

Infinite-V form factor

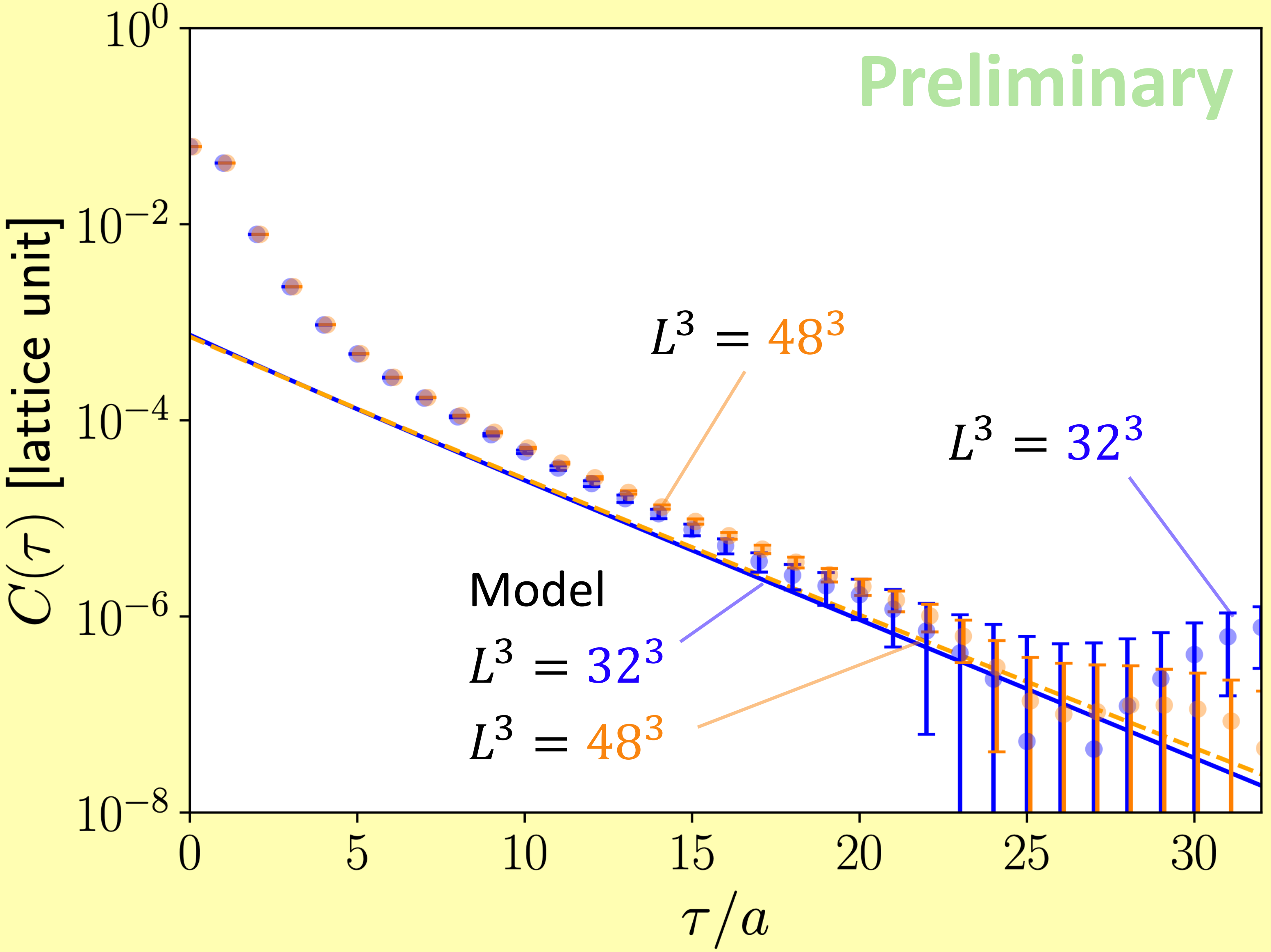
Finite-V amplitude



- Cutoff at $E < 1.5$ GeV
- 4π or higher states ignored

… only large τ should be considered

Default model compared to the lattice data



$$C(\tau) = \int d\omega \;\; \rho(\omega)\, e^{-\omega\tau}$$

Lattice data:

JLQCD, domain-wall lattice,

$32^3 \times 64$, $48^3 \times 96$,

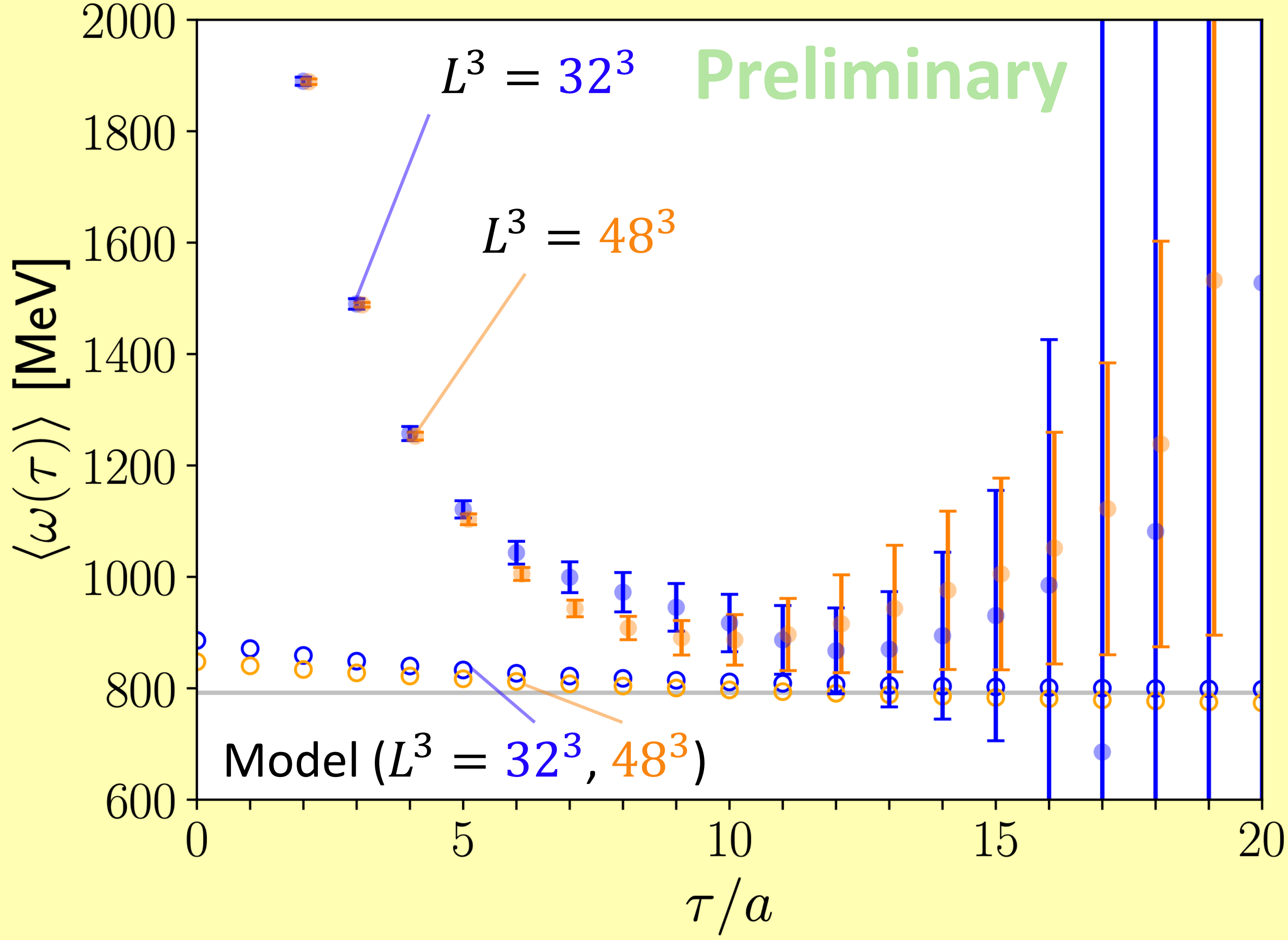
$m_\pi = 226\text{MeV}$,

$a \sim 0.08$ fm

conf = 100,

measurement
= 600 (32^3), 800 (48^3)

1st-order moment



$$\langle\omega\rangle(\tau) = \int d\omega \; \rho(\omega)\, \omega\, e^{-\omega\tau}$$

Lattice data:

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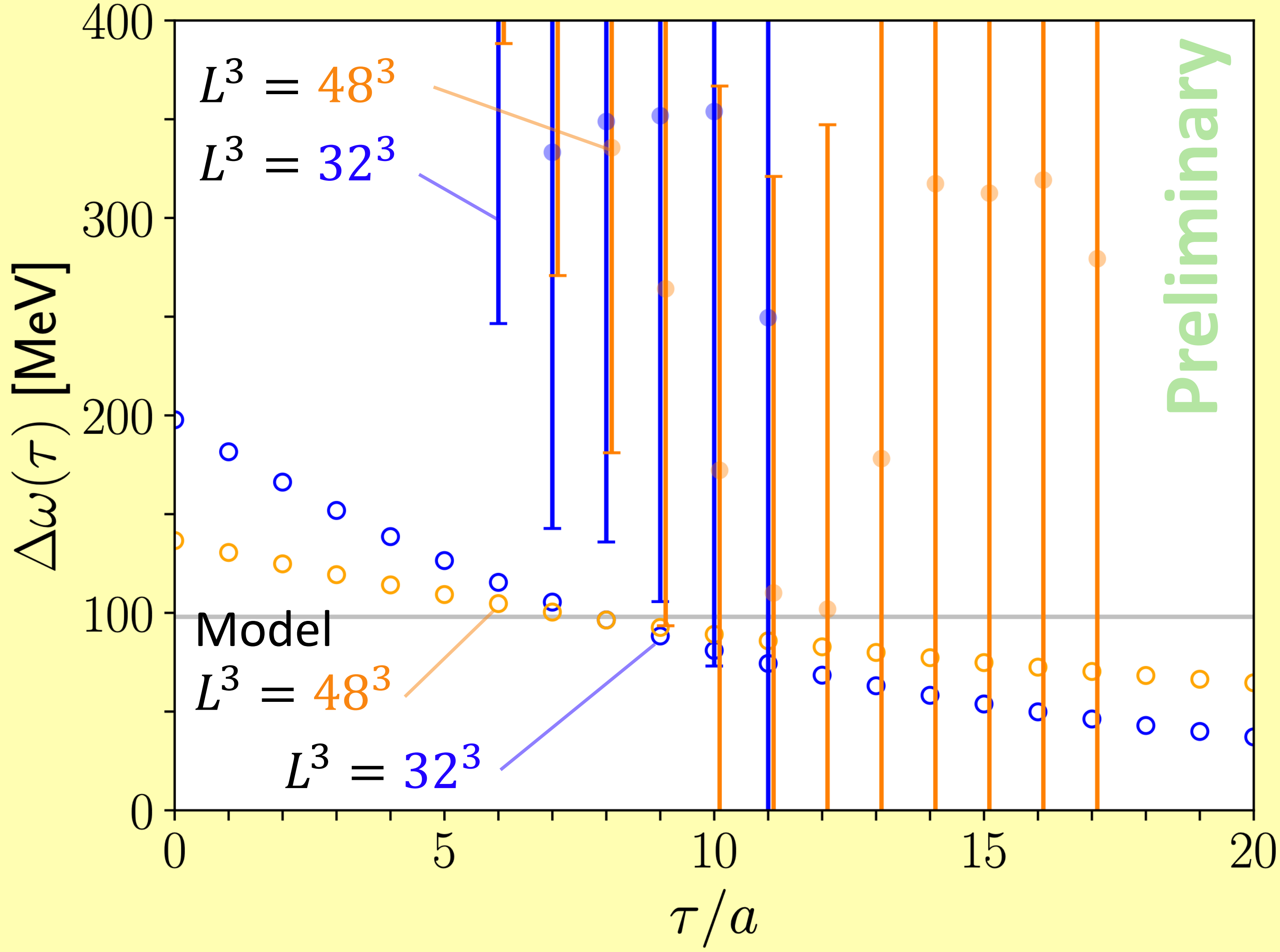
2nd-order moment



$$\Delta\omega(\tau) = \sqrt{\langle\omega^2\rangle(\tau) - \langle\omega\rangle^2(\tau)}$$

Lattice data:

JLQCD, domain-wall lattice,
$32^3{\times}64$, $48^3{\times}96$,
$m_\pi = 226$MeV,
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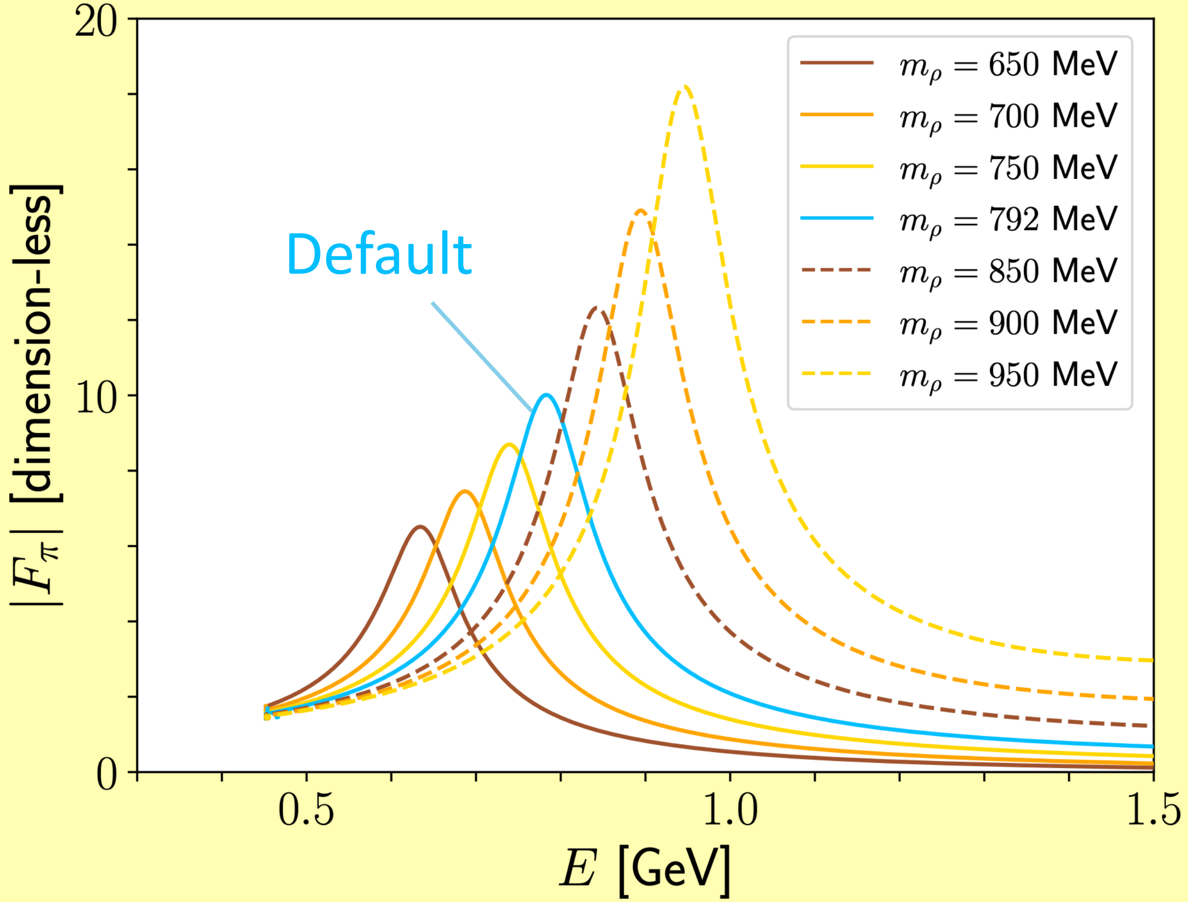
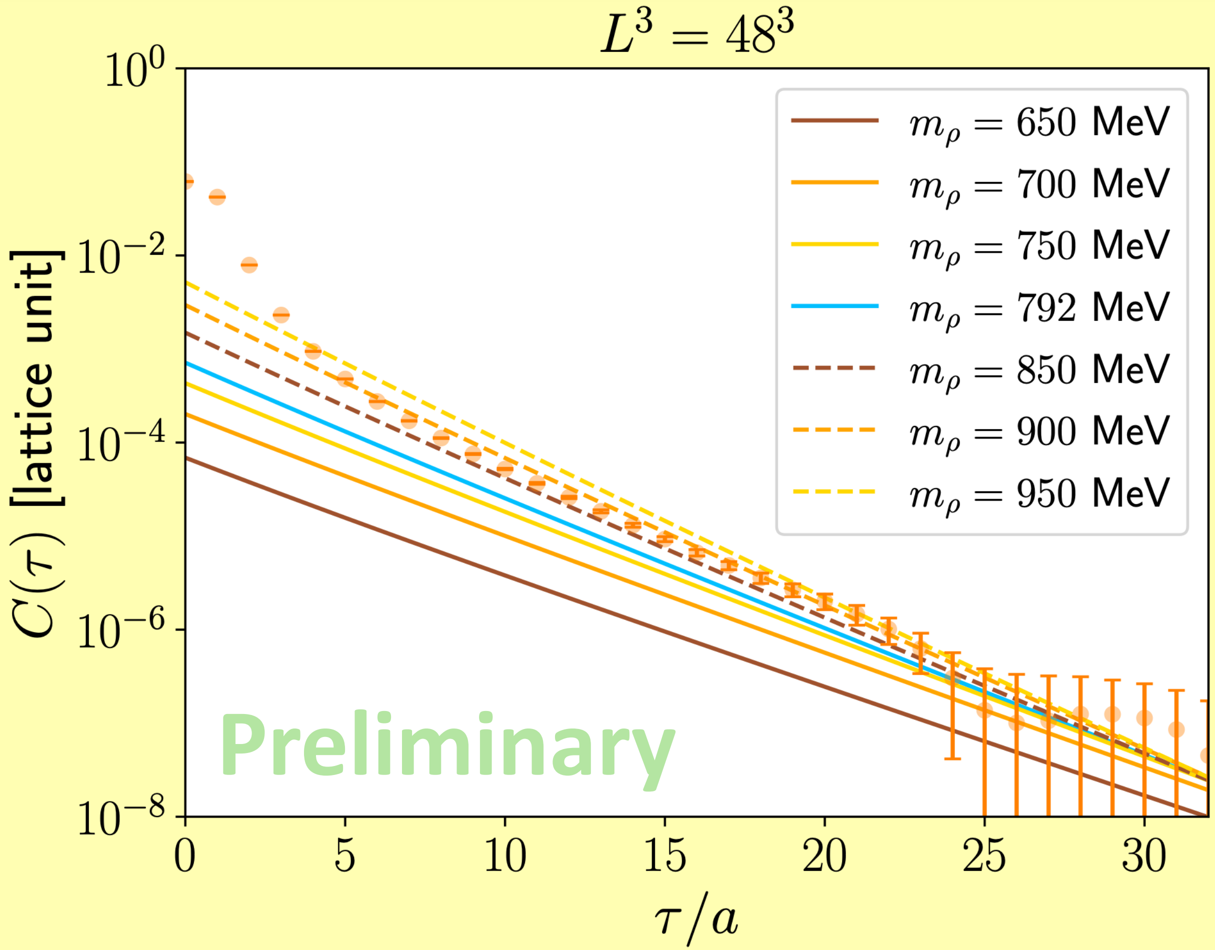
m_ρ-sensitivity of $C(\tau)$

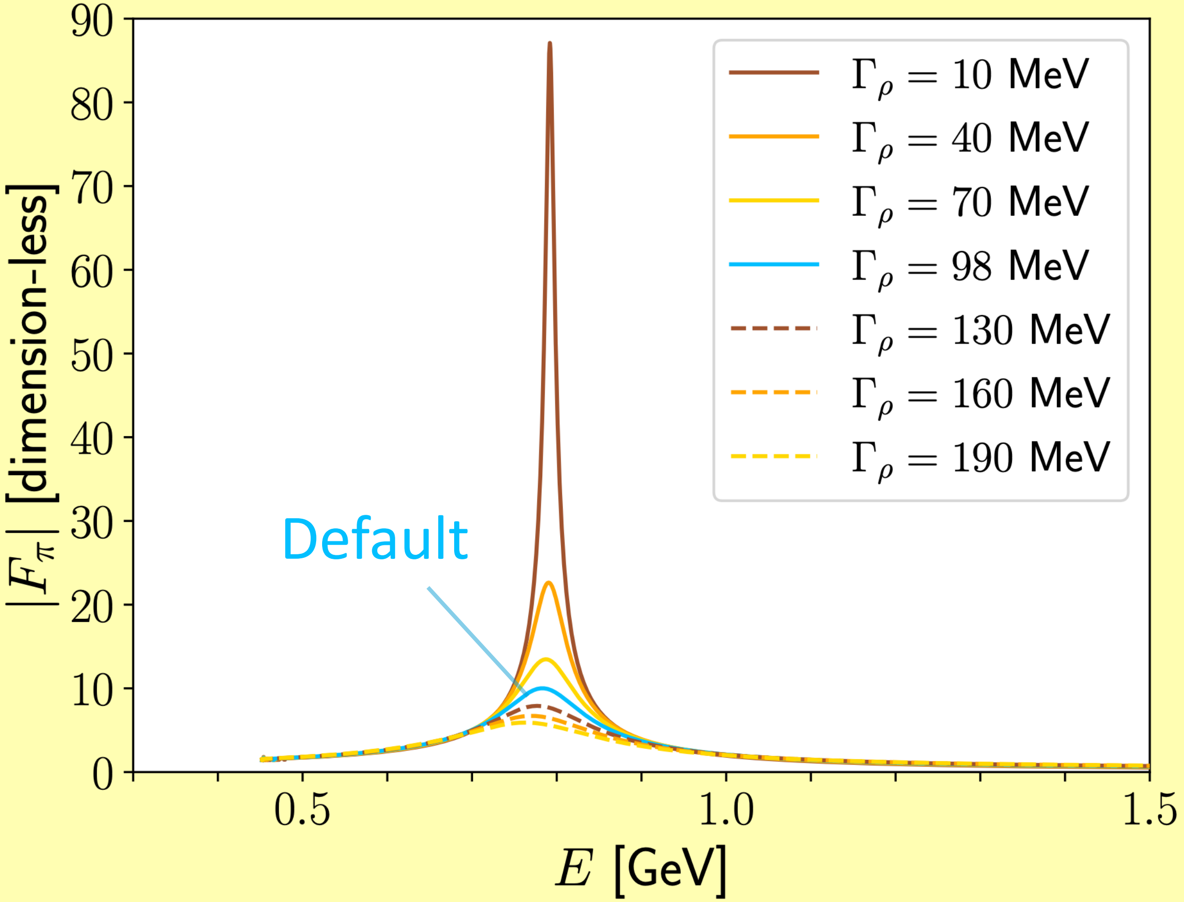


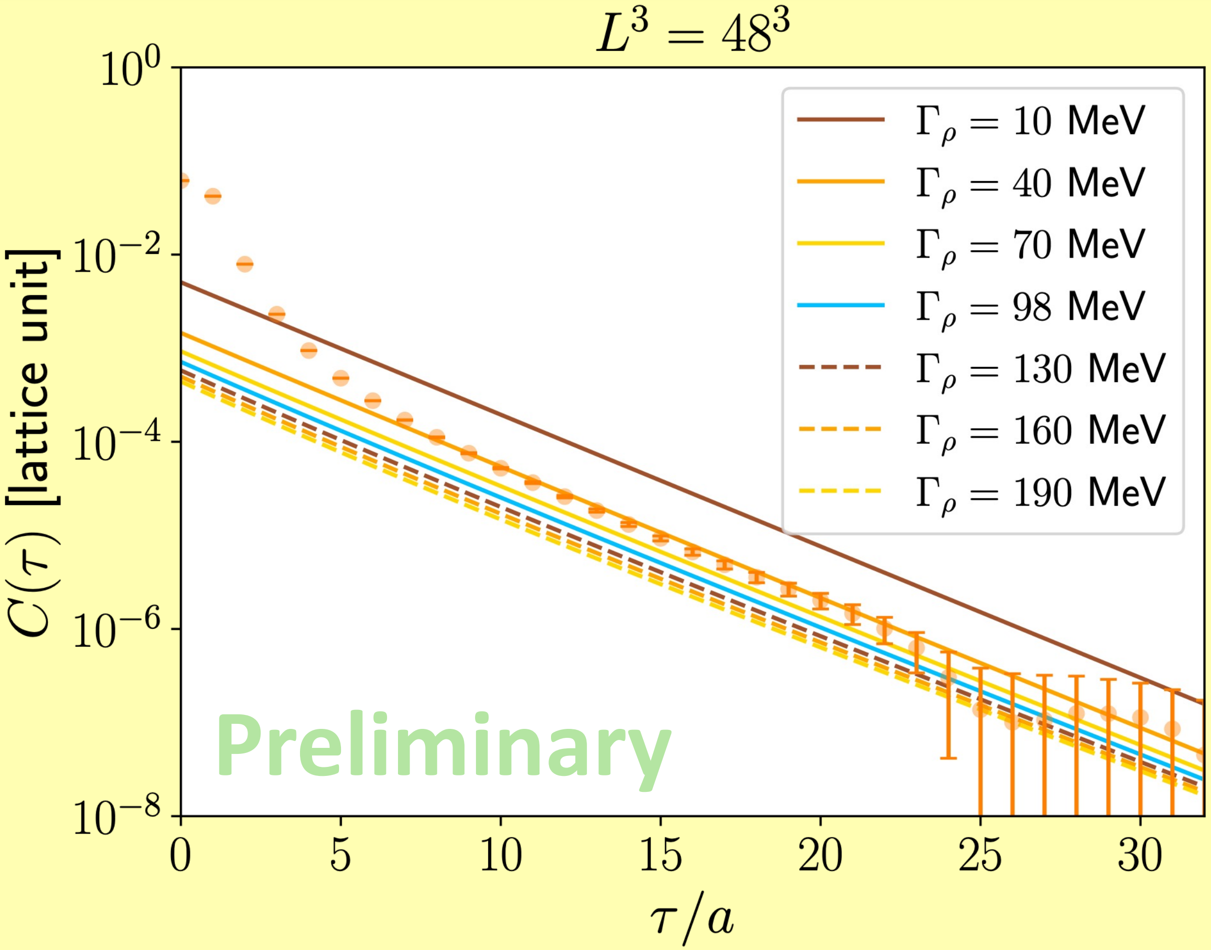


Default

m_π [MeV]	m_ρ [MeV]	Γ_ρ [MeV]	$\sqrt{\langle r_\pi^2 \rangle}$ [fm]
226	792	98	0.635

Γ_ρ-sensitivity of $C(\tau)$

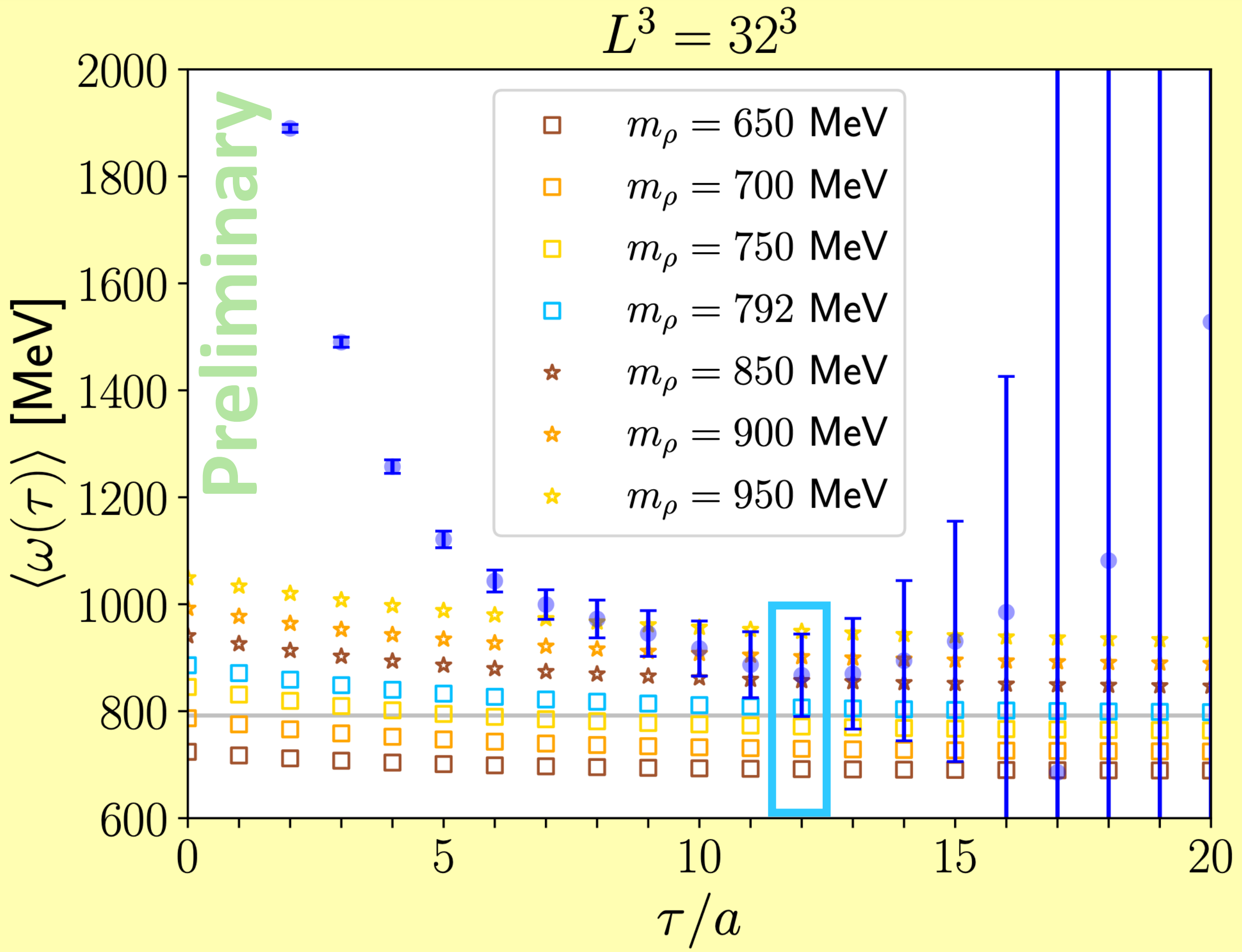


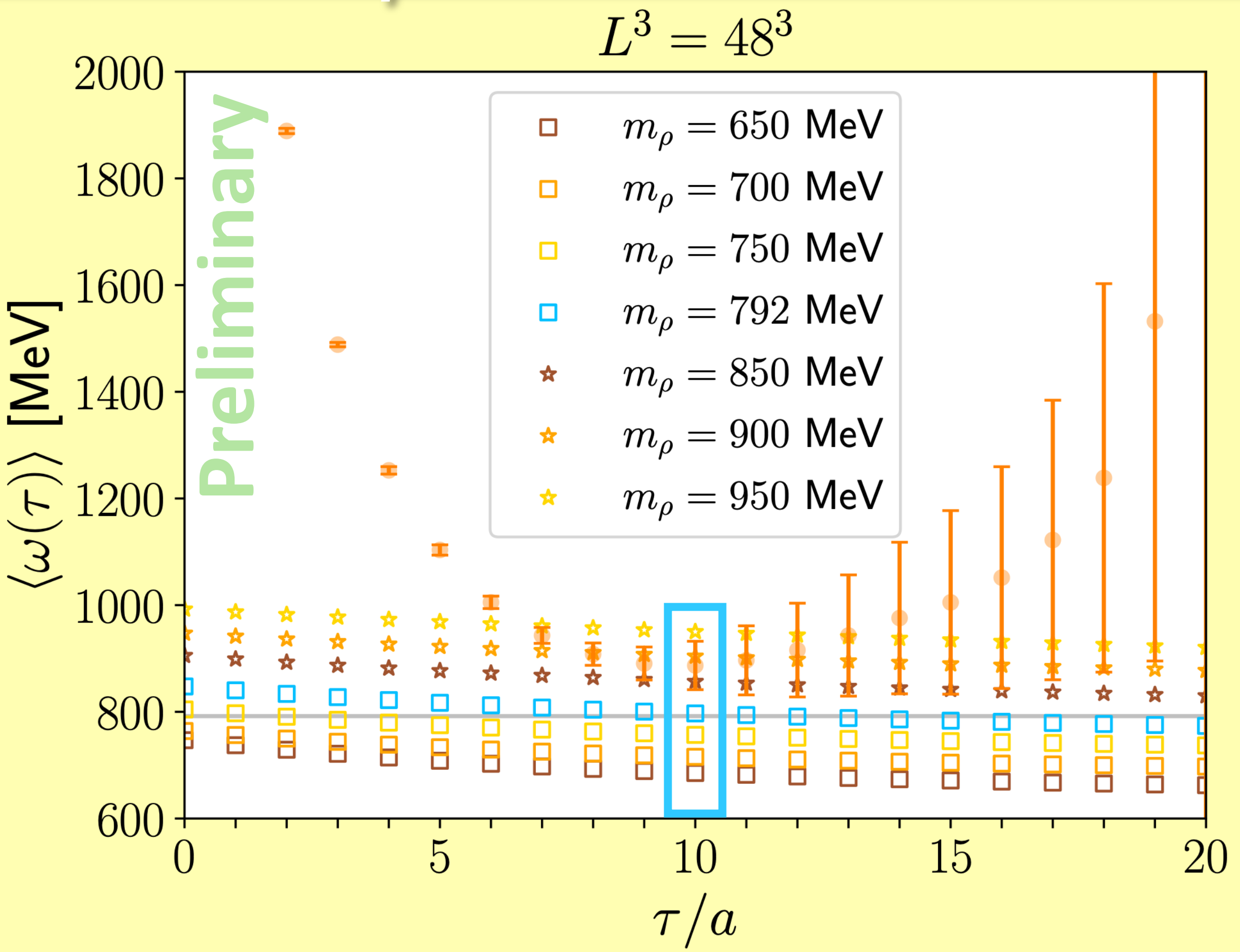


Default

m_π [MeV]	m_ρ [MeV]	Γ_ρ [MeV]	$\sqrt{\langle r_\pi^2 \rangle}$ [fm]
226	792	98	0.635

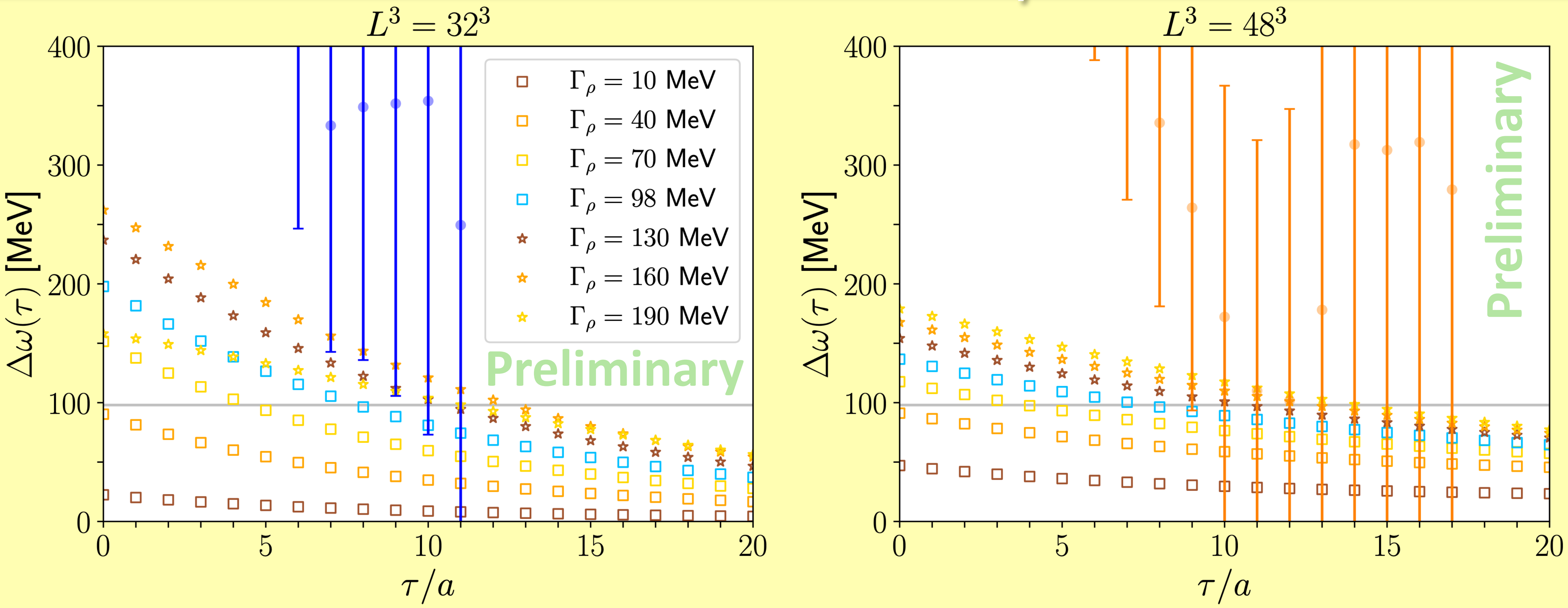
1st-order moment (different m_ρs)





Estimation: $m_\rho \sim$ 900 MeV

2nd-order moment (different Γ_ρs)



Need more statistics

Conclusion

- **Smeared spectrum ($C(\tau)$, $\langle\omega\rangle(\tau)$, $\Delta\omega(\tau)$) constructed from a model for $\pi\pi$ states**

- **SVD of $e^{-E\tau}$ to obtain $\langle\omega\rangle(\tau)$ and $\Delta\omega(\tau)$ from the lattice correlator**

- **Sensitivity for m_ρ and Γ_ρ**
  - More statistics is required to perform detailed study
  - m_ρ, Γ_ρ-fit will be performed to the lattice results

As m_ρ and Γ_ρ increase…

	m_ρ	Γ_ρ
$C(\tau)$	$\nearrow$	$\searrow$
$\langle\omega\rangle(\tau)$	$\nearrow$	
$\Delta\omega(\tau)$		$\nearrow$

Thank you for your attention.

Backup slides

Ill-posed problem

$$\boxed{C(\tau) = \int dE\, \rho(E)\, e^{-E\tau}}$$

It is hard to obtain the spectral function $\rho(E)$ with a high resolution from the lattice data of $C(\tau)$.

- Highly suppressed excited states by $e^{-E\tau}$
- A finite set of lattice data
- Finite precision

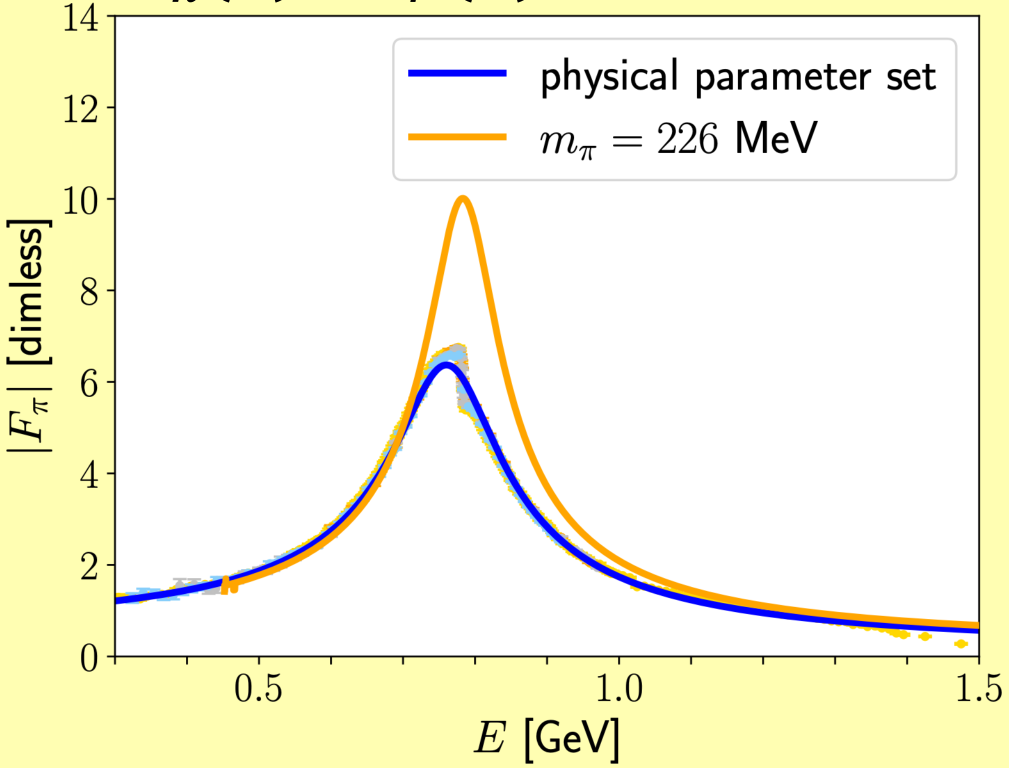
Consider a **smeared** spectral function.

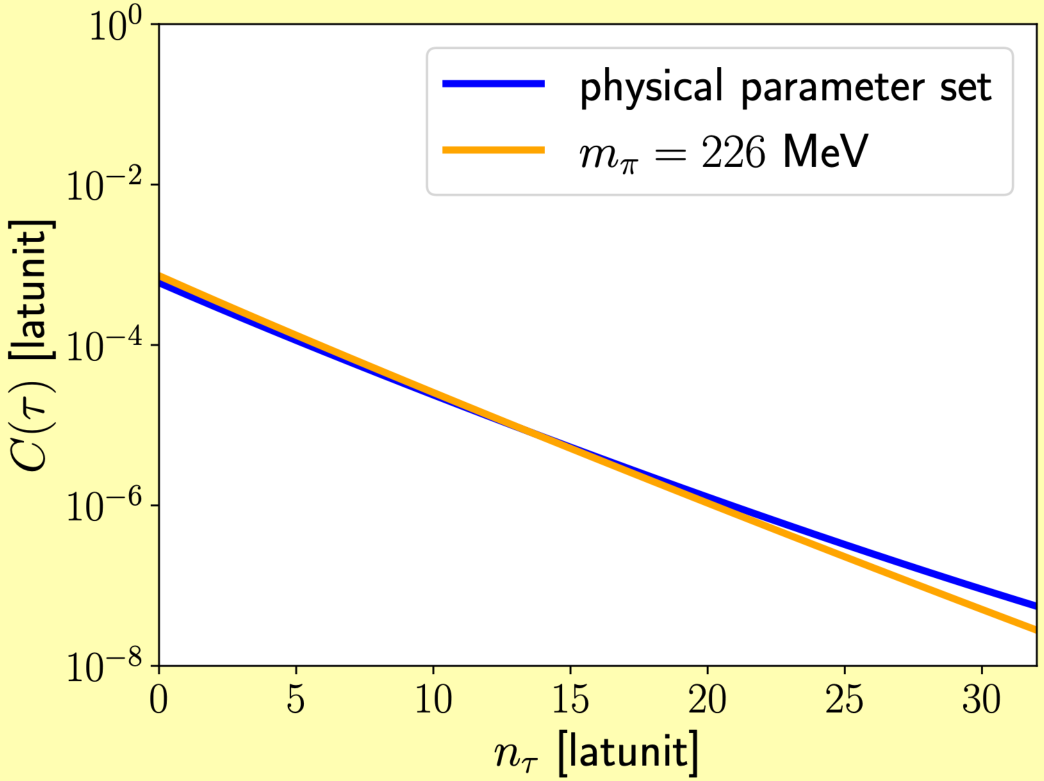
Construction of the model correlator

Infinite volume:

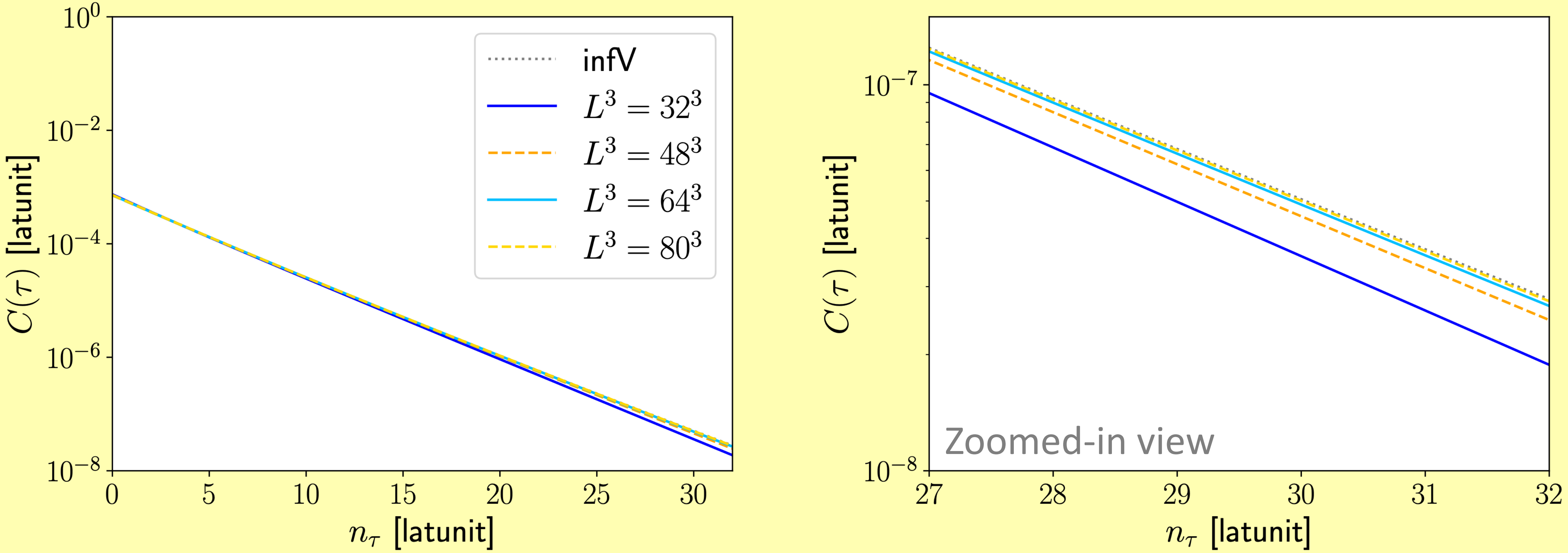
$$C(\tau) = \int dE \;\rho^{\mathrm{BW}}(E)\; e^{-E\tau}$$

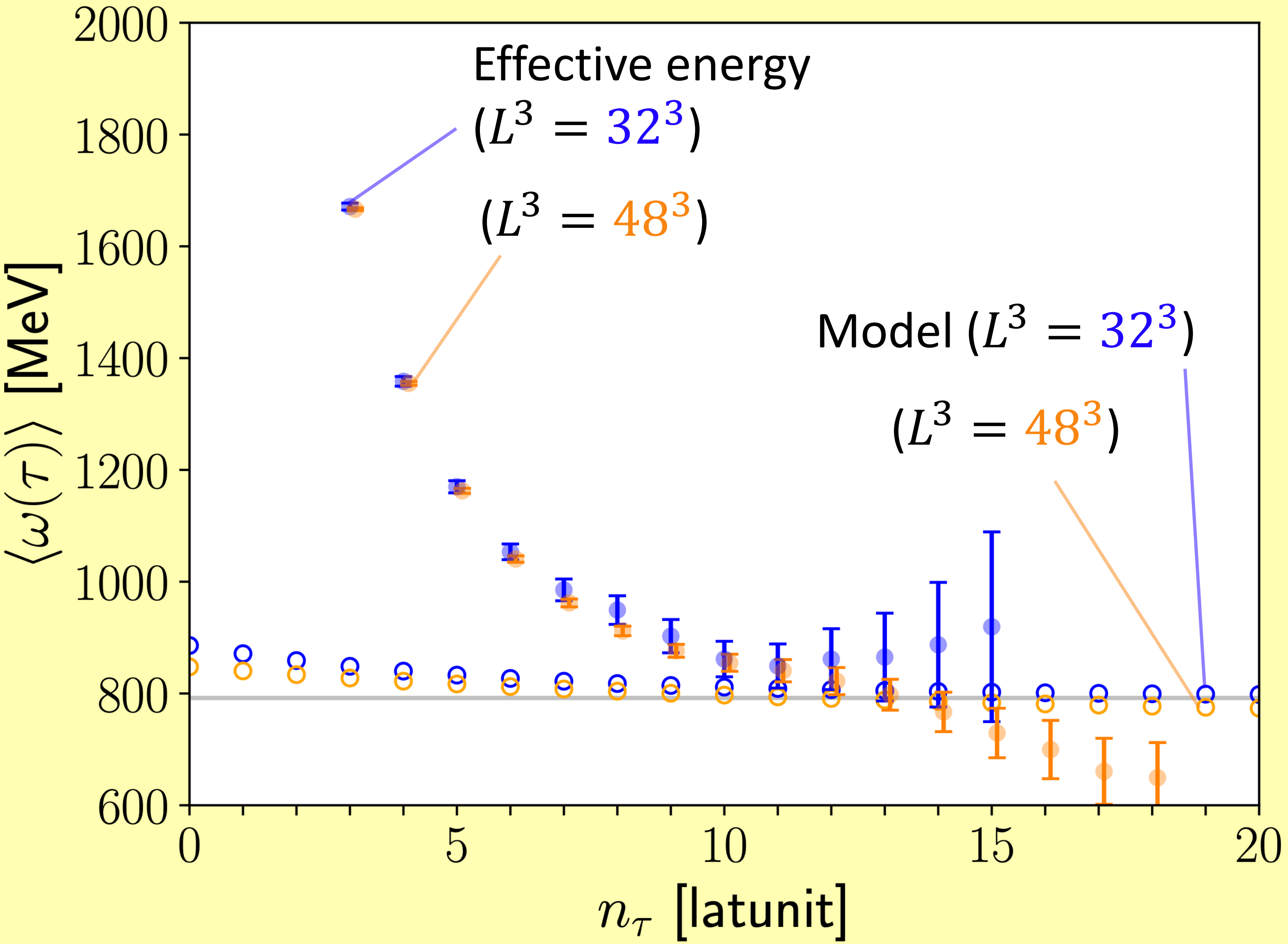
$|F_\pi(E)|^2 \sim \rho(E)$





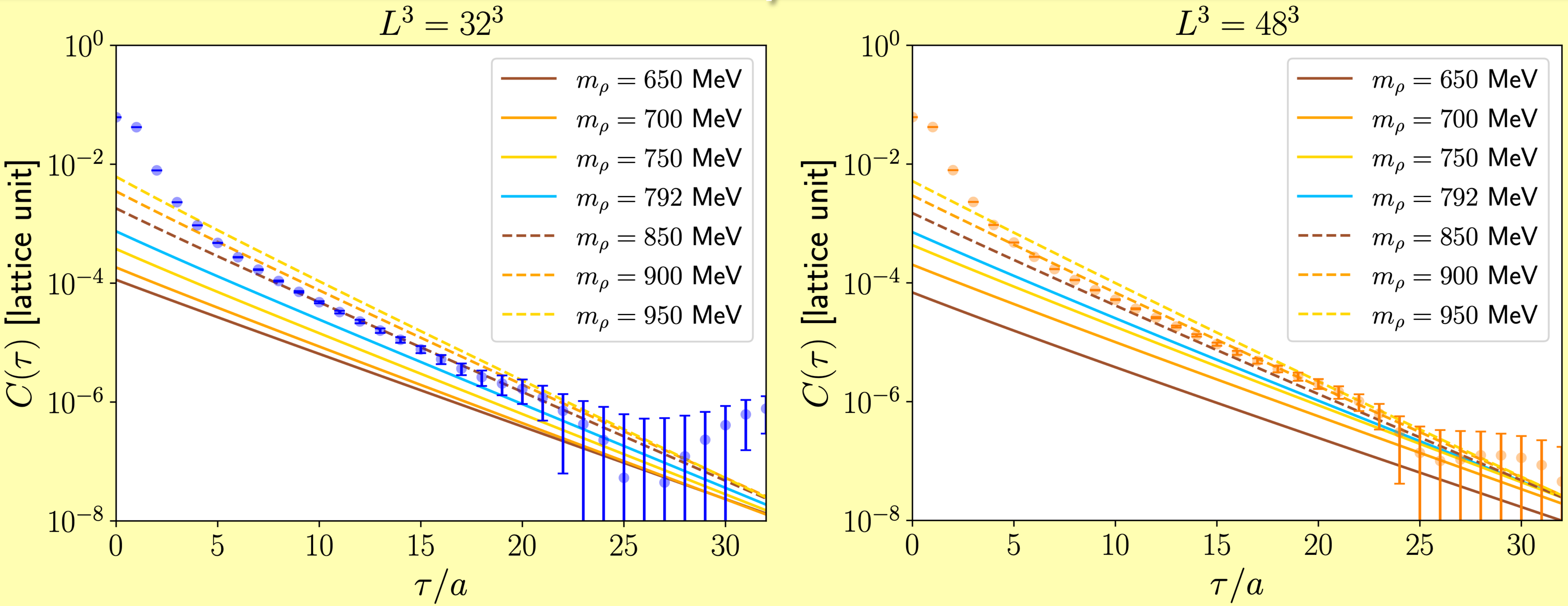
model correlators ($m_\pi = 226$ MeV)



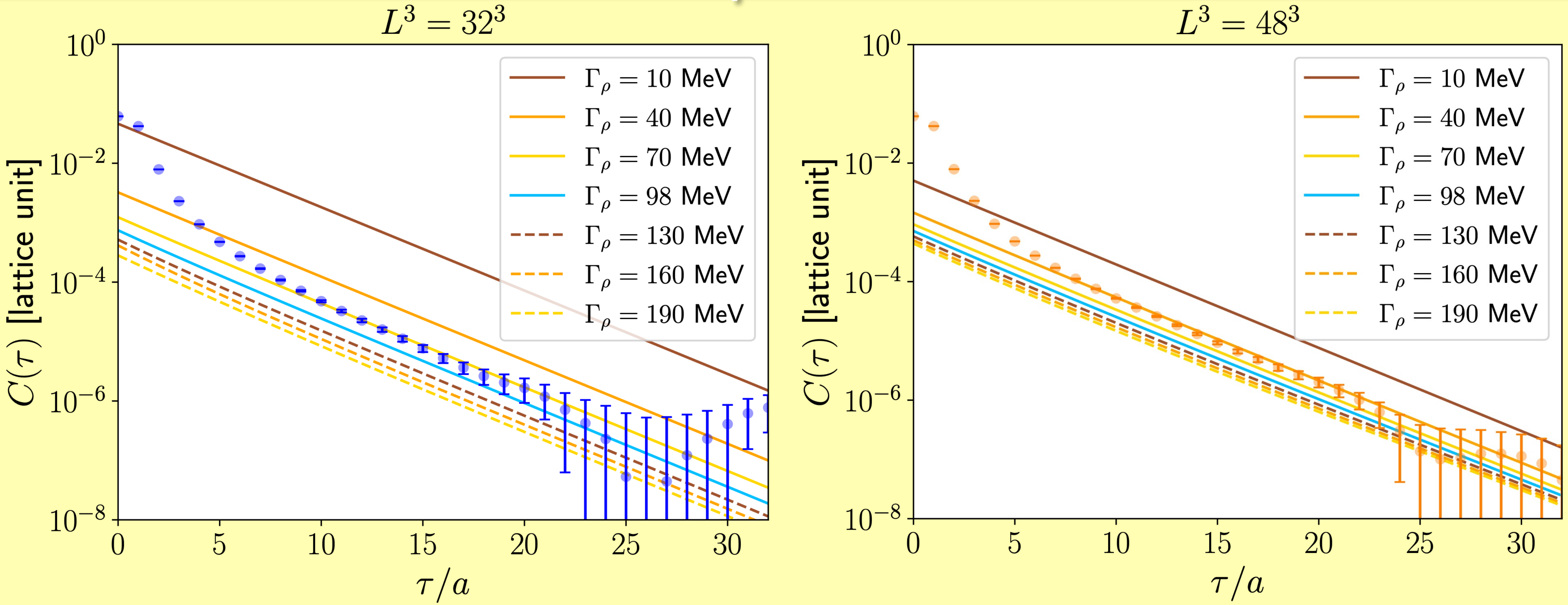


Effective energy
$(L^3 = 32^3)$
$(L^3 = 48^3)$
Model $(L^3 = 32^3)$
$(L^3 = 48^3)$
$\langle\omega(\tau)\rangle$ [MeV]
n_τ [latunit]
2000
1800
1600
1400
1200
1000
800
600
0
5
10
15
20

Correlator (different m_ρs)



Correlator (different Γ_ρs)



1st-order moment (different m_ρs)