Tensor renormalization group analysis of correlators in Yang-Mills theory

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based on [Fukuma and Kuwahara, in preparation]

Abstract

Summary

- Tensor renormalization group(TRG) is one of the candidate method for first-principles calculations of 4D finite density QCD.
- Establish an impurity method to evaluate plaquette correlation function in arbitrary gauge group in arbitrary dimensions
- Demonstrate in the 3D pure $SU(2)$ gauge theory with real/complex coupling constant and compare with HMC/worldvolume HMC results

Outline

Background

Tensor network construction for 3D gauge theory

Impurity method

Numerical results

Summary

Tensor renormalization group (TRG) [Levin and Nave, 2007]

TRG is a deterministic numerical method that expresses the partition function as a tensor network and evaluate it via coarse-graining (using SVD).

Advantages

- Free from the sign problem in principle
- Easy to calculate large volume system
  (Computational cost scales logarithmically: $\propto \log(\text{linear size})$)
- Easy to treat fermion fields

Disadvantages

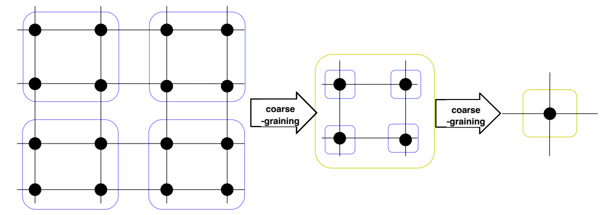
- High computational cost for higher-dimensional systems ($\propto D^{a\cdot\mathrm{dim}+b}$, where D is the bond dimension)
- Systematic error(cf. recent studies to avoid systematic errors in TRG: [Arai et al., 2023], [Todo, 2026])

TRG algorithm overview

TRG algorithm

1. Express the partition function Z as a tensor network:
$$Z = \sum_{\cdots, i, \cdots} \cdots T_{ijkl} T_{mnio} \cdots$$

2. Coarse-grain the tensors via low-rank approximation (e.g., SVD):
   $S_1 \geq S_2 \geq \cdots \geq 0$: Singular values
$$T_{ijkl} = M_{(ij),(kl)} = \sum_m U_{(ij),m} \cdot S_m \cdot V^\dagger_{m,(kl)} \sim \sum_{m=1}^{D} U_{(ij),m} \cdot S_m \cdot V^\dagger_{m,(kl)}$$



TRG studies for gauge theories

- 3D Z_2[Kuramashi and Yoshimura, 2019]
- (3 + 1)D Z_2-Higgs[Akiyama and Kuramashi, 2022]
- (3 + 1)D Z_3-Higgs[Akiyama and Kuramashi, 2023]
- 3D $O(2)$ [Bloch et al., 2021]
- 2D $U(1) + \theta$ term[Kuramashi and Yoshimura, 2020]
- (1 + 1)D U(1) gauge-Higgs with Lüscher's admissibility condition [Akiyama and Kuramashi, 2024]
- 2D non-Abelian Higgs[Bazavov et al., 2019]
- 2D SU(N)[Fukuma et al., 2021, Hirasawa et al., 2021]
- 3D pure SU(2) [Kuwahara and Tsuchiya, 2022]
- 3D pure SU(2), SU(3) [Yosprakob and Okunishi, 2024]
- (1 + 1)D SU(2) principal chiral[Luo and Kuramashi, 2023]
- (3 + 1)D 2-color QCD in strong coupling limit [Sugimoto et al., 2026a] (Sugimoto's talk on 30 July)
- (3 + 1)D QCD in strong coupling limit [Sugimoto et al., 2026b] (Sugimoto's talk on 30 July)

$SU(N_c)$ gauge theory

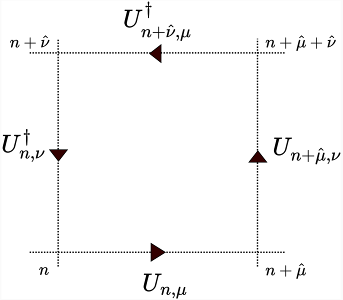
Partition function and action

$$Z = \int \prod_{n,\mu} dU_{n,\mu}\, e^{-S}$$

where $dU_{n,\mu}$ is the Haar measure on $SU(N_c)$.

$$S = -\frac{\beta}{N_c} \sum_n \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(n)$$

$$U_{\mu\nu}(n) = U_{n,\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n,\nu}$$



Random sampling method for tensor network construction

Tensor network representation for Yang–Mills theory:
character expansion or sampling

Random sampling method for tensor network construction

Tensor network representation tor Yang–Mills theory:
character expansion or sampling

Uniform sampling method [Fukuma, Kadoh, and Matsumoto, 2021]

Generate field configurations uniformly from the group manifold and approximate the integral

$$\int dU\, f(U) \sim \frac{1}{K} \sum_{i=1}^{K} f(U_i)$$

Note: Since the finite set $G = \{U_1, U_2, \cdots, U_K\}$ is generated by uniform sampling, statistical treatment is required, but the free energy density are $O(1)$ value, so no sign problem arises.

cf. Sampling with trial actions [Kuwahara and Tsuchiya, 2022]

cf. character expansion [Liu et al., 2013, Hirasawa et al., 2021, Yosprakob, 2024, Yosprakob and Okunishi, 2024]

Remark: No sign problem

- Since the finite subset $G = \{U_1, U_2, \cdots, U_K\}$ is generated by uniform sampling, statistical treatment is required
- Sign problem: to estimate an exponentially small quantity accurately, we need an exponentially large sample size in large d.o.f.
- In sampling-based tensor-network construction, the free-energy density and observables is basically O(1)

$\Rightarrow$ No sign problem even in imaginary β

Actually, we obtain an energy-density result consistent with the worldvolume HMC result for 3D SU(2) gauge theory at $\beta = 0.5i$ and $L = 2$.

Tensor network representation by random sampling

Generate field configurations $\{U_1, U_2, \cdots, U_K\}$ uniformly from the group manifold.
Then approximate the integral by

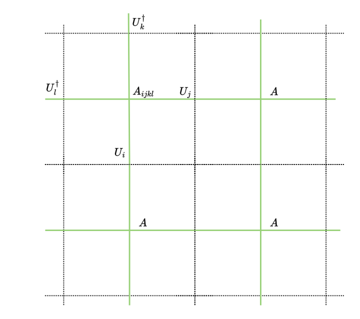
$$\int dU\, f(U) \sim \frac{1}{K}\sum_{i=1}^{K} f(U_i)\,.$$

For pure $SU(N_c)$ gauge theory, the plaquette tensor is

$$A_{ijkl} = \exp\left[\frac{\beta}{N_c}\mathrm{Re}\,\mathrm{Tr}\left(U_i U_j U_k^\dagger U_l^\dagger\right)\right]\,.$$

The partition function is represented as

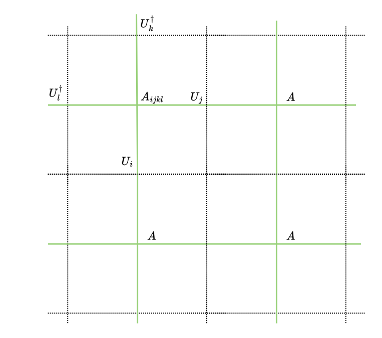
$$Z \sim \frac{1}{K^{dV}}\mathrm{tTr}\prod_{\mathrm{plaquettes}} A$$

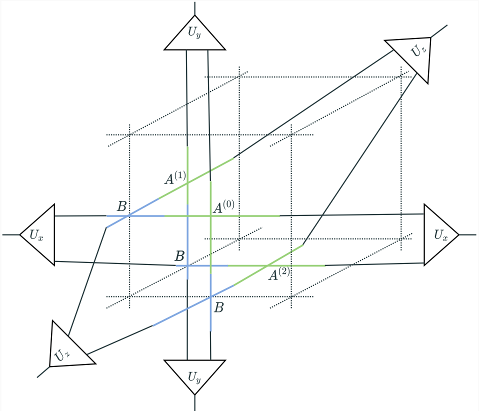


3D tensor network construction

To construct tensor network for a square lattice,
introduce the Kronecker delta $B_{ijkl} = \delta_{ij}\delta_{jk}\delta_{kl}$ on each link

initial tensor: $T = A^{(0)} \otimes A^{(1)} \otimes A^{(2)} \otimes B \otimes B \otimes B$
Exact construction requires $O((K^2)^6)$ memory.
$\Rightarrow$ Introduce isometries (using higher-order SVD) to reduce the bond dimension from K^2 to D.





Plaquette field calculation by impurity tensor

plaquette field

$$\langle W \rangle = \langle \frac{\mathrm{Tr} U_W}{N_c} \rangle = \frac{1}{Z} \int \prod_l dU_l \frac{\mathrm{Tr} U_W}{N_c} e^{-S} \equiv \frac{Z_1}{Z}$$

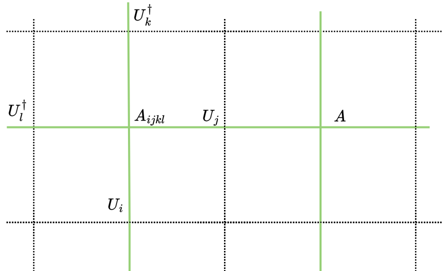
Impurity Tensor A_W (using uniform sampling)

Replace the standard tensor A on a specific plaquette with an impurity tensor A_W:

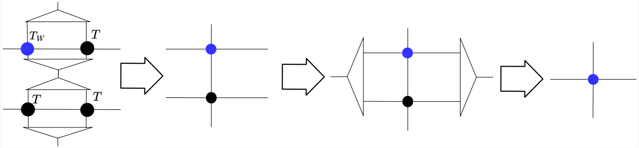
$$(A_W)_{ijkl} = \left[\frac{1}{N_c} \mathrm{Tr} \left(U_i U_j U_k^\dagger U_l^\dagger \right) \right] \exp \left[\frac{\beta}{N_c} \mathrm{Re}\, \mathrm{Tr} \left(U_i U_j U_k^\dagger U_l^\dagger \right) \right]$$

$$Z = \mathrm{tTr} \left(A_1 \otimes A_2 \otimes \cdots \otimes A_N \right)$$
$$Z_1 = \mathrm{tTr} \left(A_1 \otimes A_2 \otimes \cdots \otimes A_W \otimes \cdots \otimes A_N \right)$$

By inserting two impurity tensors, we can also evaluate the 2-point correlation function.
Good point: We can straightforwardly expand it to arbitrary Nc in arbitrary dimensions



Coarse-Graining with Impurities

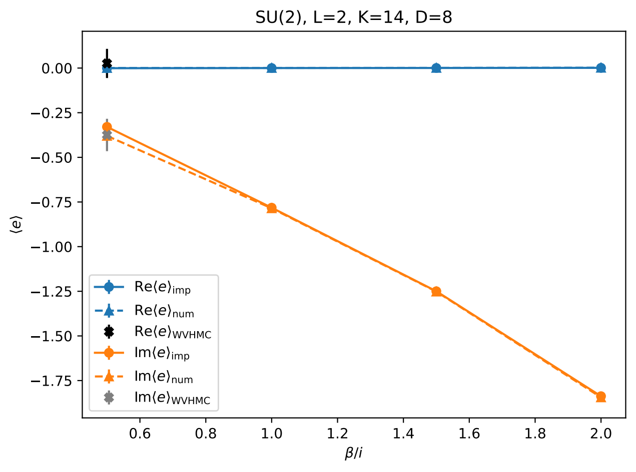


- Coarse-grain the standard tensor T and the impurity tensor T_W in parallel.(cf. [Morita and Kawashima, 2019])
- We calculate 1pt and connected 2pt function of plaquette field for 3D pure $SU(2)$ gauge theory with real/complex coupling constants

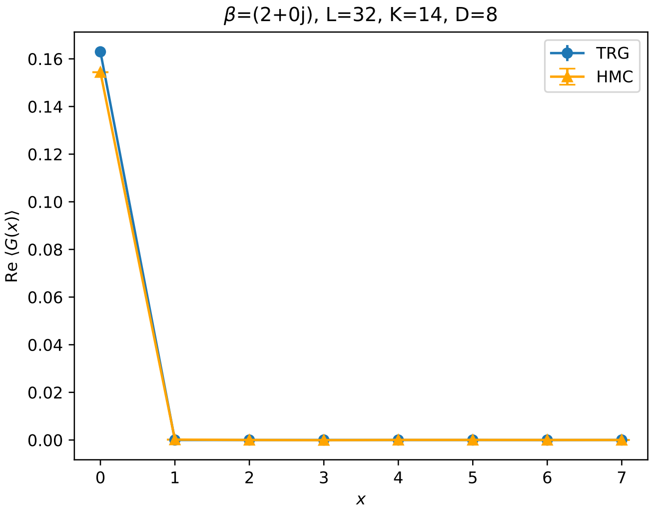
energy density in pure imaginary β

$$e = -\partial f / \partial \beta = -3W$$

Compare impurity(TRG), numerical derivative(TRG)
and worldvolume HMC[Fukuma in prep] (cf. Fukuma's talk on July 28)

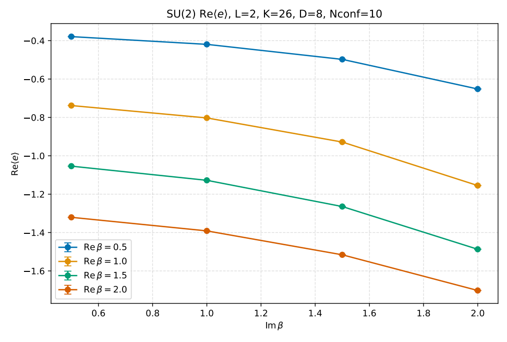


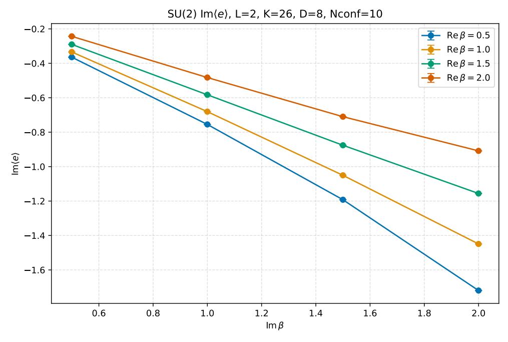
connected 2pt function of plaquette field in real β



energy density in complex β

energy density $e = -\partial f/\partial\beta = -3W$





Summary & future work

Summary

- Established an impurity method to evaluate plaquette expectation values, which is applicable to arbitrary gauge groups and dimensions
- Advantage of impurity: Enables 2pt functions calculations
- Apply this method to 3D pure SU(2) gauge theory, obtain consistent results with the one of HMC/worldvolume HMC

Ongoing and future Works

- Compare with HMC/worldvolume HMC results
- 3D pure $SU(3)$ gauge theory
- 4D pure $SU(2)$, $SU(3)$ gauge theories

References i

Akiyama, S. and Kuramashi, Y. (2022).
Tensor renormalization group study of (3+1)-dimensional $\mathbb{Z}_2$ gauge-Higgs model at finite density.
JHEP, 05:102.

Akiyama, S. and Kuramashi, Y. (2023).
Critical endpoint of (3+1)-dimensional finite density Z_3 gauge-Higgs model with tensor renormalization group.

Akiyama, S. and Kuramashi, Y. (2024).
Tensor renormalization group study of (1 + 1)-dimensional U(1) gauge-Higgs model at $\theta = \pi$ with Lüscher's admissibility condition.
JHEP, 09:086.

References ii

Arai, E., Ohki, H., Takeda, S., and Tomii, M. (2023).
All-mode renormalization for tensor network with stochastic noise.
Phys. Rev. D, 107(11):114515.

Bazavov, A., Catterall, S., Jha, R. G., and Unmuth-Yockey, J. (2019).
Tensor renormalization group study of the non-Abelian Higgs model in two dimensions.
Phys. Rev. D, 99(11):114507.

Bloch, J., Jha, R. G., Lohmayer, R., and Meister, M. (2021).
Tensor renormalization group study of the three-dimensional O(2) model.
Phys. Rev. D, 104(9):094517.

Fukuma, M., Kadoh, D., and Matsumoto, N. (2021).
Tensor network approach to two-dimensional Yang–Mills theories.
PTEP, 2021(12):123B03.

Hirasawa, M., Matsumoto, A., Nishimura, J., and Yosprakob, A. (2021).
Tensor renormalization group and the volume independence in 2D U(N) and SU(N) gauge theories.
JHEP, 12:011.

Kuramashi, Y. and Yoshimura, Y. (2019).
Three-dimensional finite temperature Z_2 gauge theory with tensor network scheme.
JHEP, 08:023.

Kuramashi, Y. and Yoshimura, Y. (2020).
Tensor renormalization group study of two-dimensional U(1) lattice gauge theory with a θ term.
JHEP, 04:089.

References iv

Kuwahara, T. and Tsuchiya, A. (2022).
Toward tensor renormalization group study of three-dimensional non-Abelian gauge theory.
PTEP, 2022(9):093B02.

Levin, M. and Nave, C. P. (2007).
Tensor renormalization group approach to 2D classical lattice models.
Phys. Rev. Lett., 99(12):120601.

Liu, Y., Meurice, Y., Qin, M. P., Unmuth-Yockey, J., Xiang, T., Xie, Z. Y., Yu, J. F., and Zou, H. (2013).
Exact Blocking Formulas for Spin and Gauge Models.
Phys. Rev. D, 88:056005.

Luo, X. and Kuramashi, Y. (2023).
Tensor renormalization group approach to (1+1)-dimensional SU(2) principal chiral model at finite density.
Phys. Rev. D, 107(9):094509.

Morita, S. and Kawashima, N. (2019).
Calculation of higher-order moments by higher-order tensor renormalization group.
Computer Physics Communications, 236:65–71.

Sugimoto, Y., Akiyama, S., and Kuramashi, Y. (2026a).
Phase structure of (3+1)-dimensional dense two-color QCD at T=0 in the strong coupling limit with the tensor renormalization group.
Phys. Rev. D, 113(3):034503.

Sugimoto, Y., Akiyama, S., and Kuramashi, Y. (2026b).
Tensor renormalization group study of cold and dense QCD in the strong coupling limit.

Todo, S. (2026).
Markov chain monte carlo in tensor network representation.

Yosprakob, A. (2024).
Reduced Tensor Network Formulation for Non-Abelian Gauge Theories in Arbitrary Dimensions.
PTEP, 2024(7):073B05.

Yosprakob, A. and Okunishi, K. (2024).
Tensor renormalization group study of the three-dimensional SU(2) and SU(3) gauge theories with the reduced tensor network formulation.