Obtaining continuum physics from dynamical simulations of Hamiltonian lattice gauge theories

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Based off work in: [arXiv:2506.16559]

Lattice 2026

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Cost of a quantum simulation

- Calculating physical observables requires taking the continuum limit $a \to 0$
- Fair comparisons between methods must include this cost
- Error from approximate time evolution can spoil continuum limit
- Procedure to remove time evolution errors when using Trotter methods developed
  [Marcela Carena, Henry Lamm, Ying-Ying Li, Wanqiang Liu, PRD, arXiv:2107.01166]
  $\to$ not applicable to other algorithms
- **Our Goal:** Develop a general framework for controlling impact of time evolution errors on continuum limit applicable to any algorithm

Renormalization: taking the continuum limit

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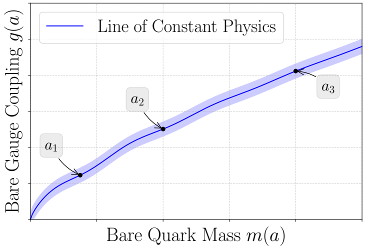

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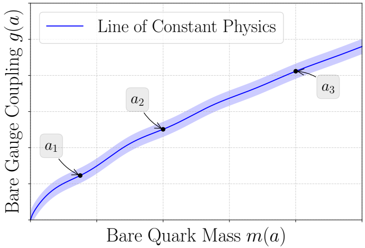

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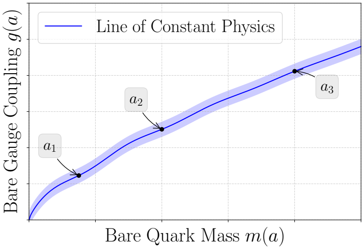

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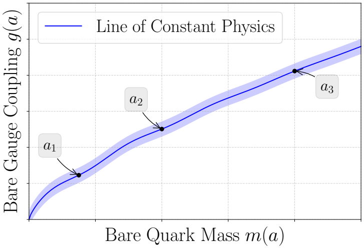

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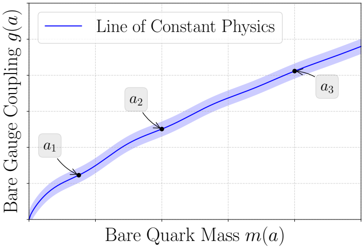

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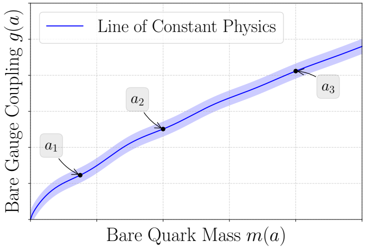

2. Calculate desired physical observable

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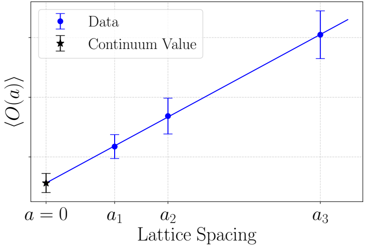

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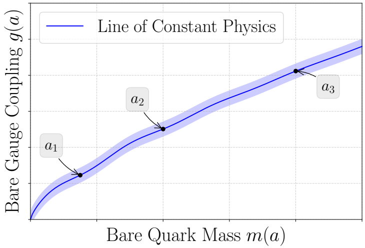

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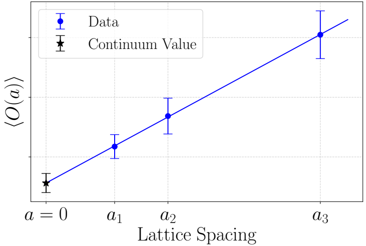

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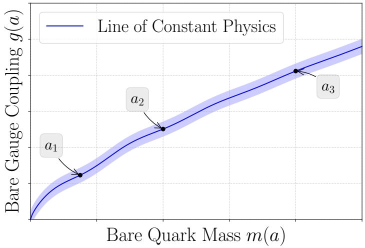

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- **a** Calculate dimensionless $\langle \hat{O}(\hat{t}, a) \rangle$ and rescale by a to convert to physical units
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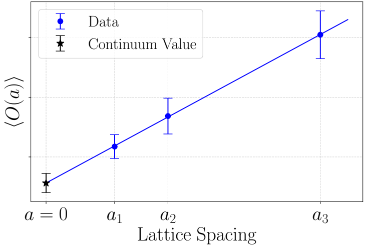
- **c** Extrapolate to continuum



Effective Hamiltonian with Trotter

*(hatted quantity $\leftrightarrow$ dimensionless)

Look at effective Hamiltonian simulated using Trotter [Marcela Carena, Henry Lamm, Ying-Ying Li, Wanqiang Liu, PRD, arXiv:2107.01166]

$$e^{-i\hat{\delta}_t(\hat{H}_E+\hat{H}_B)} \approx e^{-i\frac{\hat{\delta}_t}{2}\hat{H}_E} e^{-i\hat{\delta}_t\hat{H}_B} e^{-i\frac{\hat{\delta}_t}{2}\hat{H}_E}$$

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$$\hat{H}_{\mathrm{eff}}(\delta_t) = \hat{H}_E + \hat{H}_B + \frac{\hat{\delta}_t^2}{24}\left([\hat{H}_E, [\hat{H}_E, \hat{H}_B]] + 2[\hat{H}_B, [\hat{H}_E, \hat{H}_B]]\right) + \dots$$

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- View as "temporal lattice", treat $\hat{\delta}_t$ as parameter in the effective Hamiltonian
- $\hat{\delta}_t \neq 0$ changes physics which changes values used in tuning and scale setting

$$g(a) \to g(a,\delta_t), \qquad m(a) \to m(a,\delta_t), \qquad \dots$$

Continuum physics achieved taking simultaneous limit $\lim_{a\to 0}\lim_{\delta_t\to 0}$

(Or, work at fixed anisotropy $\xi = a/\delta_t$ and extrap $a \to 0$)

Euclidean simulations to assist renormalization

Main idea: relate H_{eff} to Euclidean transfer matrix T on anisotropic lattice

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- Including fermions introduces $\mathcal{O}(a_0)$ systematics

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- For p'th order, Trotter errors go to zero as $\mathcal{O}(a^p)$

Extension to other algorithms?

Can we apply a similar procedure to simulations done using other simulation algorithms?

Consider Quantum Signal Processing as test case

Quantum Signal Processing: high level review

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Provably optimal scaling with t and ϵ [G. Low, I. Chuang, Quantum, arXiv:1606.02685]

$$\text{Calls to } U_H = \mathcal{O}\big(\lambda t + \log\frac{1}{\epsilon}\big)$$

Breakdown of previous approach to QSP

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Approximate time evolution operator

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Previous picture relies on effective Hamiltonian formalism

Unclear how to proceed:
$\to$ approximate evolution is not unitary, not clear what H_{eff} is

We need an alternative more general approach

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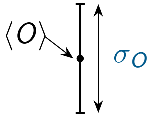
$\to$ driving time evolution error much below statistical precision effectively takes $\delta \to 0$
$\to$ renormalization procedure simplifies to exact time evolution case

Statistically-Bounded Time Evolution (SBTE) Protocol

Calculate time-dependent observable $\langle \hat{O}(t, a) \rangle$ to uncertainty σ_O
$\rightarrow$ sources of uncertainty from shot noise, device noise, etc.



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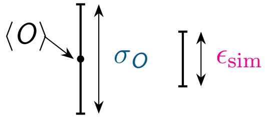

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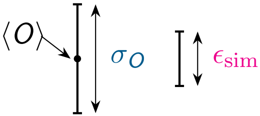

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Let α be ratio of errors:

$$\alpha = \frac{\epsilon_{\text{sim}}}{\sigma_O} = \frac{\text{time evolution error}}{\text{statistical error}}$$

- $\alpha = 1$: equal error split
- $\alpha \ll 1$: time evolution error negligible relative to statistical $\rightarrow$ SBTE

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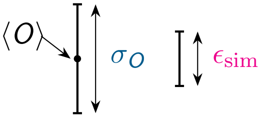


If approximate time evolution error $\epsilon_{\text{sim}} \ll \sigma_O$, we can neglect it

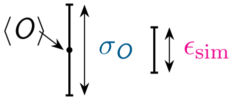
Let α be ratio of errors:

$$\alpha = \frac{\epsilon_{\text{sim}}}{\sigma_O} = \frac{\text{time evolution error}}{\text{statistical error}}$$

- $\alpha = 1$: equal error split
- $\alpha \ll 1$: time evolution error negligible relative to statistical $\rightarrow$ SBTE

Key question: what is the additional computational cost to ensure $\alpha \ll 1$?
$\rightarrow$ rigorous cost bounds in our paper [**CFK**, S. Hariprakash, C. Bauer, [arXiv:2506.16559]

Schematic cost examples



Set $\sigma_O = 10^{-2}$ and precision factor $\alpha = 0.1 \rightarrow \epsilon_{\text{sim}} = 10^{-3}$

Schematic cost examples

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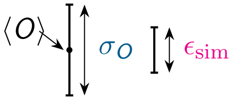

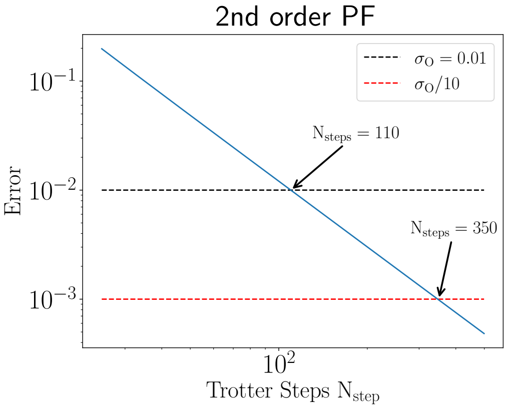

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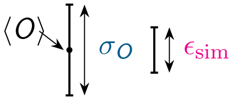
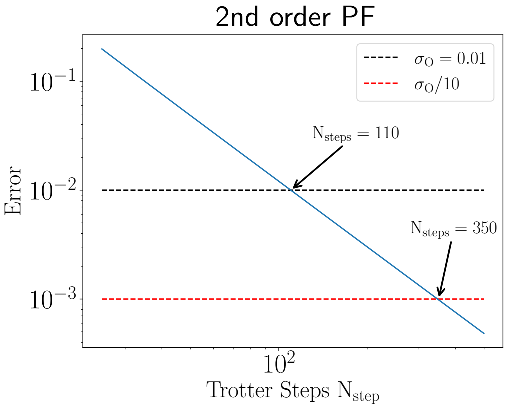
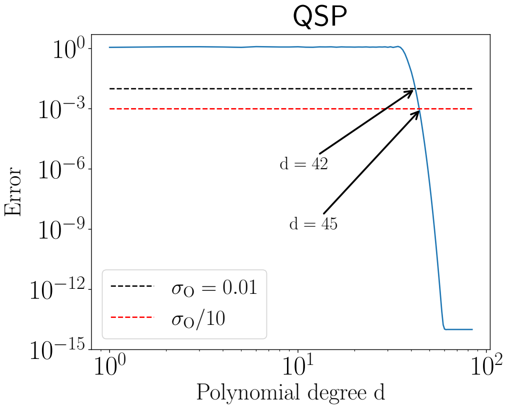






Summary and conclusions

- Full cost of a quantum simulation requires cost of $a \to 0$ limit
- Previously, controlling impact of approximate time evolution errors only understood for Trotter methods
- We introduced a simple, general approach applicable to any quantum algorithm
  $\to$ Statistically-Bounded Time Evolution Protocol
- Opens the door to performing end-to-end cost comparisons between different algorithms

Backup slides

Hamiltonian LGT

Kogut-Susskind Hamiltonian

$$H_{\mathrm{KS}} = \left[\frac{g_t^2}{a} \widetilde{H}_E - \frac{1}{a g_s^2} \widetilde{H}_B + \frac{\kappa}{a} \widetilde{H}_{\mathrm{hop}} + m \widetilde{H}_M \right]$$

Lorentz invariance broken $\longrightarrow$ gauge coupling for H_E and H_B different

- Speed of light: $c = \frac{g_t}{g_s}$
- Gauge coupling $g = \sqrt{g_s g_t}$
- Hopping coefficient κ
  $\rightarrow$ Euclidean simulations on anisotropic lattices have bare fermionic anisotropy factor γ_f
  $\rightarrow$ need γ_f so physical anisotropy $\xi = a/a_0$ "seen" by gluons and fermions is the same
  $\rightarrow$ keeping track of γ_f in Transfer matrix implies need for $\kappa(a) \neq 1$

Speed of light $c(a) \neq 1$ changes overall scale of H:

$$H_{\mathrm{KS}} = \frac{c}{a} \left[g^2 \widetilde{H}_E - \frac{1}{g^2} \widetilde{H}_B + \frac{\kappa}{c} \widetilde{H}_{\mathrm{hop}} + (\frac{a}{c} m) \widetilde{H}_M \right] \equiv \frac{1}{a_t} \hat{H}_{\mathrm{KS}}$$

Hamiltonian LGT: Two Different Scales

Hamiltonian

$$\hat{H}_{\text{KS}} = a_t H_{\mathrm{KS}}$$

Temporal scale: $a_t = \frac{a}{c}$

Momentum Operator

$$\hat{P}_{\mathrm{KS}} = a P_{\mathrm{KS}}$$

Spatial scale: a (lattice spacing)

Different "rulers" for temporal and spatial quantities

- **Speed of light c(a):** conversion factor between a and a_t

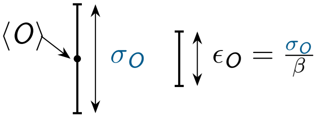
Dimensionless quantities measured on the lattice:

$$\hat{m} = a_t\, m\,, \qquad \hat{t} = \frac{1}{a_t} t\,, \qquad \hat{p} = ap, \qquad \hat{x} = \frac{1}{a} x$$

(Notation: hatted quantities are dimensionless)

Statistically-Bounded Time Evolution (SBTE) Protocol

Calculate time-dependent observable $\langle \hat{O}(t, a) \rangle$ to uncertainty σ_O
$\rightarrow$ sources of uncertainty from shot noise, device noise, etc.



If approximate time evolution error $\epsilon_O \ll \sigma_O$, we can neglect it

For some $\beta > 1$, we require

$$\epsilon_O \equiv \| e^{iHt} \hat{O}(0, a) e^{-iHt} - U_\delta^\dagger(t) \hat{O}(0, a) U_\delta(t) \| \leq \frac{\sigma_O}{\beta}$$

This is guaranteed by choosing time evolution operator error ϵ_{sim} as:

$$\epsilon_{\mathrm{sim}} = \| e^{-iHt} - U_\delta(t) \| \leq \left(\frac{\sigma_O}{2\beta \| \hat{O}(0, a) \|} \right)$$

Key question: what is the additional computational cost to ensure this?

Examples: Product formula and Quantum Signal Processing

Set simulation error $\epsilon_{\text{sim}} = \|e^{-iHt} - U_\delta(t)\| = \frac{\sigma_O}{2\beta\|\hat{O}(0,a)\|}$

$\langle O \rangle$ σ_O $\epsilon_O = \frac{\sigma_O}{\beta}$

SBTE applied to Product Formulas: extra cost not obviously negligible

$$\text{Trotter number}: \quad N_{\text{PF}} \geq \left(\frac{1}{\epsilon_{\text{sim}}}\right)^{1/p} \widetilde{\alpha}^{1/p} t^{1+1/p}$$

[A. Childs, Y. Su, M. Tran, S. Zhu, PRX arXiv:1912.08854]

$$\geq (2\beta\|O(0,a)\|)^{1/p} \times \frac{\widetilde{\alpha}^{1/p} t^{1+1/p}}{\sigma_O^{1/p}}$$

SBTE applied to QSP: extra cost negligible *a priori*

$$\text{Calls to Block Encoding}: \quad N_{\text{QSP}} \geq \frac{e\lambda t}{2} + \log\frac{1}{\epsilon_{\text{sim}}}, \qquad \lambda > \|H\|$$

$$\geq \frac{e\lambda t}{2} + \log(\frac{1}{\sigma_O}) + (\log 2\beta\|O(0,a)\|)$$

[G. Low, I. Chuang, Quantum, arXiv:1606.02685]