Canonical partition function and equation of state in the heavy quark region of finite-density lattice gauge theory

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Introduction

- We calculate thermodynamic quantities at high density through Z_C.
- Fugacity expansion and Z_3 center symmetry in finite-density QCD

$$Z_{GC}(T,\mu) = \sum_N Z_C(N,T)\, e^{\frac{N\mu}{T}}$$

Z_C: Canonical partition function

$$Z_{GC} = \int \mathcal{D}U \big(\det M(U)\big)^{N_\mathrm{f}} e^{-S_g(U)} \qquad \det M\,(\kappa,\mu) = \sum_N \det M_N\,(\kappa)\,\left(e^{\mu/T}\right)^N$$

- Hopping parameter expansion: $\det M$ can be expressed as a sum of Wilson loops.
- The N-expansion is an expansion in terms of the winding number of WL in the time dir.
- Polyakov loop Ω : order parameter of the center symmetry.

$\langle\Omega\rangle = 0$ symmetric phase

$\langle\Omega\rangle \neq 0$ broken phase

- Z_3 center transformation: $\langle\Omega\rangle \Rightarrow e^{2\pi i/3}\langle\Omega\rangle$

$$\det M_N\,(\kappa) \Rightarrow e^{2\pi i N/3} \det M_N\,(\kappa) \qquad Z_C(N,T) \Rightarrow e^{2\pi i N/3} Z_C(N,T)$$

$$Z_{GC}(T,\mu) = \sum_n Z_C(3n,T)\, e^{\frac{3n\mu}{T}} + \sum_n \underbrace{e^{\frac{2\pi i}{3}} Z_C(3n+1,T)}_{=0} e^{\frac{(3n+1)\mu}{T}} + \sum_n \underbrace{e^{\frac{4\pi i}{3}} Z_C(3n+2,T)}_{=0} e^{\frac{(3n+2)\mu}{T}}$$

In the Symmetric phase, Z_C is zero unless N is a multiple of $N_c = 3$.

- This Z_3 phase is one of the causes of the sign problem. Handling this complex phase makes the calculations of Z_{GC} and Z_C possible.
- In spontaneous symmetry breaking, applying an infinitesimal external field is crucial.
- First step: Heavy dense effective theory of 1-flavor SU(2) gauge theory

Canonical partition function of 1-flavor SU(2) gauge theory

- Fugacity expansion:

$$Z_{GC}(T,\mu) = \sum_N Z_C(N,T)\exp(N\mu/T), \qquad \det M\,(\kappa,\mu) = \sum_N \det M_N\,(\kappa)\, e^{N\mu/T}$$

- Canonical partition function

$$Z_C(N,T) = \int \mathcal{D}U \det M_N\, e^{-S_g} = \langle \det M_N \rangle_{\rm quench} Z_{\rm quench}$$

- Under the center transformation, $\det M_N \to (-1)^N \det M_N, \quad \Omega \to -\Omega$
- When N is odd, the configurations $\det M_N$ and $-\det M_N$ appear with equal probability. $\Longrightarrow$ The sign problem is serious, but $\langle \det M_N \rangle_{\rm quench} = 0$.
- When N is even, such a sign problem does not occur.

heavy fermion

- Apply an infinitesimal external field ε. $\langle \det M_N \rangle_\varepsilon = \dfrac{1}{Z_{\rm quench}} \displaystyle\int \mathcal{D}U \det M_N\, e^{-S_g}\, \underline{e^{\varepsilon V\Omega}}$

$$Z_C(N,T) = \langle \det M_N \rangle_\varepsilon Z_{\rm quench} = \frac{\int \mathcal{D}U \det M_N\, e^{-S_g}\, e^{\varepsilon V\Omega} + \int \mathcal{D}U (-1)^N \det M_N\, e^{-S_g}\, e^{-\varepsilon V\Omega}}{2}$$

- N: even, $Z_C(N,T) = \int \mathcal{D}U \det M_N \cosh(\varepsilon V\Omega)\, e^{-S_g} \approx Z_{\rm quench}\langle \det M_N \rangle_{\varepsilon=0}$
- N: odd, $Z_C(N,T) = \int \mathcal{D}U \det M_N \sinh(\varepsilon V\Omega)\, e^{-S_g} \approx Z_{\rm quench}\varepsilon V\langle \det M_N \Omega \rangle_{\varepsilon=0}$

$$\underline{\frac{Z_C(N+2,T)}{Z_C(N,T)} = \frac{\langle \det M_{N+2} \rangle_{\varepsilon=0}}{\langle \det M_N \rangle_{\varepsilon=0}}\ \text{(even)}, \qquad \frac{Z_C(N+2,T)}{Z_C(N,T)} = \frac{\langle \det M_{N+2}\,\Omega \rangle_{\varepsilon=0}}{\langle \det M_N\,\Omega \rangle_{\varepsilon=0}}\ \text{(odd)}}$$

Heavy quark and high-density effective theory

Partition function of QCD $Z_{GC} = \int \mathcal{D}U \{\det M(U)\}^{N_\mathrm{f}} e^{-S_g(U)}$

Wilson fermion kernel

$$M_{x,y} = \delta_{x,y} - \kappa \sum_{j=1}^{3} [(1-\gamma_j)\, U_{x,i} \delta_{y,x+\hat{j}} + (1+\gamma_j) U_{y,j}^\dagger\, \delta_{y,x-\hat{j}}] - \kappa(1-\gamma_4) U_{x,4} e^{\mu a} \delta_{y,x+\hat{4}} - \kappa(1+\gamma_4) U_{y,4}^\dagger\, e^{-\mu a} \delta_{y,x-\hat{4}}$$

Heavy quark, high-density region $\kappa \to 0, \quad e^{\mu a} \to \infty, \quad \kappa e^{\mu a} \to (\mathrm{finite})$

e.g. Blum, Hetrick, Toussaint, Phys. Rev. Lett. 76, 1019 (1996), Aarts, Stamatescu, JHEP 0809 (2008) 018

Ignore the spatial link terms and the $e^{-\mu a}$ terms

Heavy Dense QCD

$$M_{x,y} = \delta_{x,y} - \kappa(1-\gamma_4) U_{x,4} e^{\mu a} \delta_{y,x+\hat{4}}$$

$\boxed{\mathrm{SU}(3)}$ $$\det M = \prod_{\vec{x}} \left\{1 + 3C\Omega^{\mathrm{loc}}(\vec{x}) + 3C^2 \Omega^{\mathrm{loc}^*}(\vec{x}) + C^3\right\}^2$$

$\boxed{\mathrm{SU}(2)}$ $$\det M = \prod_{\vec{x}} \left\{1 + 2C\Omega^{\mathrm{loc}}(\vec{x}) + C^2\right\}^2$$

$C = (2\kappa)^{N_t} e^{\mu/T}$, local Polyakov loop $\Omega^{\mathrm{loc}}(\vec{x}) = \frac{1}{N_c} \mathrm{tr}(U_{(\vec{x},1),4} U_{(\vec{x},2),4} \ldots U_{(\vec{x},N_t),4})$

- We define $\det M = \sum_N f(N) C^N$, and compute $f(N)$.
- $\det M_N = f(N)(2\kappa)^{NN_t}$, $\quad Z_C(N) = \langle f(N) \rangle_{\mathrm{quench}} (2\kappa)^{NN_t} Z_{\mathrm{quench}}$

Calculation of $f(n)$

$$\det M = \prod_{\vec{x}} \left(1 + 2C\Omega^{\mathrm{loc}}(\vec{x}) + C^2\right)^2 = \sum_N f(N) C^N$$

Label $\vec{x}$ in 3D space from 1 to N_s^3, and let $f^m(n)$ be the coefficient obtained by multiplying up to the m-th point. Using $f^{m-1}(n)$, $f^m(n)$ is obtained as

$$f^m(1) = f^{m-1}(1) + f^{m-1}(0) 4\Omega^{\mathrm{loc}}(m)$$

$$f^m(2) = f^{m-1}(2) + f^{m-1}(1) 4\Omega^{\mathrm{loc}}(m) + f^{m-1}(0)\left(2 + 4{\Omega^{\mathrm{loc}}}^2(m)\right)$$

$$f^m(3) = f^{m-1}(3) + f^{m-1}(2) 4\Omega^{\mathrm{loc}}(m) + f^{m-1}(1)\left(2 + 4{\Omega^{\mathrm{loc}}}^2(m)\right) + f^{m-1}(0) 4\Omega^{\mathrm{loc}}(m)$$

For $n \geq 4$, $$f^m(n) = f^{m-1}(n) + f^{m-1}(n-1) 4\Omega^{\mathrm{loc}}(m) + f^{m-1}(n-2)\left(2 + 4{\Omega^{\mathrm{loc}}}^2(m)\right)$$
$$+ f^{m-1}(n-3) 4\Omega^{\mathrm{loc}}(m) + f^{m-1}(n-4)$$

Then, $f(n) = f^{N_s^3}(n)$. $\quad f(0) = f^{N_s^3}(0) = 1, \quad f(1) = f^{N_s^3}(1) = \sum_{\vec{x}} 4\Omega^{\mathrm{loc}}(\vec{x}) = 4N_s^3 \Omega$

By distinguishing between the cases where n is even and odd, the ratio of $Z_c(n+2,T)$ and $Z_c(n,T)$ can be calculated.

$$\ln \frac{Z_c(n+2,T)}{Z_c(n,T)} = \begin{cases} 2N_t \ln(2\kappa) + \ln \dfrac{\langle f(n+2,T)\rangle_{\mathrm{quench}}}{\langle f(n,T)\rangle_{\mathrm{quench}}} & (\mathrm{n: even}) \\ 2N_t \ln(2\kappa) + \ln \dfrac{\langle f(n+2,T)\Omega\rangle_{\mathrm{quench}}}{\langle f(n,T)\Omega\rangle_{\mathrm{quench}}} & (\mathrm{n: odd}) \end{cases}$$

$$\underset{(N:\,\mathrm{even})}{\ln Z_c(N,T)} = \ln Z_c(0,T) + \sum_{n=1}^{N/2} \ln \frac{Z_c(2n,T)}{Z_c(2n-2,T)}, \qquad \underset{(N:\,\mathrm{odd})}{\ln Z_c(N,T)} = \ln Z_c(1,T) + \sum_{n=1}^{(N-1)/2} \ln \frac{Z_c(2n+1,T)}{Z_c(2n-1,T)}$$

Grand partition function & Equation of state

- $Z_{GC}(\mu,T)=\sum_n Z_C(n,T)e^{n\mu/T}=Z_{\text{quench}}\sum_n\langle f(n,T)\rangle e^{n(\frac{\mu}{T}+N_t\ln 2\kappa)}$
- $f(0)=1,\ \ f(1)=4N_S^3\Omega,$ $\quad\langle\cdots\rangle=\langle\cdots\rangle_{\text{quench}}$

$$\frac{Z_{GC}(\mu,T)}{Z_{\text{quench}}}=\sum_{n:\text{even}}\langle f(n,T)\rangle e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)}+4N_S^3\underline{\langle\Omega(T)\rangle}\sum_{n:\text{odd}}\frac{\langle f(n,T)\Omega\rangle}{\langle f(1,T)\Omega\rangle}\,e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)}$$

- The contributions from odd numbers depends on the order parameter $\langle\Omega\rangle$.

$$\begin{cases}\text{confinement phase}\quad : Z_{\text{C}}^{\text{odd}}(n,T)=0\ \Leftrightarrow\langle\Omega(T)\rangle=0\\ \text{deconfinement phase}: Z_{\text{C}}^{\text{odd}}(n,T)\neq 0\ \Leftrightarrow\langle\Omega(T)\rangle\neq 0\end{cases}$$

  - In the confinement phase, odd terms do not contribute to Z_{GC}.
  - This means that there is no single-fermion contribution (confinement).
- The sum of n can be well approximated by only a few terms around the point where $\langle f(n,T)\rangle e^{n(\mu/T+N_t\ln 2\kappa)}$ is maximized.
- Thermodynamic quantities

Pressure: $p=\frac{T}{V}\ln Z_{GC}(\mu,T),\qquad$ Particle density: $\rho=\frac{T}{V}\frac{\partial\ln Z_{GC}(\mu,T)}{\partial\mu}$

Overlap problem

- Since $f(n) \sim \Omega^n$, configurations where Ω is near zero contribute nothing to the statistical sum in the MC method.
- We perform simulations incorporating an external Polyakov loop field.

$$\langle O \rangle_h = \frac{1}{Z} \int \mathcal{D}U \, O \, e^{-S_g + h N_S^3 \Omega}$$

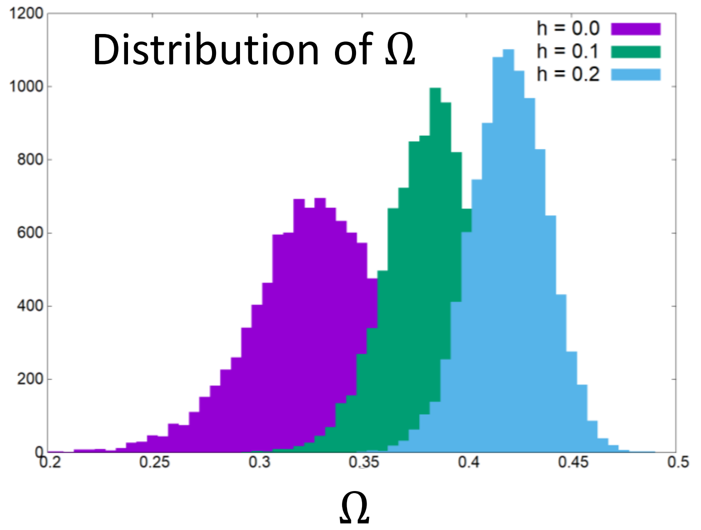
- Reweighting

$$\langle f(n) \rangle_{\text{quench}} = \left\langle f(n)\, e^{-h N_S^3 \Omega} \right\rangle_h \Big/ \left\langle e^{-h N_S^3 \Omega} \right\rangle_h$$

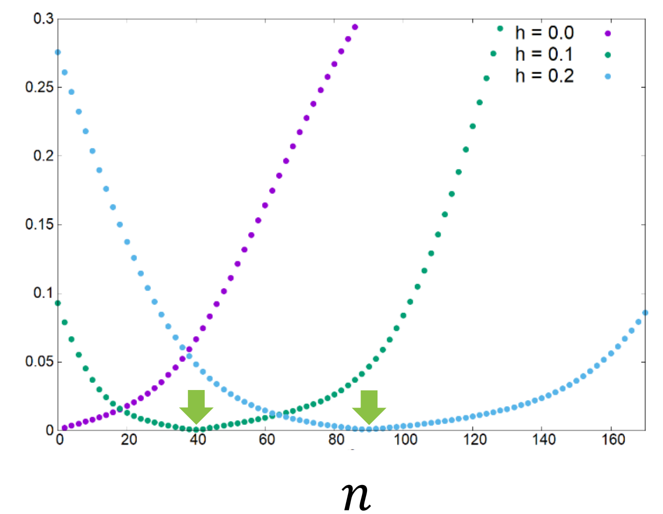
- We used the mean-field approximation to determine h such that the contributions from each configuration are equal.

$$\frac{d(\ln f(n) - h N_S^3 \Omega)}{\mathrm{d}\Omega} = 0, \qquad h = \frac{1}{N_S^3} \frac{\mathrm{d} \ln f(n)}{d\Omega}$$

- At the optimal n for each h, the statistical error is small. We determine the measurable range.



statistical error of $\langle f(n) e^{-h N_S^3 \Omega} \rangle_h$

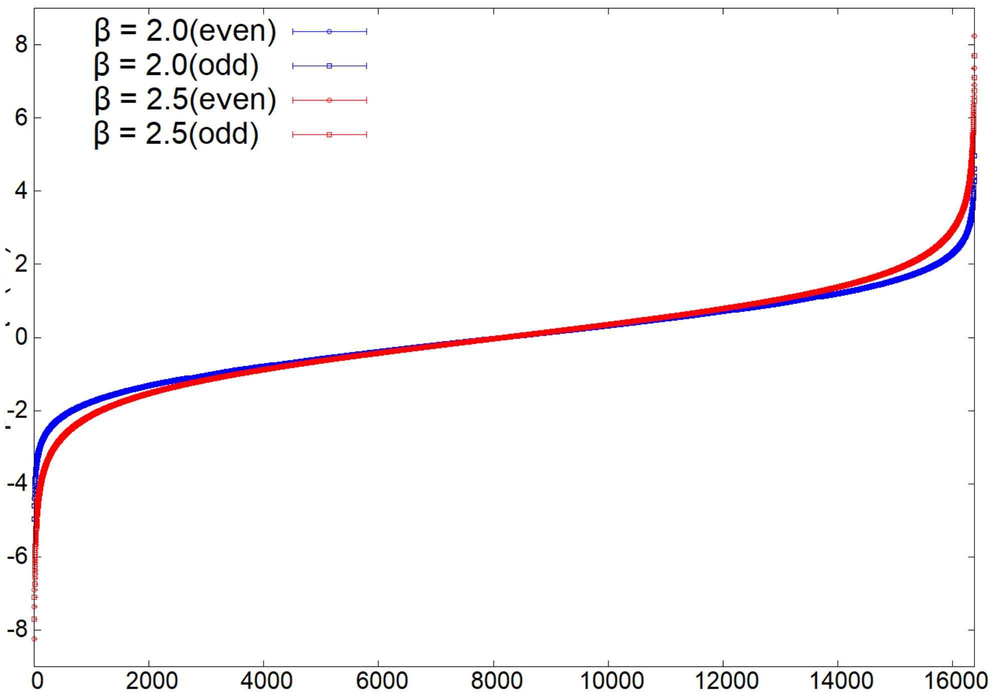


Optimal value of n for each h

Simulation results

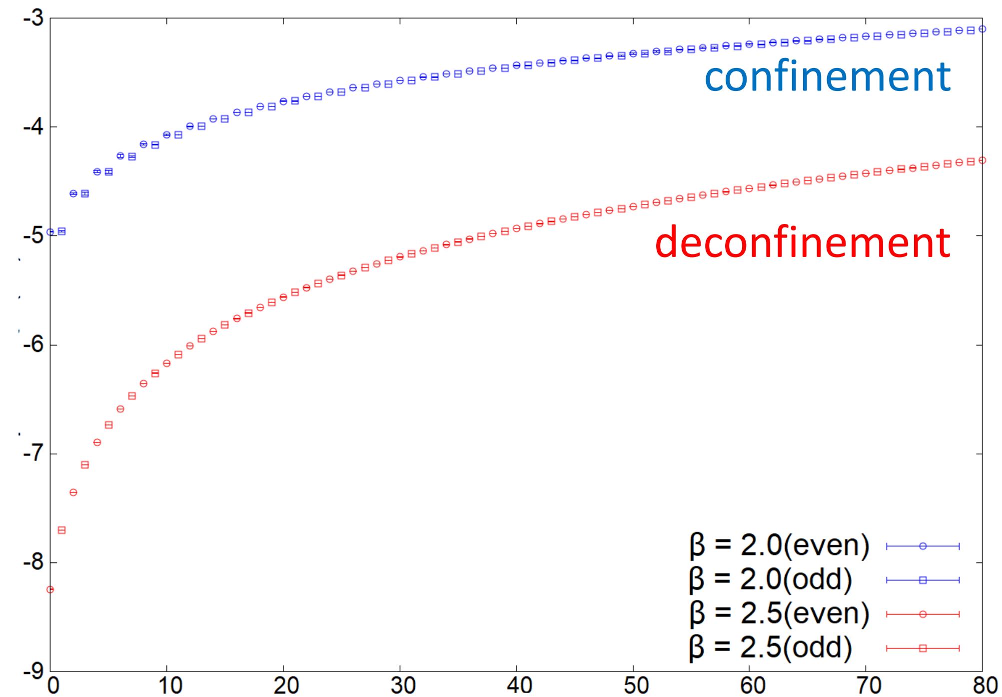
$$-\frac{\Delta \ln\langle f(n)\rangle}{\Delta n} = -\frac{\ln\langle f(n+2)\rangle - \ln\langle f(n)\rangle}{2}$$

$-\Delta \ln\langle f(n)\rangle / \Delta n$



n

$-\Delta \ln\langle f(n)\rangle / \Delta n$ (even, odd)



n

$\beta = 2.0$ (confinement)

$\beta = 2.5$ (deconfinement)

Transition point:
$\beta_c = 2.3$

Lattice size : $16^3 \times 4$

symbols
○ n: even
□ n: odd

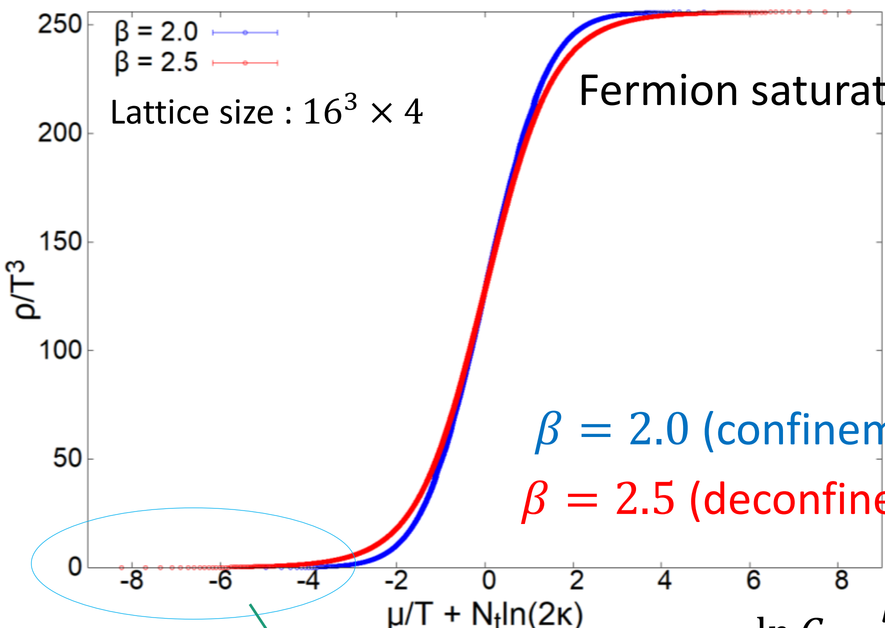
- $-\Delta \ln\langle f(n)\rangle / \Delta n$ does not differ significantly depending on whether n is even or odd.
- The heavy dense effective theory has high-density-low-density symmetry.

Particle number density

$$\rho = \frac{T}{V}\frac{\partial \ln Z_{GC}(\mu,T)}{\partial \mu}$$

$$\frac{\rho}{T^3} = \left(\frac{N_t}{N_s}\right)^3 \frac{1}{Z_{\mathrm{GC}}(\mu,T)/Z_{\mathrm{quench}}}\left\{\sum_{n:\mathrm{even}} n\langle f(n,T)\rangle e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)} + 4N_s^3 \underbrace{\langle\Omega(T)\rangle}_{\text{quenched}} \sum_{n:\mathrm{odd}} n\frac{\langle f(n,T)\Omega\rangle}{\langle f(1,T)\Omega\rangle} e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)}\right\}$$

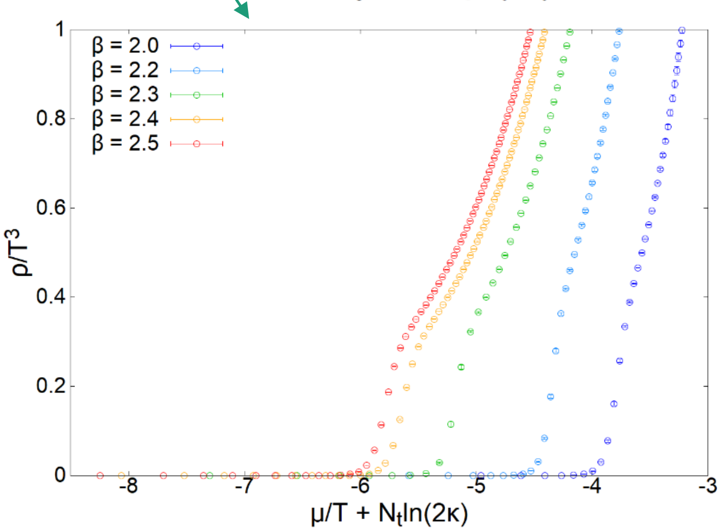
$$\frac{Z_{GC}(\mu,T)}{Z_{\mathrm{quench}}} = \sum_{n:\mathrm{even}} \langle f(n,T)\rangle e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)} + 4N_s^3 \underbrace{\langle\Omega(T)\rangle}_{\text{quenched}} \sum_{n:\mathrm{odd}} \frac{\langle f(n,T)\Omega\rangle}{\langle f(1,T)\Omega\rangle} e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)}$$



$\beta = 2.0$ (confinement)

$\beta = 2.5$ (deconfinement)

$$\ln C = \frac{\mu}{T} + 2N_t \ln(2\kappa)$$



Polyakov loop $\langle\Omega(T)\rangle$ in quenched QCD

In the confined phase, $\langle\Omega\rangle = 0$, so the sum over n involves only even values.

in the deconfined phase, $\langle\Omega\rangle$ is finite, so calculations were performed using $\langle|\Omega|\rangle$ as an upper bound.

The density increases rapidly near $\frac{\mu}{T} + N_t \ln 2\kappa = 0$.

The change is more gradual at the higher temperature ($\beta = 2.5$).

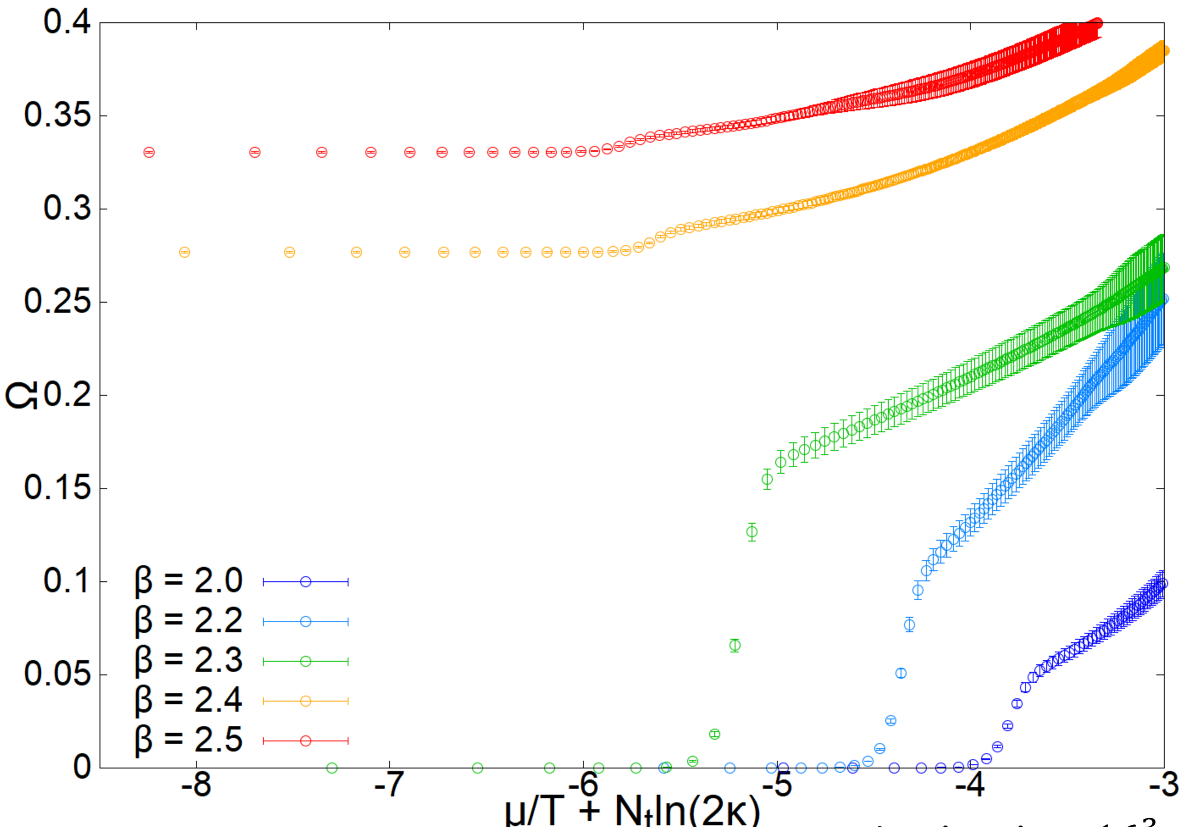
Focusing on the low-density region, the density suddenly increases from zero to a finite value at a certain μ.

Even if β is large, the density changes from zero.

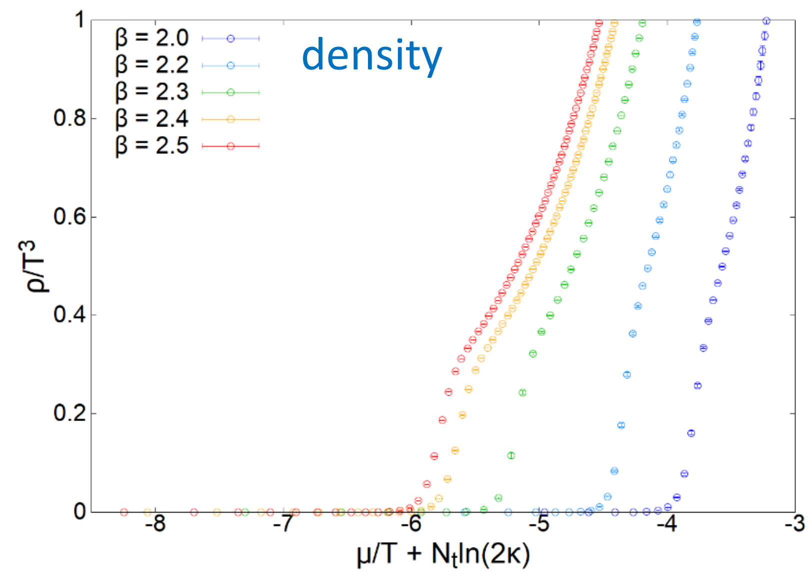
Polyakov loop

$$\langle\Omega\rangle_{(\mu,T)} = \frac{\int DU_\mu\,\Omega\det M\,e^{-S_g}}{\int DU_\mu\det M\,e^{-S_g}}$$

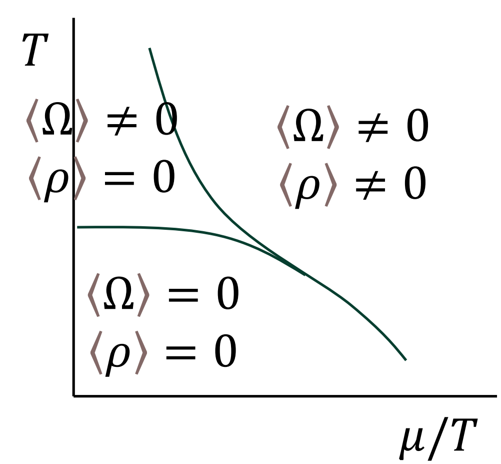
$$\langle\Omega\rangle_{(\mu,T)} = \frac{\langle\Omega\rangle\sum_{n:\mathrm{even}}\frac{\langle\Omega^2 f(n,T)\rangle}{\langle\Omega^2 f(0,T)\rangle}e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)} + \langle\Omega f(1,T)\rangle\sum_{n:\mathrm{odd}}\frac{\langle\Omega f(n,T)\rangle}{\langle\Omega f(1,T)\rangle}e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)}}{\sum_{n:\mathrm{even}}\frac{\langle f(n,T)\rangle}{\langle f(0,T)\rangle}e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)} + \langle f(1,T)\rangle\sum_{n:\mathrm{odd}}\frac{\langle\Omega f(n,T)\rangle}{\langle\Omega f(1,T)\rangle}e^{n\left(\frac{\mu}{T}+N_t\ln 2\kappa\right)}}$$



Lattice size : $16^3 \times 4$



Transition point of quench: $\beta_c = 2.30$



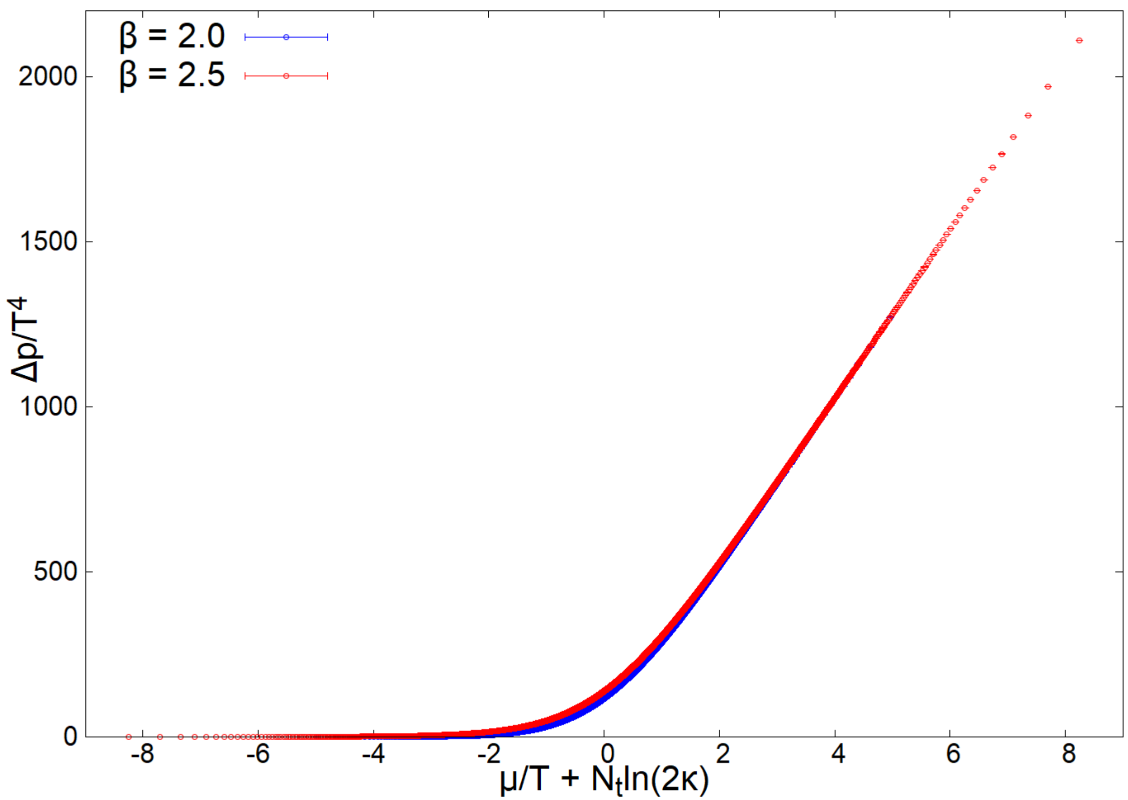
- Polyakov loop changes at $\beta_c = 2.3$.
- Density is zero even at $\beta > 2.3$.

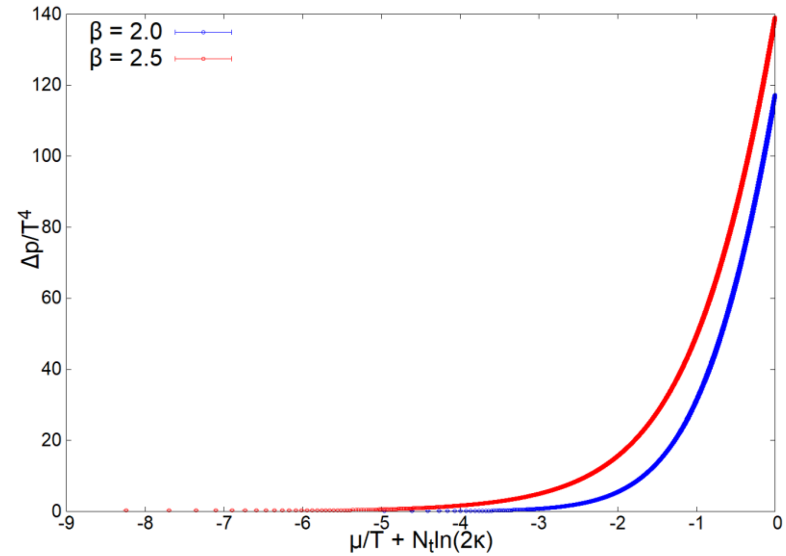
Pressure

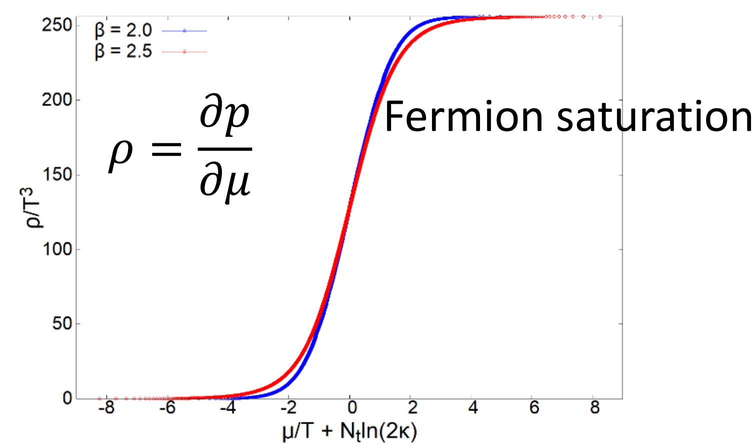
$$\frac{\Delta p}{T^4} = \frac{p}{T^4} - \frac{p_{\mathrm{quench}}}{T^4}$$

$$\frac{p}{T^4} = \frac{T}{T^4 V} \ln Z_{GC}(\mu, T) = \left(\frac{N_t}{N_s}\right)^3 \ln \frac{Z_{\mathrm{GC}}(\mu, T)}{Z_{\mathrm{quench}}} + \left(\frac{N_t}{N_s}\right)^3 \ln Z_{\mathrm{quench}} = \frac{\Delta p}{T^4} + \frac{p_{\mathrm{quench}}}{T^4}$$

Here, Δp is the deviation from the pressure of quenched QCD.

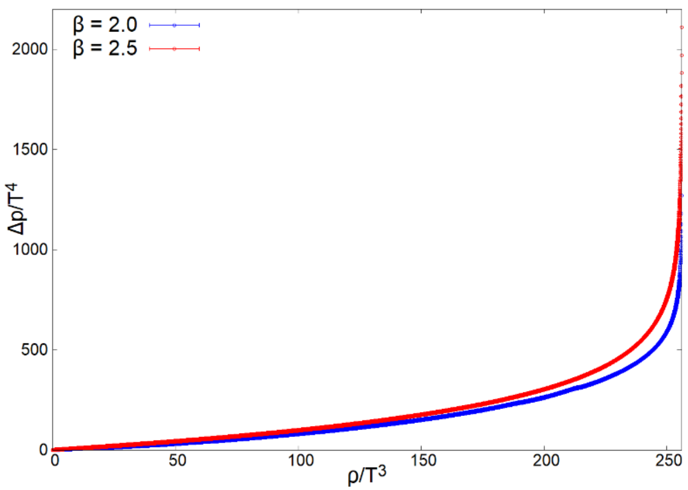






When $\frac{\mu}{T} + N_t \ln(2\kappa)$ exceeds zero, the density saturates, and the pressure increases linearly.

Pressure and density (softness), Speed of sound



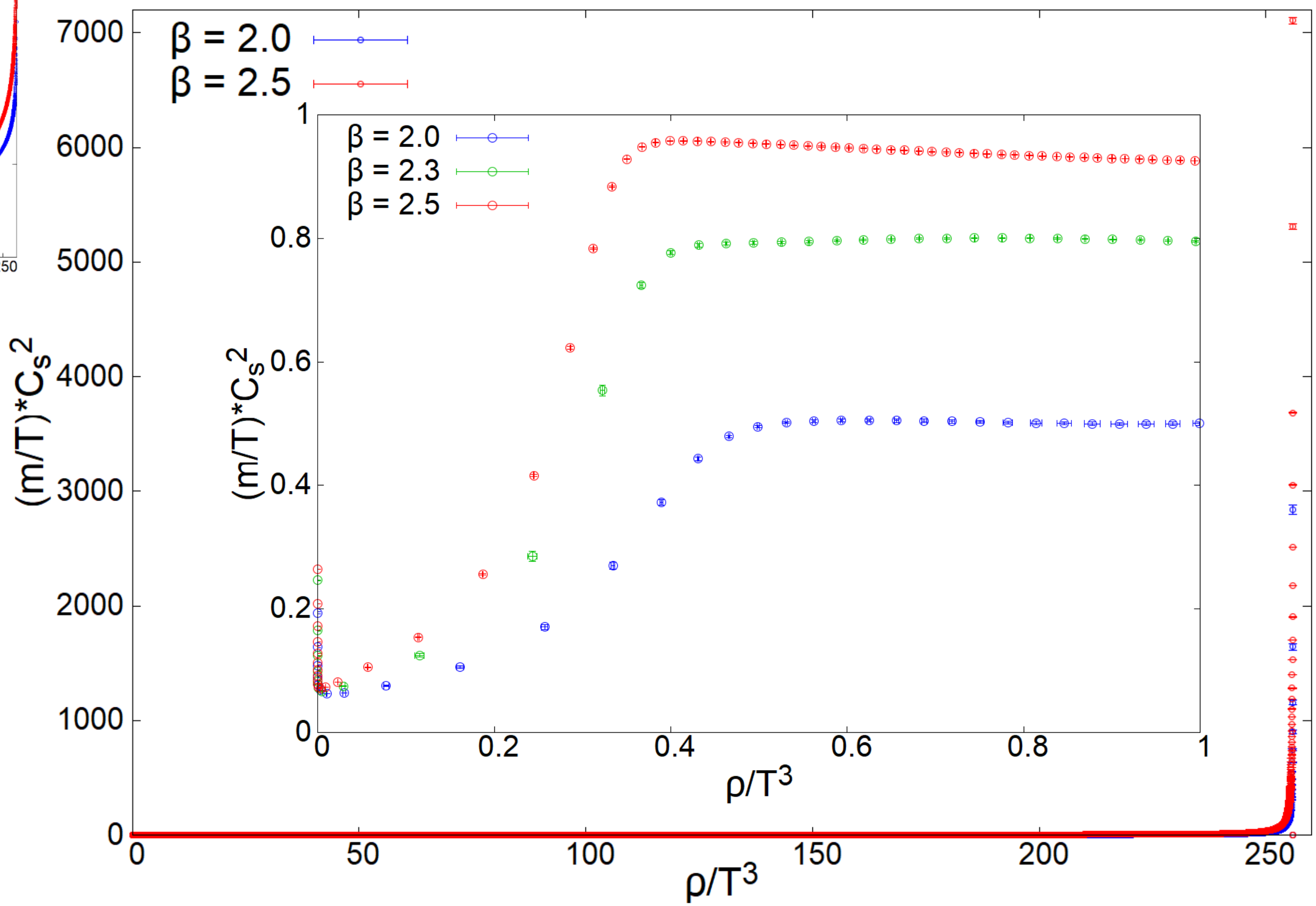
Pressure and particle number density are plotted with μ/T as an internal variable.

Lattice size : $16^3 \times 4$

Transition point of quench: $\beta_c = 2.30$

Non-relativistic speed of sound: C_s

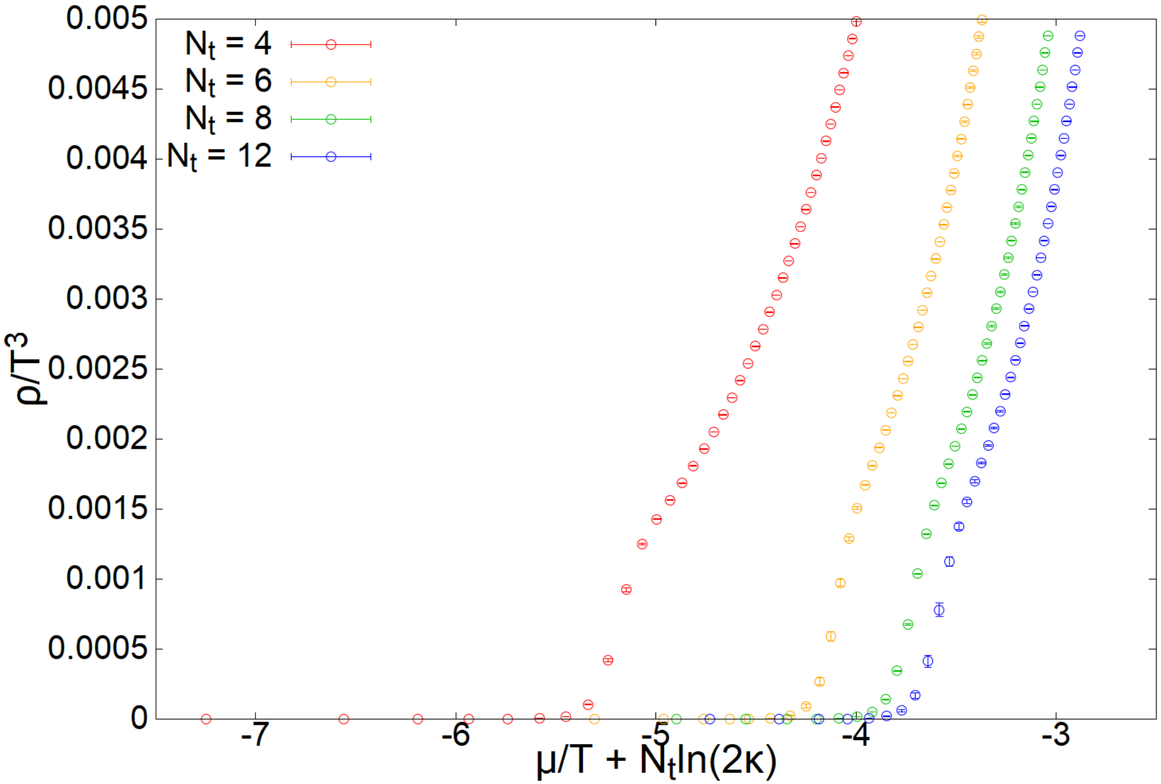
$$C_s^2 = \frac{dp}{d\rho_m} = \frac{T}{m}\frac{d(\Delta p/T^4)}{d(\rho/T^3)}, \qquad \frac{m}{T}C_s^2 = \frac{d(\Delta p/T^4)}{d(\rho/T^3)}$$

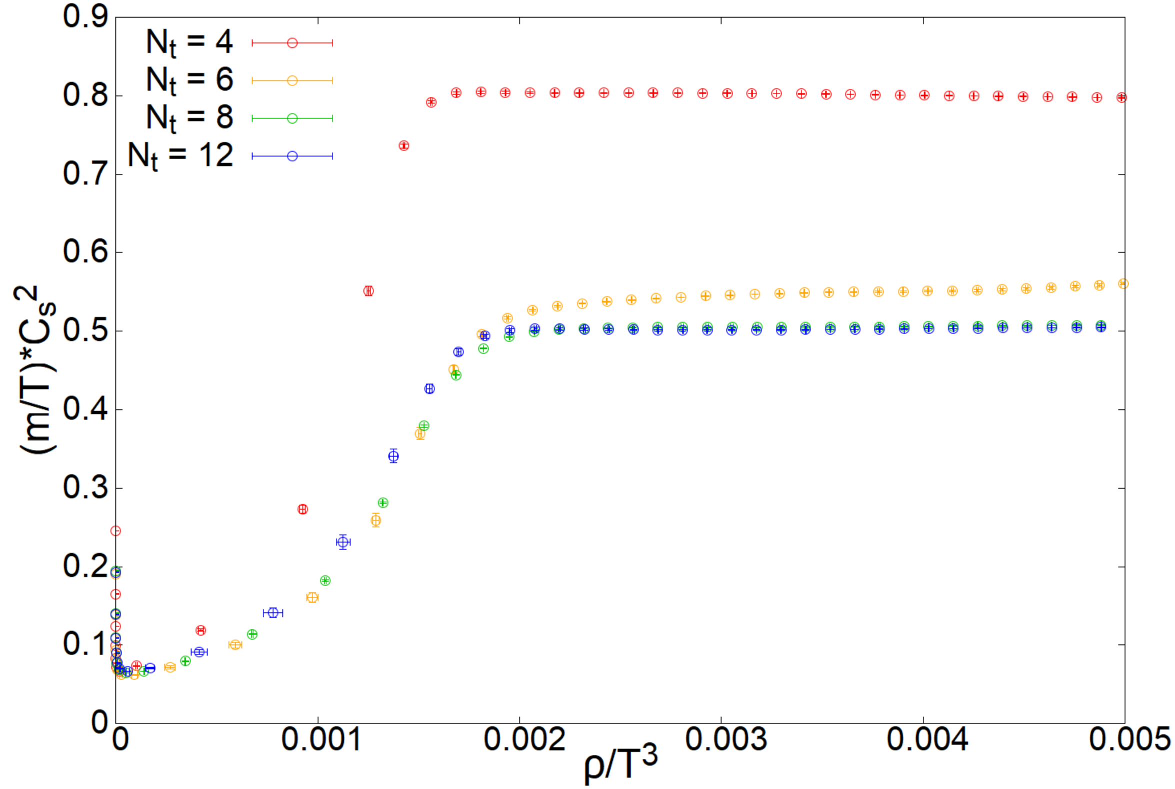


The speed of sound slows down when density changes from zero to a finite value.

Density and speed of sound

- Simulation point: Transition point of $N_t = 4,\ \beta = 2.30$
- Temperature
  - $N_t = 4$: $T = T_c$
  - $N_t = 6$: $T = \frac{2}{3} T_c$
  - $N_t = 8$: $T = \frac{1}{2} T_c$
  - $N_t = 12$: $T = \frac{1}{3} T_c$
- In the low-temperature phase, the temperature dependence of the speed of sound is small.



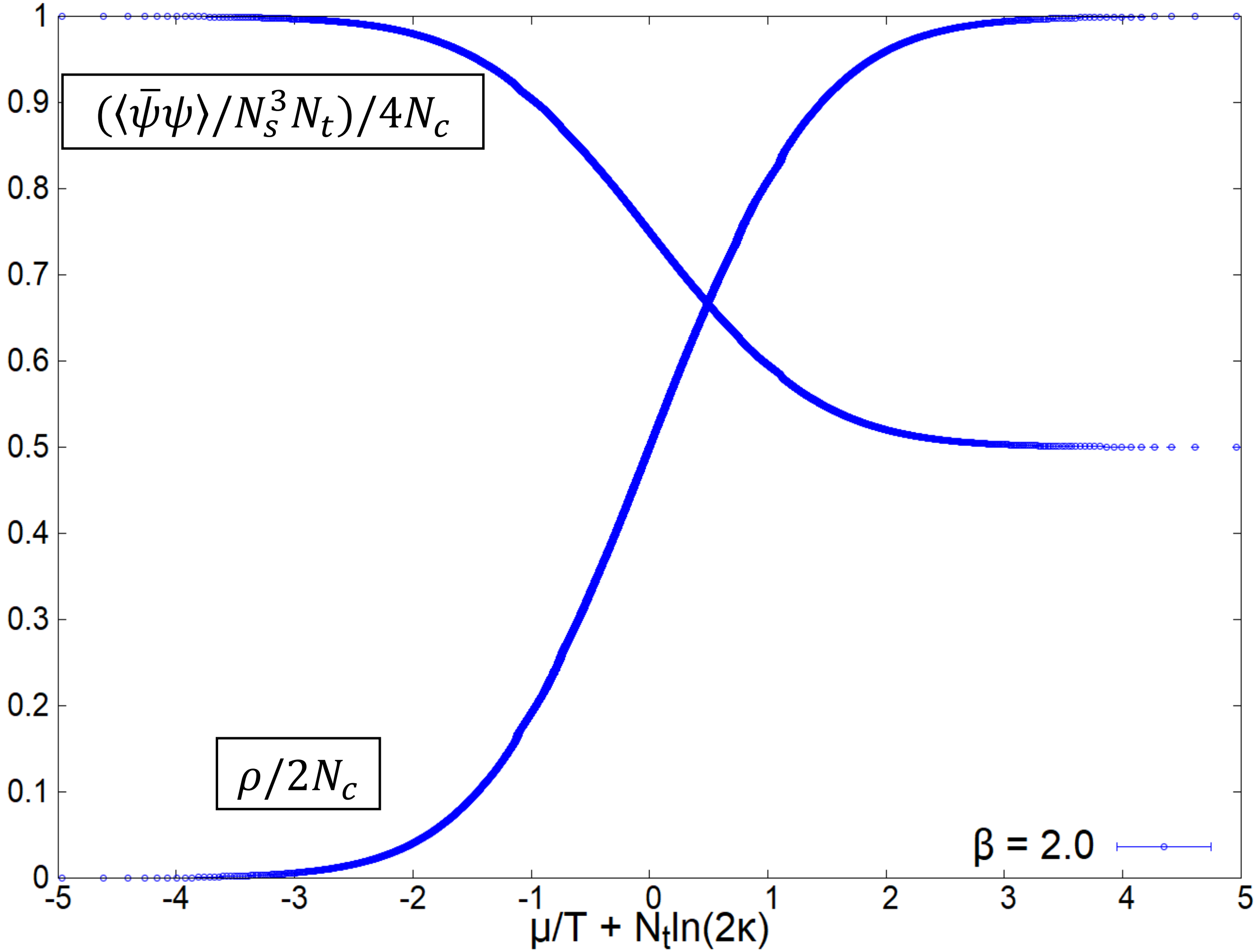


Chiral condensation

Heavy Dense effective theory

$$\langle\bar{\psi}\psi\rangle = \langle \mathrm{tr}\, M^{-1}\rangle = 4N_c N_s^3 N_t - \frac{Z_{\mathrm{quench}}}{Z_{GC}} \kappa \left\langle \frac{\partial \det M}{\partial \kappa} \right\rangle_{\mathrm{quench}} = N_s^3 N_t (4N_c - \langle\rho\rangle)$$

Particle number density in the lattice unit and chiral condensation



Maximum value normalized to 1

Because the fermions are heavy, chiral condensation does not become zero.

This result is similar to the result obtained using the tensor renormalization group at $\beta = 0, m = 0$. [Sugimoto, Akiyama, Kuramashi, Phys. Rev. D 113, 034503 (2026)]

Summary

- We investigate the phase structure and equations of state in lattice gauge theory at finite temperature and finite density, focusing on the canonical partition function.
- we propose an algorithm to calculate the canonical partition function for heavy-quark, high-density effective theory and first tested it using SU(2) gauge theory with one-flavor.
- For SU(2), it can be seen that the sign problem is serious in the part with an odd number of particles, while it is not so serious in the part with an even number of particles.
- The sign problem was solved by employing regularization that introduces an infinitesimal external field.
- We calculated the equation of state in SU(2) gauge theory with heavy one flavor at high density.
- As a future research, applying this method to SU(3) gauge theory is very interesting.