Step-scaling determination of the SU(N) Λ-parameter in the twisted gradient flow scheme

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29 July 2026

Based on:

Phys.Rev.D 113 (2026) 7, 07450

arXiv:2607.07176

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Motivation – SU(N) Λ–parameter

Λ-parameter: RG-invariant, scheme-dependent energy scale

$$\Lambda_s = \mu[b_0\lambda_s(\mu)]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\lambda_s(\mu)}} \exp\left\{-\int_0^{\lambda_s(\mu)} dx \left(\frac{1}{2\beta_s(x)} + \frac{1}{2b_0^2 x^2} - \frac{b_1}{2b_0^2 x}\right)\right\}$$

with $\lambda_s(\mu) = N g_s^2(\mu)$ 't Hooft coupling in scheme s

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Pure-gauge theory: determination for $\mathrm{SU}(N = 3, 5, 8)$ Yang-Mills

- Theoretical interest in *large-N limit* and *N-dependence* of confinement scale
- Phenomenological relevance of $N = 3$ for relation to Λ_{QCD} via *decoupling* strategies (Dalla Brida et al., 2026)

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Step-scaling strategy: coupling runs with ℓ^{-1}, RG-flow integrated in discrete steps

- finite-volume twisted gradient flow (TGF) scheme (Bribián et al., 2021): twisted boundary conditions (Arroyo, Okawa, 1983), gradient-flow coupling, known matching to $\overline{\mathrm{MS}}$
- Other works rely on *asymptotic scaling* (coupling runs with a^{-1}) for $N > 3$

Strategy – Outline of computation of Λ

Step-scaling function $\sigma : \lambda_{\mathrm{TGF}}(\mu) \to \lambda_{\mathrm{TGF}}(\mu/2)$

Choose u_0, implicitly define $u_0 \equiv \lambda_{\mathrm{TGF}}(\mu_{\mathrm{had}})$

Sequence of couplings:

$$u_{k+1} = \sigma^{-1}(u_k)$$

Determine $u_n = \lambda_{\mathrm{TGF}}(\mu_{\mathrm{pt}})$ with $\mu_{\mathrm{pt}} = 2^n \mu_{\mathrm{had}}$

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Gradient-flow scale t_0 as reference

$$\Lambda_{\overline{\mathrm{MS}}}\sqrt{8t_0} = \mu_{\mathrm{had}}\sqrt{8t_0} \cdot \frac{\mu_{\mathrm{pt}}}{\mu_{\mathrm{had}}} \cdot \left.\frac{\Lambda_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{pt}}}\right|_{\mathrm{pt}}$$

μ_{had} determined explicitly with scale setting

Equal to 2^n after n steps

PT expression at $u_n = \lambda_{\mathrm{TGF}}(\mu_{\mathrm{pt}})$ for several n, extrapolation $u_n \to 0$ and match TGF to $\overline{\mathrm{MS}}$

Strategy – Outline of computation of Λ

Step I

Determination of σ^{-1} from u_0 to small couplings by fitting lattice data for $\lambda(b, L)$ with $b = \beta/(2N^2)$. Finite-volume scheme: $\mu^{-1} \propto a(b)L$

Change of behaviour around $1.5\mu_{\rm had}$, no single simple parameterization of σ^{-1}:

A. $u_1 = \lambda_{\rm TGF}(2\mu_{\rm had})$ determined directly from chosen $u_0 \equiv \lambda_{\rm TGF}(\mu_{\rm had})$

B. Global fit of σ^{-1} and its lattice artifacts for $u \leq u_1$

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Step 2

Construction of step-scaling sequence $u_{k+1} = \sigma^{-1}(u_k)$ to flow up to small couplings

Extrapolation $u_n \to 0$ of perturbative expression for $\Lambda_{\overline{\rm MS}}/\mu_{\rm had}$

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Step 3

Scale setting for $\mu_{\rm had}\sqrt{8t_0}$ through dedicated simulations (Bonanno et al., 2026)

Conversion of $\Lambda_{\overline{\rm MS}}/\mu_{\rm had}$ to $\Lambda_{\overline{\rm MS}}\sqrt{8t_0}$

Step 1A – u_l via fits of λ(b,L)

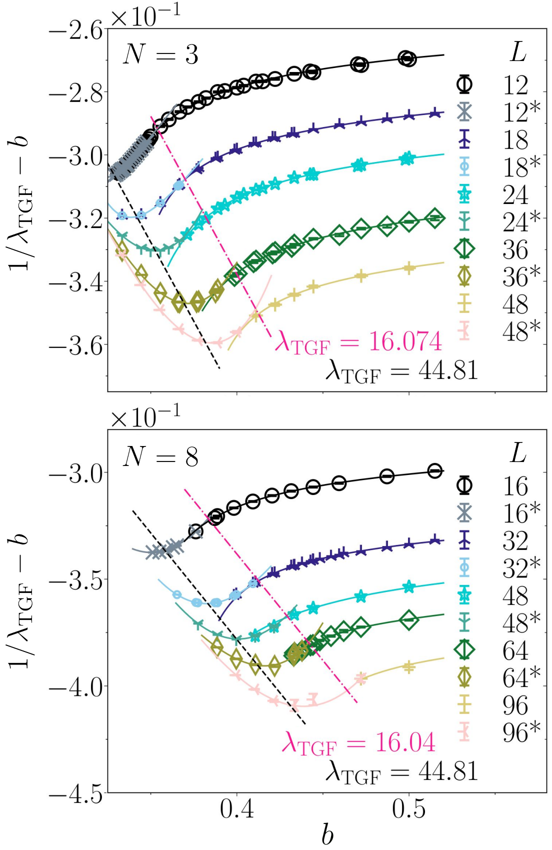
Fit $\lambda(b,L)$, $\lambda(b,2L)$ vs b with Padé approximants:

$$b\lambda(b,L) = \frac{a_0 + a_1 b + b^2}{a_2 + a_3 b + b^2}$$

two fits: strong and weak-intermediate coupling range

Chosen $u_0 = 44.81$ gives for $N = 3,5,8$

- $\mu_{\rm had}$ small enough for feasible scale setting
- $u_1 = \lambda_{\rm TGF}(2\mu_{\rm had})$ in weak-intermediate coupling region



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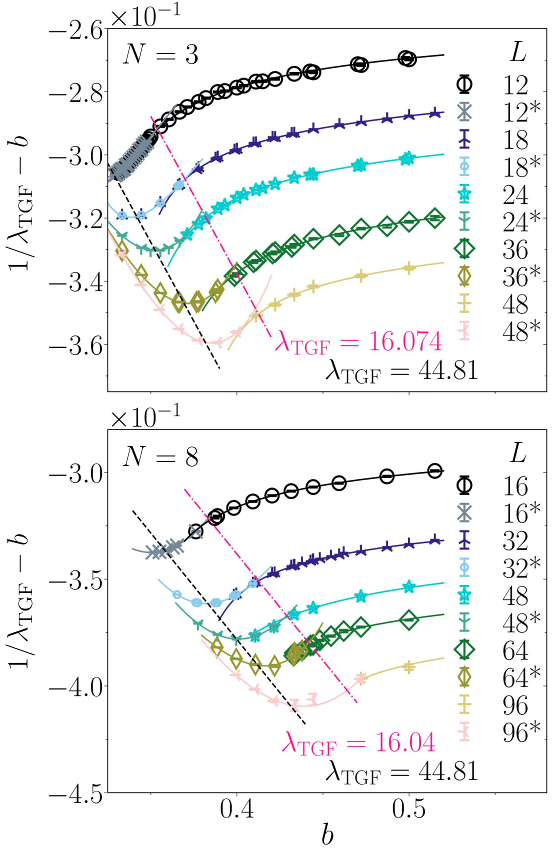
Interpolation of $b_0 = b(u_0, 2L)$ giving LCP $u_0 = \lambda(b_0, 2L)$

Since $\mu_{\text{had}}^{-1} = 0.3 \cdot a(b_0) \cdot 2L$, scale doubled with $2L \to L$

$$u_1 = \lim_{1/L \to 0} \lambda(b_0(2L), L)$$

using fit results to interpolate. Crosscheck: compatible $\lambda(b_0(2L), L)$ from direct simulations at some b_0 for $N = 5$

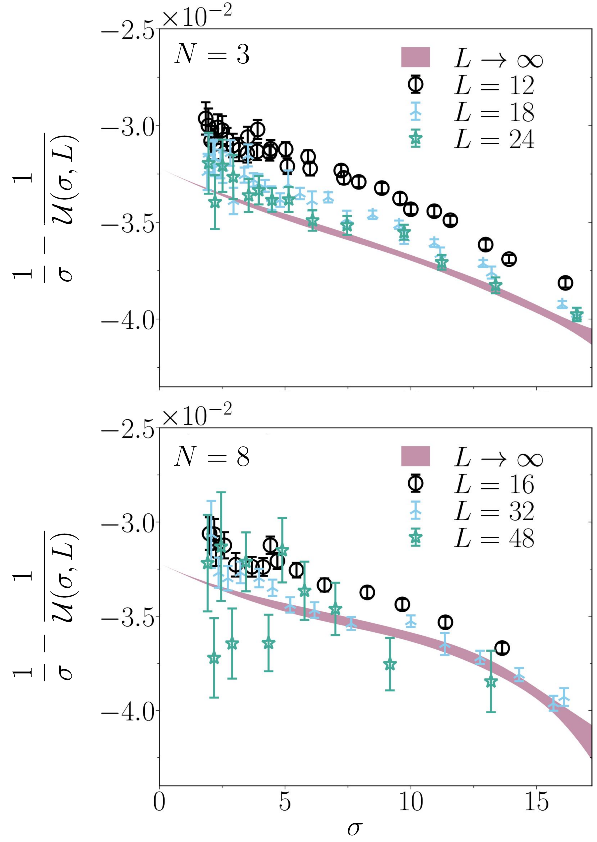


Step 1B – u(σ) via global fit of λ

Parameterization of inverse step-scaling function $u(\sigma)$ for weak-intermediate couplings:

$$\frac{1}{u(\sigma)} = \frac{1}{\sigma} + \sum_{n=0}^{d_1} p_n \sigma^n$$

$p_0,\ p_1$ fixed by perturbation theory



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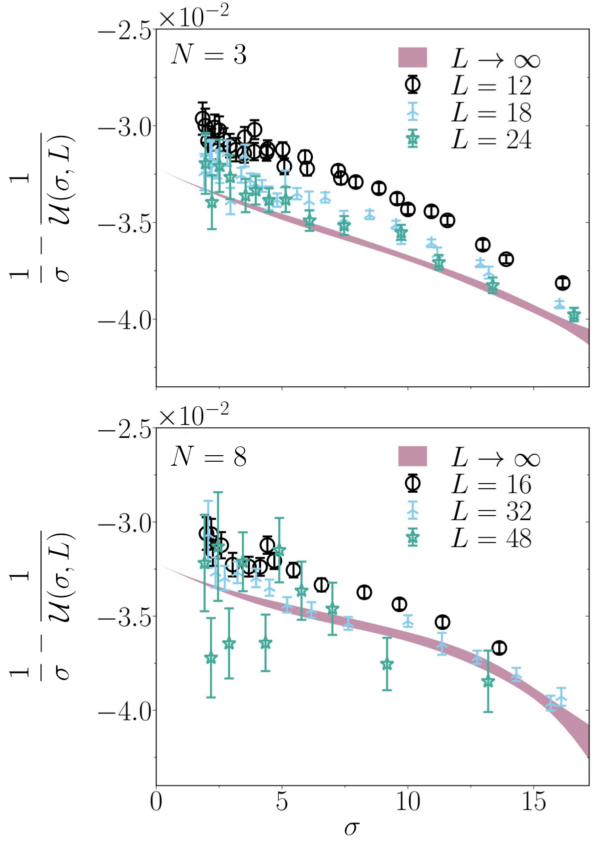
On the lattice with $\mathcal{O}(a^2\mu^2) = \mathcal{O}(L^{-2})$ artifacts:

$$\frac{1}{\mathcal{U}(\sigma, L)} - \frac{1}{\sigma} = \sum_{n=0}^{d_1} p_n \sigma^n + \frac{1}{L^2}\sum_{n=0}^{d_2} \rho_n \sigma^n$$

$$\sigma \equiv \lambda(b, 2L) \implies \mathcal{U}(\sigma, L) = \lambda(b, L)$$

Global fit stability checked by varying $d_1,\ d_2$ between 4 and 6
Crosscheck: continuum extrapolation of $\mathcal{U}(\sigma, L)$ on $N = 3$ LCP at fixed σ gives compatible result (Bribián et al., 2021)



Step 2 – Step-scaling sequence for $\Lambda / \mu_{\rm had}$

Generate step-scaling sequence with $u(\sigma)$:

$$u_k = \lambda_{\rm TGF}(\mu_k)\,, \;\; \mu_k = 2^k \mu_{\rm had}$$

$$k_{\rm max} = 15 \;:\;\; \mu_{\rm had} \simeq 500 \text{ MeV}\,, \;\; \mu_{\rm pt} \simeq 16 \text{ TeV}$$

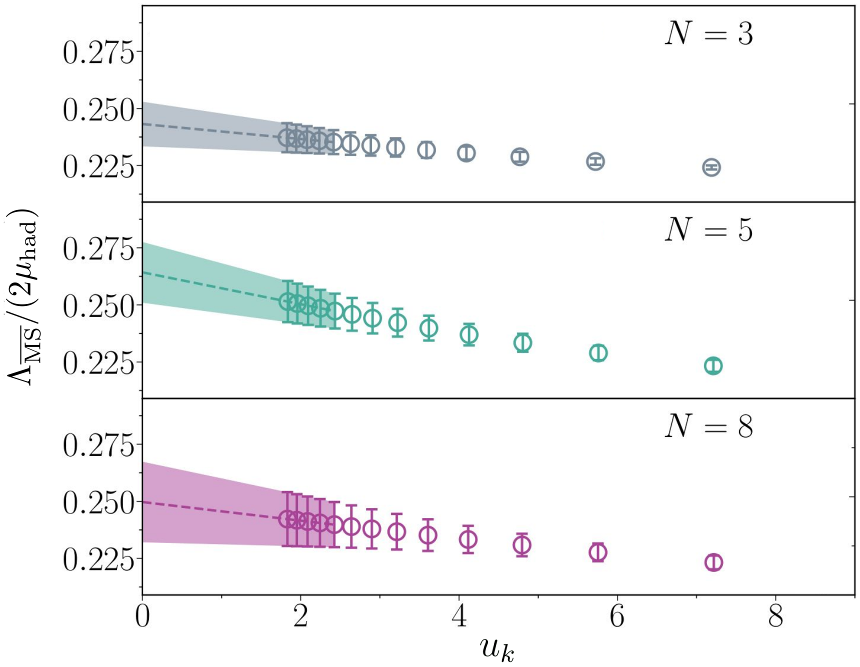
Perturbative expression and conversion to $\overline{\rm MS}$ with 1-loop factor c_1 (Bribián, Pérez, 2019):

$$\left.\frac{\Lambda_{\overline{\rm MS}}}{2\mu_{\rm had}}\right|_k = e^{c_1} \cdot 2^{k-1} \cdot \left.\frac{\Lambda_{\rm TGF}}{\mu_k}\right|_{\rm pt}$$

$$\left.\frac{\Lambda_{\rm TGF}}{\mu_k}\right|_{\rm pt} = [b_0 u_k]^{-\frac{b_1}{2b_0^2}} \exp\left(-\frac{1}{2b_0 u_k}\right) + \mathcal{O}[u_k]$$

Zero coupling extrapolation:

$$\frac{\Lambda_{\overline{\rm MS}}}{2\mu_{\rm had}} = \lim_{u_k \to 0} \left.\frac{\Lambda_{\overline{\rm MS}}}{2\mu_{\rm had}}\right|_k$$

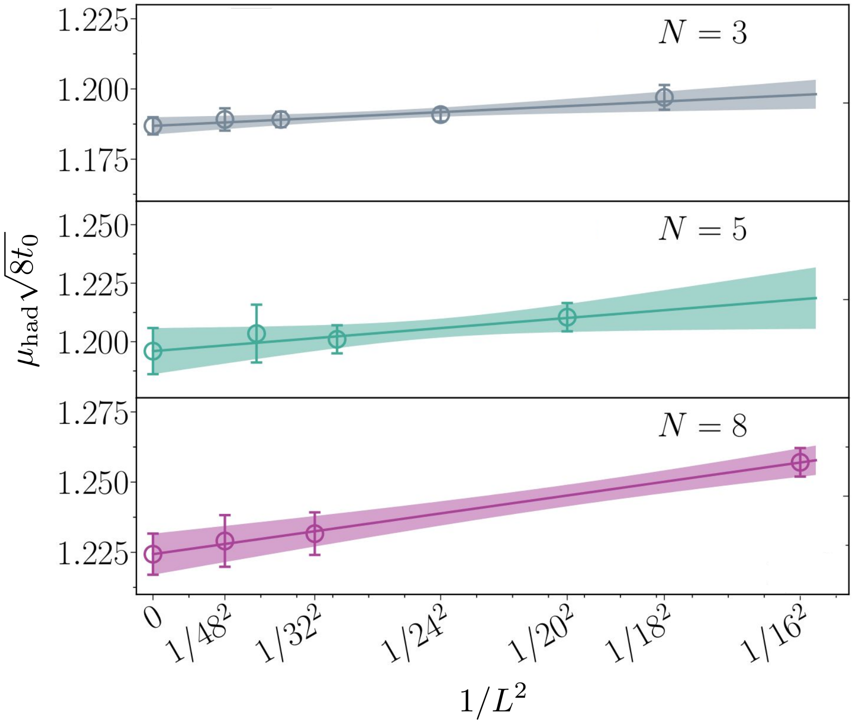


Step 3 – Setting μ_{had} in units of t_0

Gradient-flow scale defined through energy density:

$$\frac{N}{N^2-1}\langle t^2 E(t)\rangle\Big|_{t=t_0} = 0.1125 \implies \sqrt{8t_0} \simeq 0.5 \text{ fm}$$

Dedicated $N = 3, 5, 8$ study (Bonanno et al., 2026) and $N = 3$ results of (Giusti, Lüscher, 2019) to interpolate t_0/a^2 on LCP at μ_{had}



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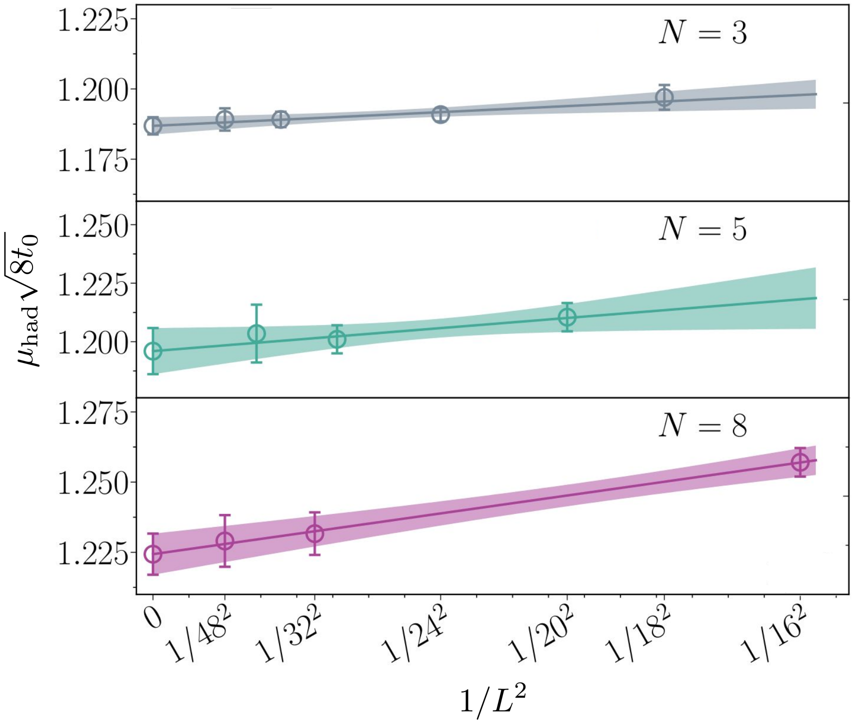
Continuum limit assuming $\mathcal{O}(a^2\mu_{\text{had}}^2) = \mathcal{O}(L^{-2})$ artifacts:

$$\mu_{\text{had}}\sqrt{8t_0} = \lim_{a\to 0} a\mu_{\text{had}} \cdot \sqrt{\frac{8t_0}{a^2}}$$

Bootstrap analysis to propagate uncertainty on tuning of LCP



Final result obtained as:

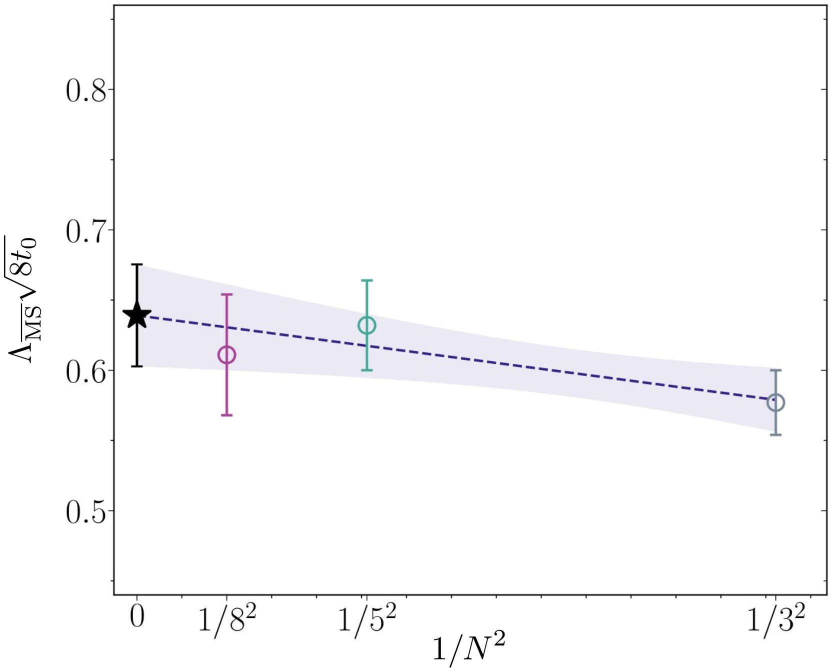
$$\Lambda_{\overline{\mathrm{MS}}}\sqrt{8t_0} = 2 \cdot \frac{\Lambda_{\overline{\mathrm{MS}}}}{2\mu_{\mathrm{had}}} \cdot \mu_{\mathrm{had}}\sqrt{8t_0}$$

Large-N extrapolation:

$$\sqrt{8t_0}\Lambda_{\overline{\mathrm{MS}}}(N) = \sqrt{8t_0}\Lambda_{\overline{\mathrm{MS}}}(\infty)\left[1 + \frac{A}{N^2}\right]$$

N-dependence is mild: $A = -0.85(62)$

N	$\Lambda_{\overline{\mathrm{MS}}}\sqrt{8t_0}$	uncertainty
3	0.577(23)	4.0%
5	0.632(32)	5.1%
8	0.611(43)	7.0%
∞	0.639(36)	5.6%



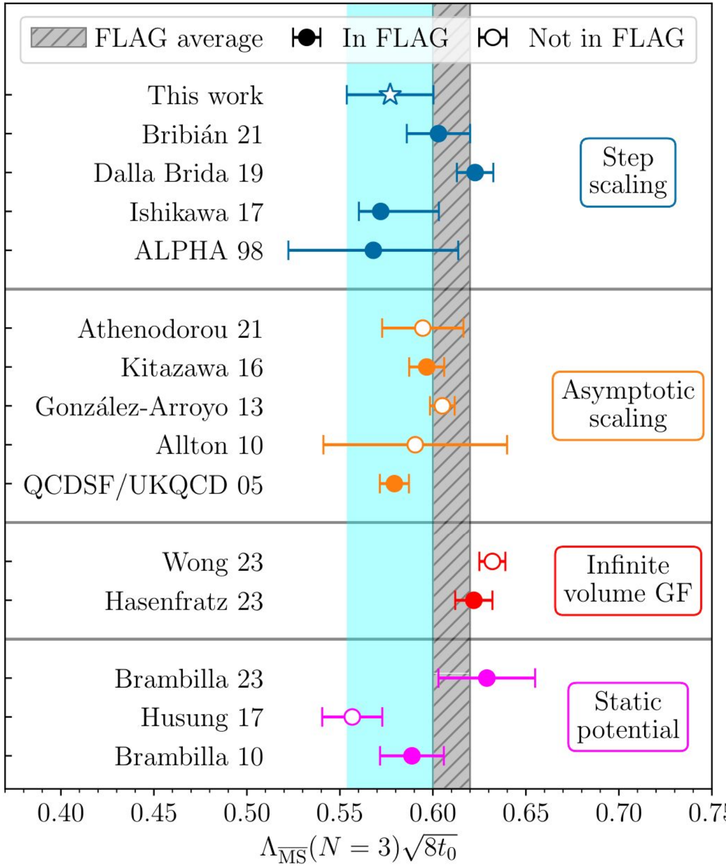
Results – Comparison with literature for N = 3

Results classified by employed strategy:

- Asymptotic scaling: short-distance observables, $\mu \propto a^{-1}$
- Infinite-volume GF: energy density $E(t)$, $\mu \propto t^{-\frac{1}{2}}$
- Static potential: potential $V(r)$ or force $F(r)$, $\mu \propto r^{-1}$

Results converted in units of $\sqrt{8t_0}$:

- $\sqrt{8t_0\sigma} = 1.0962(46)$ (Bonanno, 2026)
- $\sqrt{8t_0}/r_0 = 0.9435(97)$ (Dalla Brida, Ramos, 2019)
- $t_1/t_0 = 0.4204(11)\,, w_0^2/t_0 = 1.0425(54)$ (Bonanno et al., 2026)



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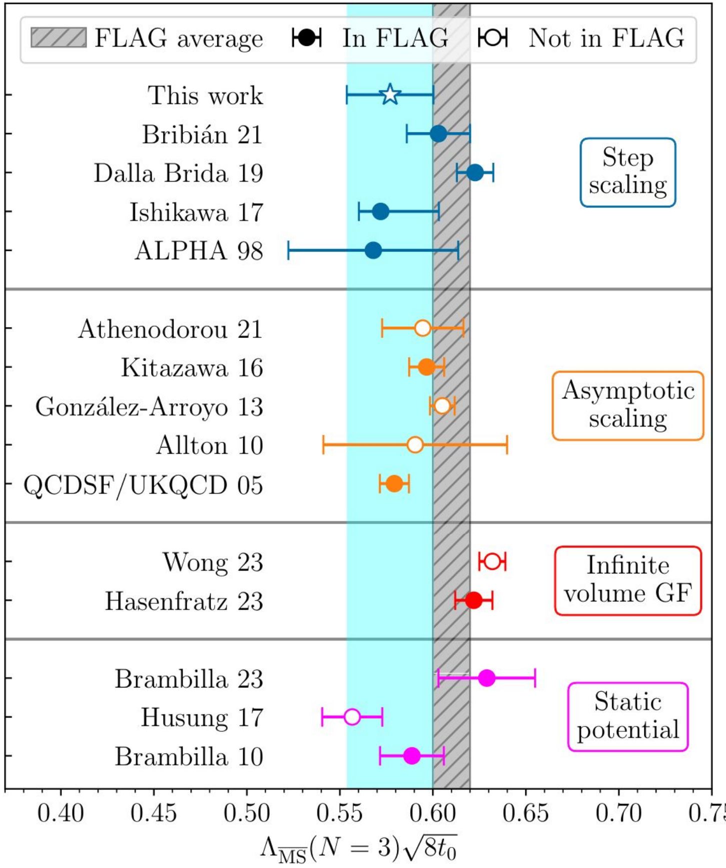
Result in agreement (4% uncertainty) with other determinations (Bribián 21 not independent) and FLAG mean (1.7% uncertainty)

Not precise enough to address possible tensions between Dalla Brida 19 and QCDSF/UKQCD 05 (future work)



Results – Comparison with literature for N = 5, 8

Results converted in units of $\sqrt{8t_0}$ (Bonanno, 2026):

- $\sqrt{8t_0\sigma} = 1.1648(48) \quad (N = 5)$
- $\sqrt{8t_0\sigma} = 1.1903(37) \quad (N = 8)$

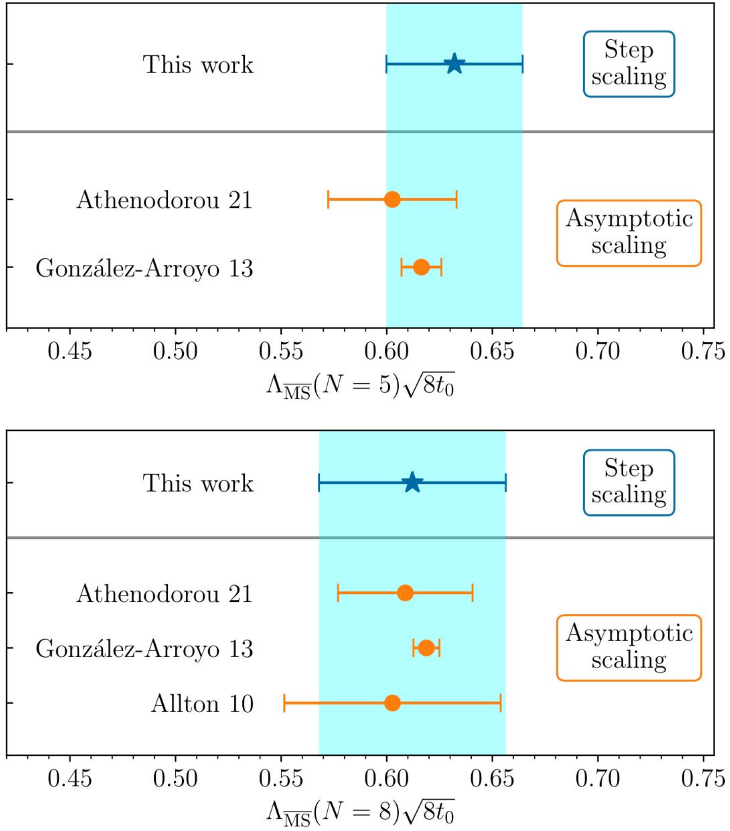
Other $N > 3$ works use asymptotic scaling, except one $N = 4$ step scaling result (Lucini, Moraitis, 2008):

$$\sqrt{8t_0}\Lambda_{\overline{\mathrm{MS}}} = 0.603(24)_{\mathrm{stat}}(71)_{\mathrm{syst}}$$

In agreement with $N = 4$ interpolation of our fit:

$$\sqrt{8t_0}\Lambda_{\overline{\mathrm{MS}}} = 0.605(18)$$

Result in agreement (5%, 7% uncertainty) with other determinations



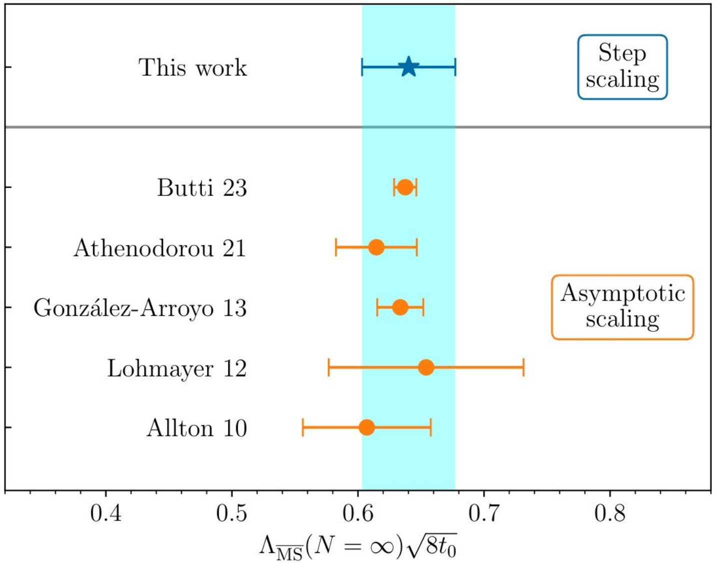
Results – Comparison with literature at large-N

Results converted in units of $\sqrt{8t_0}$:

- $\sqrt{8t_0\sigma}\Big|_{N\to\infty} = 1.2068(46)$ (Bonanno, 2026)
- $t_1'/t_0 = 0.3278(12)$ (Bonanno et al., 2026)

Other works use asymptotic scaling

Result in agreement (5.6% uncertainty) with other determinations



Conclusions

Summary

Step-scaling determination of the Λ–parameter of SU(N) Yang–Mills theories for N = 3, 5, 8 and in the large-N limit

Determination of step-scaling function in finite-volume TGF scheme to connect non-perturbative and perturbative regimes on the lattice

First determination not relying on asymptotic scaling for N > 4,
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Future outlooks

N = 3 dedicated study within this framework to improve accuracy. Increasing statistics is not enough: match to another scheme such as SF for better-behaved matching to PT

Infinite-volume GF determination using data generated for scale setting

Step-scaling determination of Λ directly at large-N using the twisted Eguchi–Kawai formulation. Coupling in (Perez et al., 2014), missing $\overline{\mathrm{MS}}$ conversion

Backup – Lattice setup and TGF scheme

Geometry: $L^2 \times L_s^2$ lattice with $L_s = L/N$ for directions 1, 2

Twisted Boundary Conditions (TBC) on the short plane (1,2) (Arroyo, Okawa, 1983):

- same as Periodic BC up to a phase factor
- effective volume $V_{\text{eff}} = N^2 V = L^4$ for *twisted volume reduction* (Arroyo, Okawa, 2010)

Action: Wilson action with phase factor $Z_{\mu\nu}(n)$ implementing TBC

$$S_{\mathrm{W}}[U] = -Nb \sum_{x,\mu \neq \nu} Z^*_{\mu\nu}(x)\,\mathrm{Tr}\left[P_{\mu\nu}(x)\right] \qquad b = \beta/(2N^2)$$

Coupling: defined with energy density after gradient flow and projected to trivial topological sector

$$\lambda_{\mathrm{TGF}}\left(\mu = \frac{1}{c\ell}\right) = \frac{1}{\mathcal{N}(c,L)} \left.\frac{\langle t^2 E(t)\delta_{Q,0}\rangle}{N\,\langle \delta_{Q,0}\rangle}\right|_{\sqrt{8t}=\mu^{-1}} \qquad (c = 0.3)$$

$Q = 0$ projection proven to bypass topological freezing in this setup using Parallel Tempering on Boundary Conditions (PTBC) algorithm (Bonanno et al., 2024)