Renormalizing the two-photon contribution to $K_{\mathrm{L}} \to \mu^+\mu^-$ decay II

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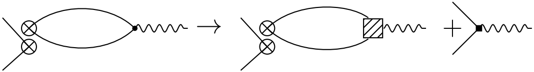
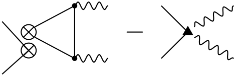
Based on arXiv:2607.14077

Outline

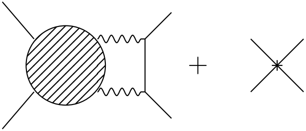
- Introduction
- Implementation
  - Class-A



  - Class-B



  - Class-C

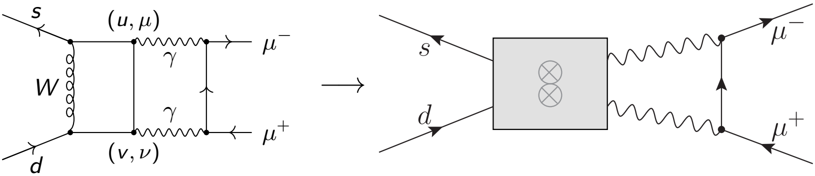


- Renormalization of $\mathrm{Im}A^{K\mu\mu}$
- Summary and outlook

Introduction

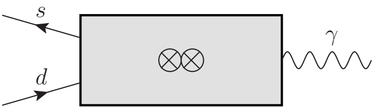
- $N_f = 3$ calculation with current-current operators (arXiv:2509.04346)

$$\begin{aligned}
A^{K\mu\mu}(\delta_{\max}, R_{\max}) &= \int_{-\frac{T}{2}}^{\delta_{\max}} dv_0 \int_{v_0}^{v_0+R_{\max}} du_0 \int_V d^3\mathrm{u} \int_V d^3\mathrm{v} \\
&\quad \langle 0|J_\mu(u)J_\nu(v)H_w^{N_f=3}(0)|K_{\mathrm{L}}\rangle K_{\mu\nu}(u,v) &(1)\\
H_w^{N_f=3} &= C_1 Q_1^u + C_2 Q_2^u &(2)\\
Q_1^q &= (\bar{s}_a q_b)_{\mathrm{V-A}}\,(\bar{q}_b d_a)_{\mathrm{V-A}} &(3)\\
Q_2^q &= (\bar{s}_a q_a)_{\mathrm{V-A}}\,(\bar{q}_b d_b)_{\mathrm{V-A}} &(4)
\end{aligned}$$

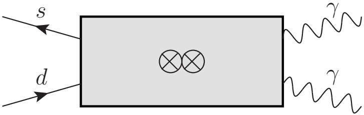


Introduction

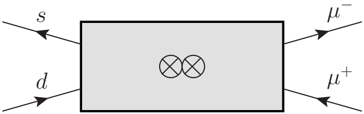
- Weak operator + external legs subdiagram at $\mathcal{O}\left(\alpha_s^0(m_c)\right)$:
  - Class-A : HVP-like
  - Class-B : Adler ambiguity
  - Class-C : Whole graph



Class – A

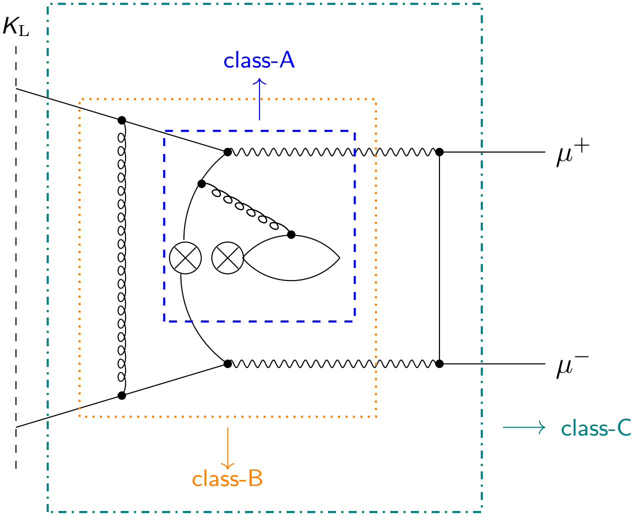


Class – B



Class – C

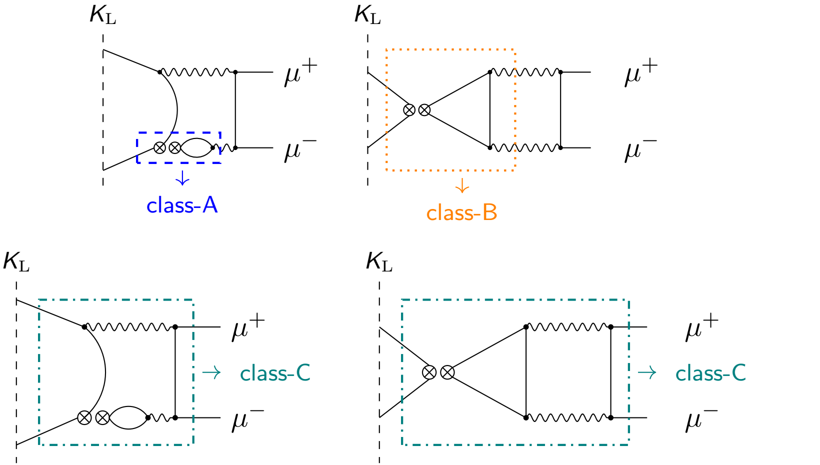
Introduction



Implementation

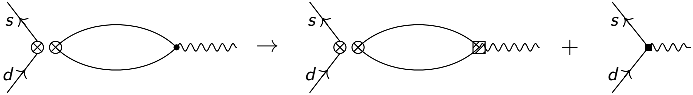
Implementation

- $N_f = 3$, 24ID($a^{-1} = 1.023$ GeV) $\longrightarrow$ $N_f = 4$, 32IF($a^{-1} = 3.148$ GeV)

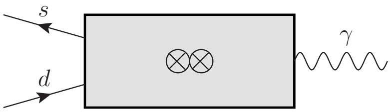


- Conserved current with one counterterm:

$$Q_1^A = \left(\bar{s}\gamma^\mu d + \bar{d}\gamma^\mu s\right) \partial^\nu F_{\mu\nu} \tag{5}$$



- NPR scheme:
  - $\mu \ll m_c$, e.g. $\mu \sim \mathcal{O}(500)$ MeV
  - heavy quarks → suppress gauge noise + small-volume calculation



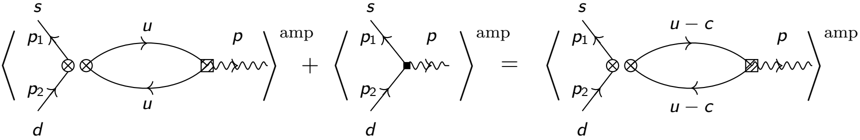
Class − A

- Matching condition $(p_1^2 = p_2^2 = (p_1 - p_2)^2 = \mu^2)$

$$\mathrm{Tr}\left\{\gamma^\mu \Lambda_\mu^{N_f=4}\right\} = \mathrm{Tr}\left\{\gamma^\mu \Lambda_\mu^{N_f=3}\right\} + C_1^A \mathrm{Tr}\left\{\gamma^\mu \Gamma_\mu\right\} \tag{6}$$

$$\Gamma_{\beta\alpha,\mu}(p_1, p_2) = \int d^4x e^{i(p_1-p_2)x} \langle \tilde{s}_\alpha(p_1) Q_1^A(0) \bar{\tilde{d}}_\beta(p_2) A_\mu(x)\rangle^{\mathrm{amp}} \tag{7}$$

$$\Lambda_{\beta\alpha,\mu}^{N_f}(p_1, p_2) = \int d^4x e^{i(p_1-p_2)x} \langle \tilde{s}_\alpha(p_1) H_w^{N_f}(0) J_\mu^{\mathrm{con}}(x) \bar{\tilde{d}}_\beta(p_2)\rangle^{\mathrm{amp}} \tag{8}$$



Preliminary results

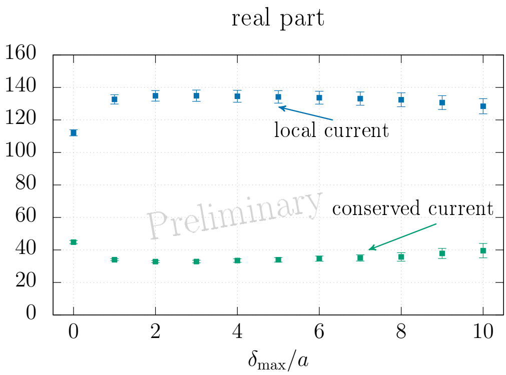
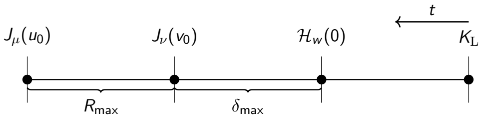


Figure 1: Results for local & conserved vector current results.

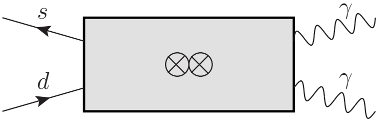


Class-B

- Class-B diagram is finite but ambiguous
- Ambiguity contributes as a local counterterm

$$\zeta \times \epsilon_{\mu\nu\rho\sigma}(\partial^\rho A^\mu)A^\nu J_{\mathrm{A}}^\sigma, \;\; J_{\mathrm{A}}^\sigma = \bar{s}\gamma^\sigma\gamma_5 d + \bar{d}\gamma^\sigma\gamma_5 s \tag{9}$$

- ζ is independent of quark mass and external momenta



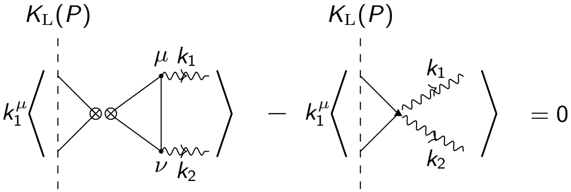
Class-B

- ζ determination

$$k_1^\mu R_{\mu\nu}(k_1,k_2,P) - \zeta k_1^\mu A_{\mu\nu}(k_1,k_2,P) = 0 \tag{10}$$

$$R_{\mu\nu}(k_1,k_2,P) = \langle \tilde{J}_\mu(k_1)\tilde{J}_\nu(k_2)H_w(0)|K_\mathrm{L}(P)\rangle \tag{11}$$

$$A_{\mu\nu}(k_1,k_2,P) = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}(k_1-k_2)^\rho \langle 0|J_\mathrm{A}^\sigma(0)|K_\mathrm{L}(P)\rangle \tag{12}$$



- kinematics comparison ($q = P - k_1 - k_2$)
  - kinematics 1 (lattice momentum)

$$a\vec{k_1} = \frac{2\pi a}{L}(-1, 1, 1), a\vec{k_2} = \frac{2\pi a}{L}(-1, -2, 0)$$

  - kinematics 2 (non-lattice momentum)

$$a\vec{k_1} = \frac{2\pi a}{L}(\frac{1}{2}, \frac{3}{2}, \frac{\sqrt{2}}{2}), a\vec{k_2} = \frac{2\pi a}{L}(-1.3849, -0.9121, \frac{3}{2})$$

$$a^2|\vec{k_1}|^2 = 0.2056, a^2|\vec{k_2}|^2 = 0.3427, a^2|\vec{q}|^2 = 0.4112$$

- quark mass comparison
  - mass 1: $am_u = 0.085$
  - mass 2: $am_u = 0.147$

Preliminary results

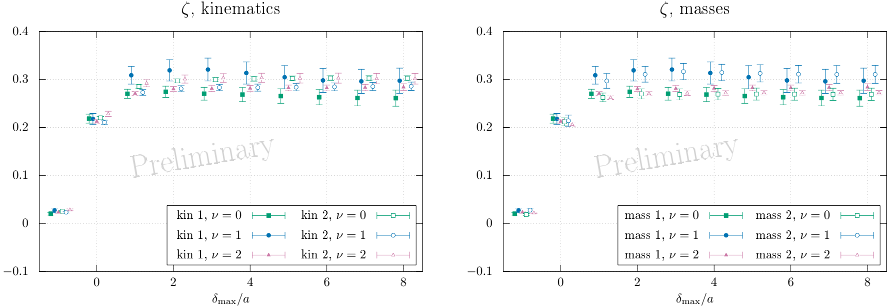


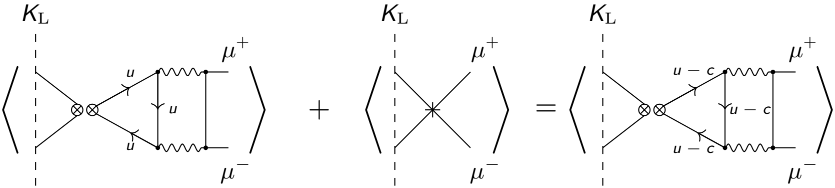
Figure 2: Comparison ζ between different kinematics and loop quark masses.

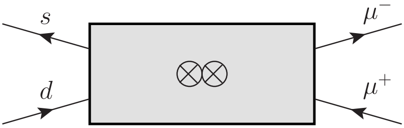
Class-C

- Work with matrix element instead of amputated Green's function

$$O^{AA} = \left(\bar{s}\gamma_\nu\gamma_5 d + \bar{d}\gamma_\nu\gamma_5 s\right)\bar{\mu}\gamma^\nu\gamma^5\mu \tag{13}$$

$$\langle\mu^+\mu^-|H_w^{N_f=4}|K_{\rm L}\rangle = \langle\mu^+\mu^-|H_w^{N_f=3}|K_{\rm L}\rangle + \lambda\langle\mu^+\mu^-|O_{\rm RI}^{AA}|K_{\rm L}\rangle \tag{14}$$





Class − C

Preliminary results

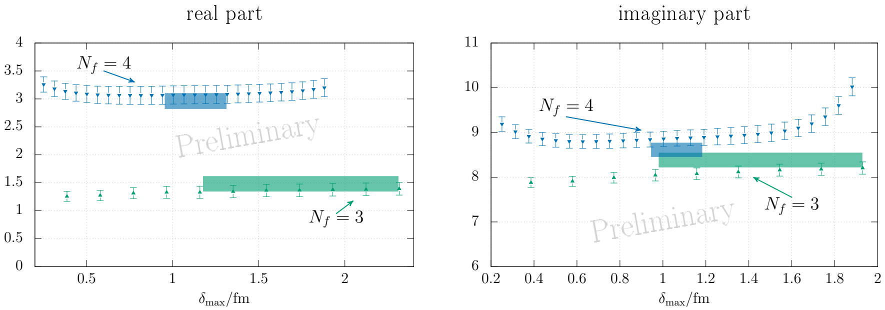
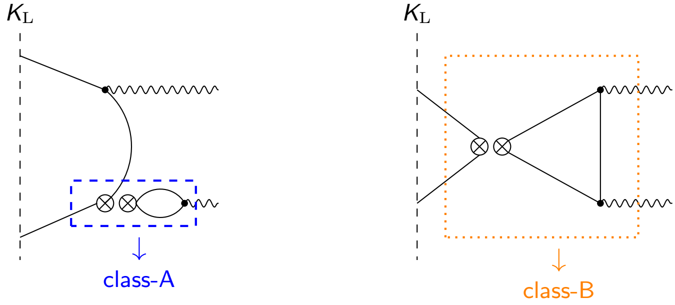


Figure 3: Class-C results of Type3 for $N_f = 3$ and $N_f = 4$ calculation. Color bands are from asymptotic fit and discrete points are from explicit unphysical state subtraction.

Renormalization of $\mathrm{Im} A^{K\mu\mu}$

- Two diagrams need renormalization at $\mathcal{O}\left(\alpha_s^0(m_c)\right)$



Preliminary results

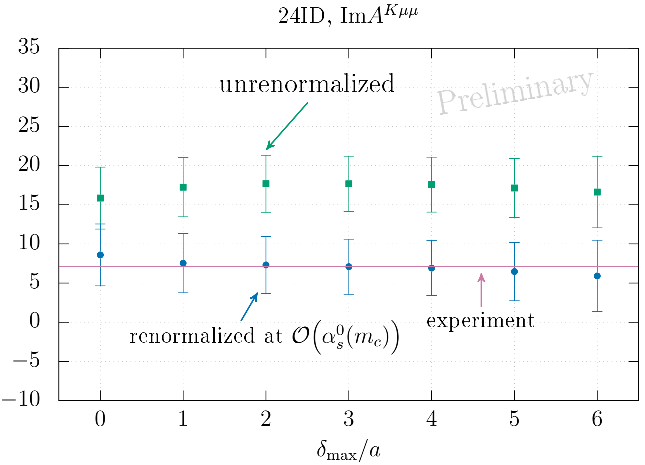


Figure 4: Results of imaginary part of $K_{\mathrm{L}} \to \mu^+\mu^-$ amplitude.

Summary and outlook

Summary and outlook

- $N_f = 3$ calculation of $K_{\mathrm{L}} \to \mu^+\mu^-$ with $\Delta S = \pm 1$ current-current operator needs renormalization

- At $\mathcal{O}\left(\alpha_s^0(m_c)\right)$, 3 classes of subdiagrams can be renormalized by matching between $N_f = 3$ and $N_f = 4$ effective theories:
  - Class-A : HVP like
  - Class-B : Adler ambiguity
  - Class-C : Whole graph

- For the calculation on 24ID, the preliminary result of renormalized imaginary part shows $\sim 50\%$ correction to unrenormalized

- For the future work on 48I($a^{-1} = 1.730$ GeV) and 64I($a^{-1} = 2.359$ GeV), renormalization to all orders in $\alpha_s(m_c)$ is possible