Reconstructing spectral densities from integral transforms II

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*Speaker

Outline

- Spectral densities via Mehler-Fock and Kontorovich-Lebedev transforms
- Incomplete Mehler-Fock transform
- Incomplete smeared spectral densities
- Bounds on the systematic errors
- Examples of reconstructions

Motivation

- Spectral densities $\rho(\omega)$ in QCD (smeared with kernels $\kappa(\omega)$) $\implies$ Relevant phenomenological observables
  1. R-ratio
  2. Inclusive semi-leptonic decays, cross-sections
  3. Properties of the QGP
  4. ...
- In Lattice QCD we compute $C(t)$, but thanks to the Källen-Lehmann decomposition:

$$C(t) = \int_0^{+\infty} \rho(\omega) e^{-\omega t} \, d\omega$$

$\implies$ Inverse Laplace Transform

Smeared spectral densities from integral transforms

Smeared spectral densities from integral transforms

- Target definition, $C(t)$ with $t \in [t_0, +\infty)$:

$$\bar{C}_s = \int_{t_0}^{\infty} C(t)\, \bar{u}_s(t, t_0)\, dt, \quad \hat{\kappa}_s = \int_0^{+\infty} \kappa(\omega)\, \hat{u}_s(\omega, t_0)\, d\omega$$

$$\rho_\kappa = \int_0^{+\infty} \rho(\omega)\kappa(\omega)\, d\omega = \int_0^{\infty} \frac{\bar{C}_s}{|\Gamma(1/2 + is)|} \hat{\kappa}_s\, ds$$

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- Lattice definition, $\mathcal{C}(t_0 + an)$ with $n = 0, 1, \dots, +\infty$:

$$\bar{\mathcal{C}}_s = a \sum_{n=0}^{\infty} \mathcal{C}(t_0 + an)\, \bar{v}_s(n, t_0, a), \quad \hat{\kappa}_s = \int_0^{+\infty} \kappa(\omega)\, \hat{v}_s(\omega, t_0, a)\, d\omega$$

$$\varrho_\kappa = \int_0^{+\infty} \varrho(\omega)\kappa(\omega)\, d\omega = \int_0^{\infty} \frac{\bar{\mathcal{C}}_s}{|\Gamma(1/2 + is)|} \hat{\kappa}_s\, ds$$

Smeared spectral densities from integral transforms

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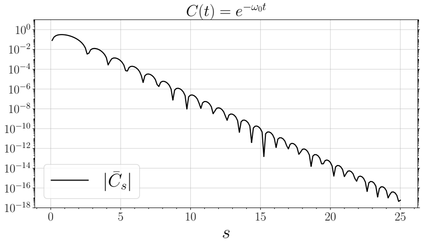
- Formulae need to be modified for $C(t)$ with $t \in [t_0, t_{\max}]$

Incomplete Mehler-Fock transform

$$\bar{C}_s = \int_{t_0}^{\infty} C(t)\, \bar{u}_s(t, t_0)\, dt$$

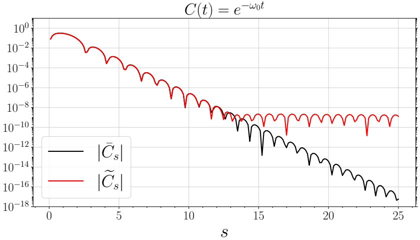


- The Mehler-Fock transform $\bar{C}_s$ decays exponentially with s

Incomplete Mehler-Fock transform

$$\bar{C}_s = \int_{t_0}^{\infty} C(t)\, \bar{u}_s(t, t_0)\, dt$$

$$\widetilde{C}_s(t_{\max}) = \int_{t_0}^{t_{\max}} C(t)\, \bar{u}_s(t, t_0)\, dt$$



- The Mehler-Fock transform $\bar{C}_s$ decays exponentially with s
- The *incomplete* Mehler-Fock transform $\widetilde{C}_s(t_{\max})$ approximates $\bar{C}_s$ for $s < s_{\max}$

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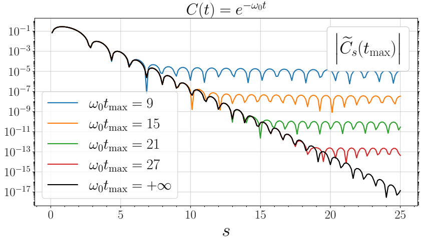
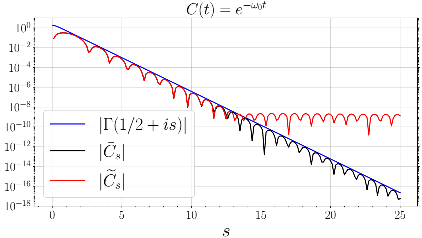


- The Mehler-Fock transform $\bar{C}_s$ decays exponentially with s
- The *incomplete* Mehler-Fock transform $\widetilde{C}_s(t_{\max})$ approximates $\bar{C}_s$ for $s < s_{\max}$
- $s_{\max} \propto t_{\max}$: longer lattices $\implies$ $\tilde{C}_s(t_{\max})$ approximates $\bar{C}_s$ for higher values of s

Incomplete Mehler-Fock transform

$$\rho_\kappa = \int_0^\infty \frac{\bar{C}_s}{|\Gamma(1/2+is)|}\hat{\kappa}_s\, ds$$



- The Mehler-Fock transform $\bar{C}_s$ decays exponentially with s
- The *incomplete* Mehler-Fock transform $\widetilde{C}_s(t_{\max})$ approximates $\bar{C}_s$ for $s < s_{\max}$
- $s_{\max} \propto t_{\max}$: longer lattices $\implies$ $\tilde{C}_s(t_{\max})$ approximates $\bar{C}_s$ for higher values of s
- $\widetilde{C}_s(t_{\max})/|\Gamma(1/2+is)|$ well behaved up to $s_{\max}$

The solution: incomplete smeared spectral densities

Definition

This suggests us to introduce:

- the *incomplete spectral density*:

$$\rho_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\bar{C}_s}{|\Gamma(1/2 + is)|} \hat{\kappa}_s \, ds$$

- the *approximated, incomplete spectral density*:

$$\tilde{\rho}_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\widetilde{C}_s(t_{\max})}{|\Gamma(1/2 + is)|} \hat{\kappa}_s \, ds$$

For $C(t)$ with $t \in [t_0, t_{\max}]$, we can compute only $\tilde{\rho}_\kappa(s_{\max})$
$\implies$ We need to bind the systematic errors

Spectral density from the incomplete ones

Strategy

Decomposition of the difference between ρ_κ and $\tilde{\rho}_\kappa(s_{\max})$:

$$\rho_\kappa - \tilde{\rho}_\kappa(s_{\max}) = [\rho_\kappa(s_{\max}) - \tilde{\rho}_\kappa(s_{\max})] + [\rho_\kappa - \rho_\kappa(s_{\max})]$$

Two systematic errors:

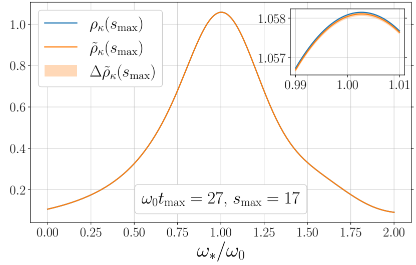
1. Difference between incomplete spectral density and approximated one $|\rho_\kappa(s_{\max}) - \tilde{\rho}_\kappa(s_{\max})| \leq \Delta\tilde{\rho}_\kappa(s_{\max})$
2. Difference between spectral density and incomplete one $|\rho_\kappa - \rho_\kappa(s_{\max})| \leq \Delta\rho_\kappa(s_{\max})$

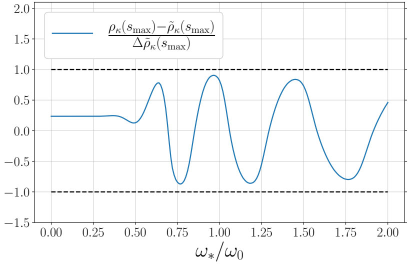
$$|\rho_\kappa - \tilde{\rho}_\kappa(s_{\max})| \leq \Delta\tilde{\rho}_\kappa(s_{\max}) + \Delta\rho_\kappa(s_{\max})$$

1. Bound on $[\rho_\kappa(s_{\max}) - \tilde{\rho}_\kappa(s_{\max})]$

$$|\rho_\kappa(s_{\max}) - \tilde{\rho}_\kappa(s_{\max})| \leq \Delta\tilde{\rho}_\kappa(s_{\max})$$

$$\Delta\tilde{\rho}_\kappa(s_{\max}) = |C(t_{\max})| \int_0^{s_{\max}} b_s(t_{\max}) \frac{|\hat{\kappa}_s|}{|\Gamma(1/2+is)|}\, ds$$



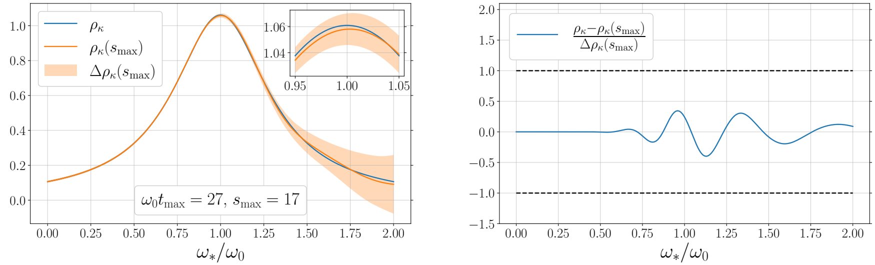


$$\kappa(\omega) = \frac{1}{\pi}\frac{\sigma}{(\omega-\omega_*)^2+\sigma^2}$$

2. Bound on $[\rho_\kappa - \rho_\kappa(s_{\max})]$

$$|\rho_\kappa - \rho_\kappa(s_{\max})| \leq \Delta\rho_\kappa(s_{\max})$$

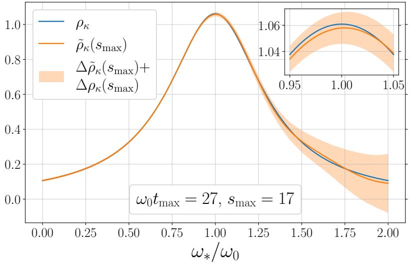
$$\Delta\rho_\kappa(s_{\max}) = r \int_{s_{\max}}^{\infty} \frac{\sqrt{2s\sinh(\pi s)}}{\pi} \max_{\omega\in\mathbb{R}^+} |K_{is}(\omega t_0)|\, |\hat{\kappa}_s|\, ds$$

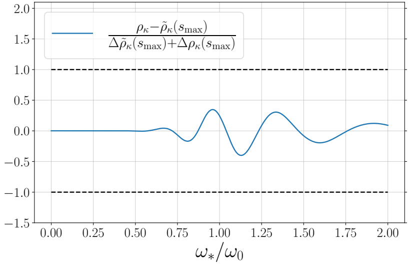


$$\kappa(\omega) = \frac{1}{\pi} \frac{\sigma}{(\omega-\omega_*)^2 + \sigma^2}$$

Full reconstruction

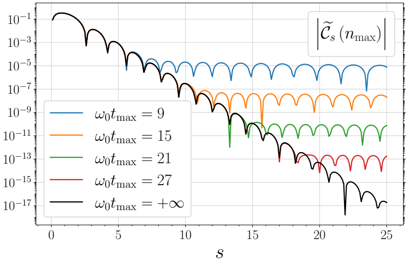
$$|\rho_\kappa - \tilde{\rho}_\kappa(s_{\max})| \leq \Delta\tilde{\rho}_\kappa(s_{\max}) + \Delta\rho_\kappa(s_{\max})$$

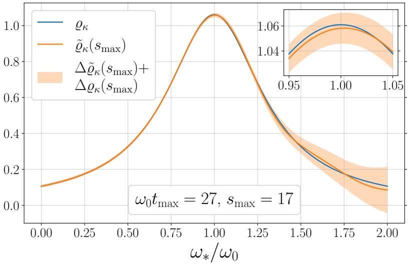




- We have access only to $\tilde{\rho}_\kappa(s_{\max})$, but we can put bounds on the unknowns $\implies$ *rigorous estimate* of ρ_κ
- $\tilde{\rho}_\kappa(s_{\max})$, $\Delta\rho_\kappa(s_{\max})$ and $\Delta\tilde{\rho}_\kappa(s_{\max})$ can be obtained from the knowledge of $C(t)$ for $t \leq t_{\max}$

Discrete sampling

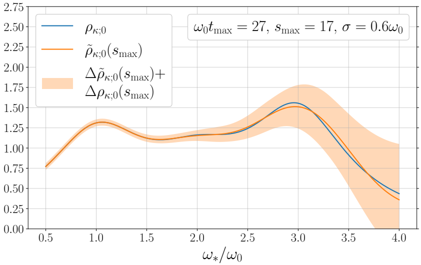


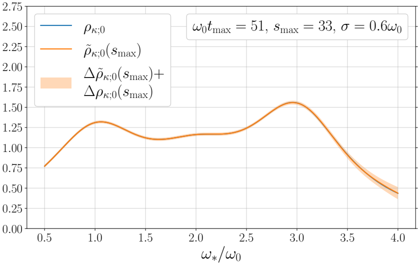


- The *incomplete discrete transform* has the same behaviour of the continuous one $\implies$ We can apply the same strategy
- We have access only to $\tilde{\varrho}_\kappa(s_{\max})$, but we can put bounds on the unknowns $\implies$ *rigorous estimate* of ϱ_κ

Dependence on t_{max}

$$C(t) = c_0 e^{-\omega_0 t} + c_1 e^{-\omega_1 t} + c_2 e^{-\omega_2 t}$$

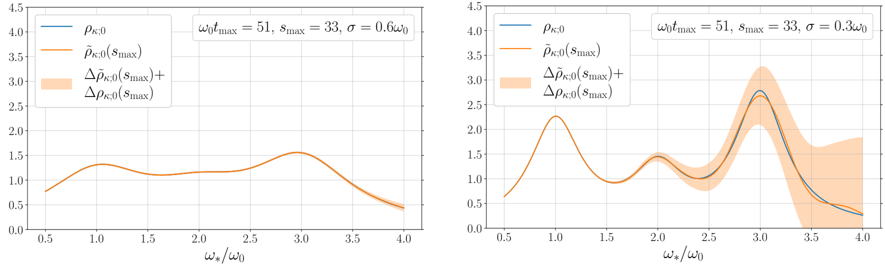




$$\text{Higher } t_{\mathrm{max}} \implies \text{Higher } s_{\mathrm{max}} \implies \text{Better reconstructions}$$

Dependence on the width of the kernel: constant σ

$$C(t) = c_0 e^{-\omega_0 t} + c_1 e^{-\omega_1 t} + c_2 e^{-\omega_2 t}$$

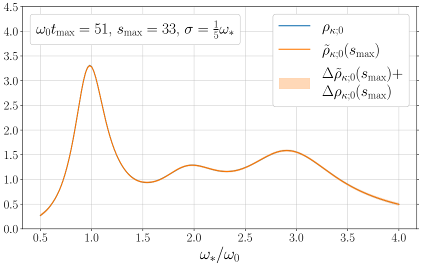


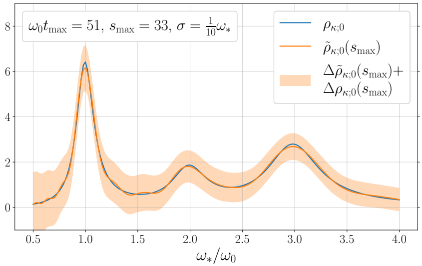
If we lower the width σ

- Emergence of peak structure
- Higher systematic errors $\implies$ We need higher $t_{\max}$

Dependence on the width of the kernel: variable σ

$$C(t) = c_0 e^{-\omega_0 t} + c_1 e^{-\omega_1 t} + c_2 e^{-\omega_2 t}$$





Width $\sigma \propto \omega_* \implies$ Uniform systematic error

Real data: the vector-vector correlator

- Light, connected component of the vector-vector correlator

$$C(an) = -\frac{a^3}{3} \sum_{\mathbf{n},k} \langle V_k^{ud}(an, a\mathbf{n}) V_k^{du}(0, \mathbf{0}) \rangle$$

- Data obtained via multi-level Monte Carlo simulations
  $N_f = 2$, $T \times L^3 = 96 \times 48^3$, $a = 0.065\,\mathrm{fm}$, $M_\pi = 270\,\mathrm{MeV}$
  Data from [M. Dalla Brida, L. Giusti, T. Harris, M. Pepe, 2007.02973]

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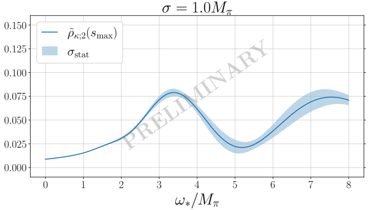
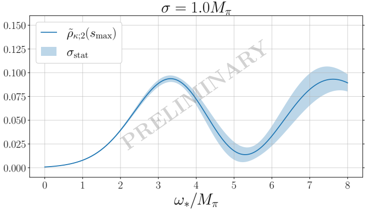
$$\kappa(\omega) = \frac{1}{\pi}\frac{\sigma}{(\omega-\omega_*)^2+\sigma^2} \qquad\qquad \kappa(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\omega-\omega_*)^2}{2\sigma^2}}$$





$$a\omega_0 \approx 0.3,\ t_{\max} = 40a,\ \omega_0 t_{\max} \approx 12$$

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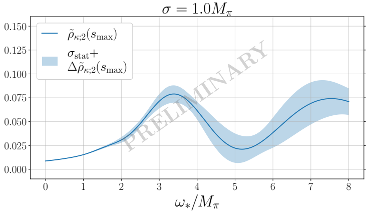

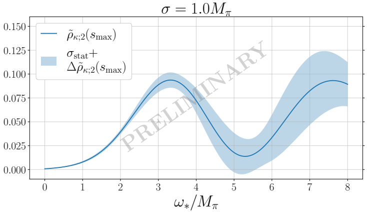

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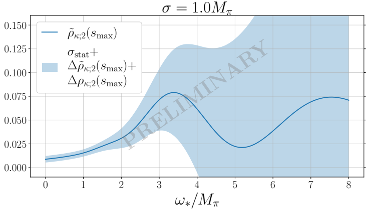

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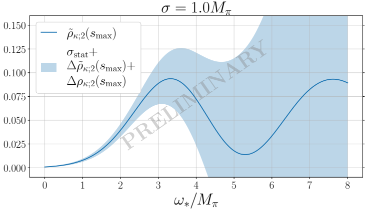

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Conclusion and outlook

- Integral transforms offer a theoretical framework that allows to extract spectral densities, in the continuum and on the lattice, with a controlled continuum limit

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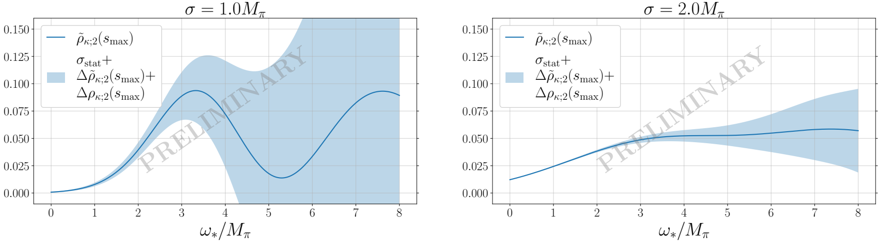
Conclusion and outlook

- Integral transforms offer a theoretical framework that allows to extract spectral densities, in the continuum and on the lattice, with a controlled continuum limit
- If $C(t)$ is known for $t \in [t_0, t_{\max}]$, incomplete spectral densites with the associated bounds give a rigorous estimate of the spectral density
- We are limited by the systematic error, so we need to:
  - Choose suitable kernels $\kappa(\omega)$
  - Improve the bounds (possibly depending on the specific application)

Conclusion and outlook

- Integral transforms offer a theoretical framework that allows to extract spectral densities, in the continuum and on the lattice, with a controlled continuum limit
- If $C(t)$ is known for $t \in [t_0, t_{\max}]$, incomplete spectral densites with the associated bounds give a rigorous estimate of the spectral density
- We are limited by the systematic error, so we need to:
  - Choose suitable kernels $\kappa(\omega)$
  - Improve the bounds (possibly depending on the specific application)
- Eventually, we need to resolve longer lattices


$\sigma = 1.0 M_\pi$
$\tilde{\rho}_{\kappa;2}(s_{\max})$
$\sigma_{\rm stat}+$
$\Delta\tilde{\rho}_{\kappa;2}(s_{\max})+$
$\Delta\rho_{\kappa;2}(s_{\max})$
PRELIMINARY
ω_*/M_π
$\sigma = 2.0 M_\pi$
$\tilde{\rho}_{\kappa;2}(s_{\max})$
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$\Delta\tilde{\rho}_{\kappa;2}(s_{\max})+$
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PRELIMINARY
ω_*/M_π

Thank you!

Backup slides

Continuum solution

1. The Mehler-Fock and Kontorovich-Lebedev bases are:

$$\bar{u}_s(t, t_0) = \sqrt{\frac{s \tanh(\pi s)}{t_0}} \, {}_2F_1\left(\frac{1}{2} + is, \frac{1}{2} - is, 1; \frac{t_0 - t}{2t_0}\right)$$

$$\hat{u}_s(\omega, t_0) = \frac{\sqrt{2s \sinh(\pi s)}}{\pi} \frac{K_{is}(\omega t_0)}{\sqrt{\omega}}, \quad t \geq t_0 > 0, \quad s \in \mathbb{R}$$

related by:

$$\left|\Gamma\left(\frac{1}{2} + is\right)\right| \hat{u}_s(\omega, t_0) = \int_{t_0}^{+\infty} e^{-\omega t} \bar{u}_s(t, t_0) \, dt$$

Continuum solution

2. We define the Mehler-Fock and Kontorovich-Lebedev transforms as:

$$\bar{C}_s = \int_{t_0}^{+\infty} C(t)\bar{u}_s(t, t_0)\, dt; \quad \hat{\rho}_s = \int_0^{+\infty} \rho(\omega)\hat{u}_s(\omega, t_0)\, d\omega$$

The relation between the bases leads to:

$$\bar{C}_s = |\Gamma(1/2 + is)|\hat{\rho}_s$$

Using the completeness of $\hat{u}_s(\omega, t_0)$:

$$\rho(\omega) = \int_0^{\infty} \frac{\bar{C}_s}{|\Gamma(1/2 + is)|}\hat{u}_s(\omega, t_0)\, ds$$

Continuum solution

3. For smeared spectral densities:

$$\rho_\kappa = \int_0^{+\infty} \rho(\omega)\kappa(\omega) = \int_0^{\infty} \frac{\bar{C}_s}{|\Gamma(1/2+is)|} \hat{\kappa}_s \, ds$$

$$\hat{\kappa}_s = \int_0^{+\infty} \kappa(\omega) \hat{u}_s(\omega, t_0) \, d\omega$$

Discrete solution

1. The discrete Mehler-Fock and Kontorovich-Lebedev bases are:

$$\bar{v}_s(n,t_0,a) = \sqrt{\frac{s\tanh(\pi s)}{t_0}}\,\frac{\left|\Gamma(2t_0/a-1/2+is)\right|}{\sqrt{a/(2t_0)}\,\Gamma(2t_0/a)}\times$$

$$\times {}_3F_2\Big(-n,\frac{1}{2}+is,\frac{1}{2}-is;1,\frac{2t_0}{a};1\Big)$$

$$\hat{v}_s(\omega,t_0,a) = \frac{\sqrt{2s\sinh(\pi s)}}{\pi}\,\frac{\left|\Gamma(2t_0/a-1/2+is)\right|}{\sqrt{a/(2t_0)}\,\Gamma(2t_0/a)}\times$$

$$\times\sqrt{\frac{\pi}{2t_0}}\,e^{-\omega t_0}\,\frac{a}{1-e^{-a\omega}}\;{}_2F_1\Big(\frac{1}{2}+is,\frac{1}{2}-is;\frac{2t_0}{a};-\frac{e^{-a\omega}}{1-e^{-a\omega}}\Big)$$

related by:

$$|\Gamma(1/2+is)|\hat{v}_s(\omega,t_0,a) = a\sum_{n=0}^{\infty} e^{-\omega a n}\bar{v}_s(n,t_0,a)$$

Discrete solution

2. We define the discrete Mehler-Fock and Kontorovich-Lebedev transforms as:

$$\bar{\mathcal{C}}_s = a \sum_{n=0}^{\infty} \mathcal{C}(t_0 + an)\bar{v}_s(n, t_0, a); \quad \hat{\varrho}_s = \int_0^{+\infty} \rho(\omega)\hat{v}_s(\omega, t_0, a)\, d\omega$$

The relation between the bases leads to:

$$\bar{\mathcal{C}}_s = |\Gamma(1/2 + is)|\hat{\varrho}_s$$

Using the completeness of $\hat{v}_s(\omega, t_0, a)$:

$$\varrho(\omega) = \int_0^{\infty} \frac{\bar{\mathcal{C}}_s}{|\Gamma(1/2 + is)|}\hat{v}_s(\omega, t_0, a)\, ds$$

Discrete solution

3. For smeared spectral densities:

$$\varrho_\kappa = \int_0^{+\infty} \varrho(\omega)\kappa(\omega) = \int_0^{\infty} \frac{\bar{\mathcal{C}}_s}{|\Gamma(1/2+is)|}\hat{\kappa}_s \, ds$$

$$\hat{\kappa}_s = \int_0^{+\infty} \kappa(\omega)\hat{v}_s(\omega, t_0, a)\, d\omega$$

Bound on $[\rho_\kappa(s_{\max}) - \tilde{\rho}_\kappa(s_{\max})]$: derivation

1. We look at $\left|\bar{C}_s - \widetilde{C}_s(t_{\max})\right|$:

$$\left|\bar{C}_s - \widetilde{C}_s(t_{\max})\right| = \int_{t_{\max}}^{+\infty} C(t)\bar{u}_s(t, t_0)$$

We have, for $t \geq t_{\max}$:

$$|C(t)| \leq |C(t_{\max})|\, e^{-\omega_0(t-t_{\max})}$$

So we have:

$$\left|\bar{C}_s - \widetilde{C}_s(t_{\max})\right| \leq |C(t_{\max})|\, b_s(t_{\max})$$

$$b_s(t_{\max}) = \int_{t_{\max}}^{+\infty} e^{-\omega_0(t-t_{\max})} |\bar{u}_s(t, t_0)|$$

Bound on $[\rho_\kappa(s_{\max}) - \tilde{\rho}_\kappa(s_{\max})]$: derivation

2. We use the definitions of the incomplete spectral densities:

$$\rho_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\bar{C}_s}{|\Gamma(1/2+is)|} \hat{\kappa}_s \, ds,$$

$$\tilde{\rho}_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\widetilde{C}_s(t_{\max})}{|\Gamma(1/2+is)|} \hat{\kappa}_s \, ds$$

To obtain:

$$|\rho_\kappa(s_{\max}) - \tilde{\rho}_\kappa(s_{\max})| \leq \int_0^{s_{\max}} \left| \bar{C}_s - \widetilde{C}_s(t_{\max}) \right| \frac{|\hat{\kappa}_s|}{|\Gamma(1/2+is)|} \, ds$$

$$\leq |C(t_{\max})| \int_0^{s_{\max}} b_s(t_{\max}) \frac{|\hat{\kappa}_s|}{|\Gamma(1/2+is)|} \, ds = \Delta\tilde{\rho}_\kappa(s_{\max})$$

Bound on $[\rho_\kappa - \rho_\kappa(s_{\max})]$: derivation

1. We consider the following ratio:

$$\left| \frac{\bar{C}_s}{\Gamma(1/2 + is)} \right| \leq \int_0^{+\infty} |\rho(\omega)||\hat{u}_s(\omega, t_0)| \, d\omega$$

Using the definition:

$$\hat{u}_s(\omega, t_0) = \frac{\sqrt{2s \sinh(\pi s)}}{\pi} \frac{K_{is}(\omega t_0)}{\sqrt{\omega}}$$

we get:

$$\left| \frac{\bar{C}_s}{\Gamma(1/2 + is)} \right| \leq \frac{\sqrt{2s \sinh(\pi s)}}{\pi} \max_{\omega \in \mathbb{R}^+} |K_{is}(\omega t_0)| \int_0^{+\infty} \left| \frac{\rho(\omega)}{\sqrt{\omega}} \right|$$

Bound on $[\rho_\kappa - \rho_\kappa(s_{\max})]$: derivation

2. Using the fact that:

$$\frac{1}{\sqrt{\omega}} = \frac{1}{\Gamma(1/2)} \int_0^{+\infty} e^{-\omega t} t^{-1/2}\, dt$$

and assuming $\rho(\omega) \geq 0$, we write:

$$\begin{aligned} r &= \int_0^{+\infty} d\omega\, \frac{\rho(\omega)}{\sqrt{\omega}} = \\ &= \frac{1}{\Gamma(1/2)} \int_0^{+\infty} dt\, t^{-1/2} \int_0^{+\infty} d\omega\, \rho(\omega) e^{-\omega t} = \\ &= \frac{1}{\Gamma(1/2)} \int_0^{+\infty} C(t) t^{-1/2}\, dt \end{aligned}$$

Bound on $[\rho_\kappa - \rho_\kappa(s_{\max})]$: derivation

3. For $t > t_{\max}$, we use:

$$|C(t)| \leq |C(t_{\max})|\, e^{-\omega_0(t-t_{\max})}$$

so we obtain:

$$r \leq \frac{1}{\Gamma(1/2)} \int_0^{t_{\max}} C(t) t^{-1/2}\, dt + \\ + \frac{e^{\omega_0 t_{\max}}}{\sqrt{\omega_0}} \frac{\Gamma(1/2, \omega_0 t_{\max})}{\Gamma(1/2)} |C(t_{\max})|$$

Bound on $[\rho_\kappa - \rho_\kappa(s_{\max})]$: derivation

4. To conclude, using the definitions:

$$\rho_\kappa = \int_0^{+\infty} \frac{\bar{C}_s}{|\Gamma(1/2+is)|}\hat{\kappa}_s\, ds$$
$$\rho_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\bar{C}_s}{|\Gamma(1/2+is)|}\hat{\kappa}_s\, ds$$

we have:

$$|\rho_\kappa - \rho_\kappa(s_{\max})| \leq \int_{s_{\max}}^{+\infty} \left|\frac{\bar{C}_s}{\Gamma(1/2+is)}\right| |\hat{\kappa}_s|\, ds \leq$$
$$\leq r \int_{s_{\max}}^{\infty} \frac{\sqrt{2s\sinh(\pi s)}}{\pi} \max_{\omega\in\mathbb{R}^+} |K_{is}(\omega t_0)|\, |\hat{\kappa}_s|\, ds = \Delta\rho_\kappa(s_{\max})$$

Discrete incomplete smeared spectral densities

Definition

- the *incomplete, discrete Mehler-Fock transform*:

$$\tilde{\mathcal{C}}(n_{\max})_s = a \sum_{n=0}^{n_{\max}} \mathcal{C}(t_0 + an)\bar{v}_s(n, t_0, a), \quad n_{\max} = (t_{\max} - t_0)/a$$

- the *incomplete spectral density*:

$$\varrho_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\bar{\mathcal{C}}_s}{|\Gamma(1/2 + is)|} \hat{\kappa}_s \, ds$$

- the *approximated, incomplete spectral density*:

$$\tilde{\varrho}_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\widetilde{\mathcal{C}}_s(t_{\max})}{|\Gamma(1/2 + is)|} \hat{\kappa}_s \, ds$$

Bound on $[\varrho_\kappa(s_{\max}) - \tilde{\varrho}_\kappa(s_{\max})]$: derivation

1. We look at $\left|\bar{\mathcal{C}}_s - \widetilde{\mathcal{C}}_s(n_{\max})\right|$:

$$\left|\bar{\mathcal{C}}_s - \widetilde{\mathcal{C}}_s(n_{\max})\right| = a \sum_{n_{\max}+1}^{+\infty} \mathcal{C}(t_0 + an)\bar{v}_s(n, t_0, a)$$

We have, for $n \geq n_{\max}$:

$$|\mathcal{C}(t_0 + an)| \leq |\mathcal{C}(t_0 + an_{\max})|\, e^{-a\omega_0(n-n_{\max})}$$

So we have:

$$\left|\bar{\mathcal{C}}_s - \widetilde{\mathcal{C}}_s(n_{\max})\right| \leq |\mathcal{C}(t_0 + an_{\max})|\, \mathfrak{b}_s(n_{\max})$$

$$\mathfrak{b}_s(t_{\max}) = a \sum_{n_{\max}+1}^{+\infty} e^{-a\omega_0(n-n_{\max})} |\bar{v}_s(n, t_0, a)|$$

Bound on $[\varrho_\kappa(s_{\max}) - \tilde{\varrho}_\kappa(s_{\max})]$: derivation

2. We use the definitions of the incomplete spectral densities:

$$\varrho_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\bar{\mathcal{C}}_s}{|\Gamma(1/2+is)|} \hat{\kappa}_s \, ds$$

$$\tilde{\varrho}_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\widetilde{\mathcal{C}}_s(n_{\max})}{|\Gamma(1/2+is)|} \hat{\kappa}_s \, ds$$

To obtain:

$$|\varrho_\kappa(s_{\max}) - \tilde{\varrho}_\kappa(s_{\max})| \leq \int_0^{s_{\max}} \left| \bar{\mathcal{C}}_s - \widetilde{\mathcal{C}}_s(n_{\max}) \right| \frac{|\hat{\kappa}_s|}{|\Gamma(1/2+is)|} \, ds$$

$$\leq |\mathcal{C}(t_0 + a n_{\max})| \int_0^{s_{\max}} \mathfrak{b}_s(n_{\max}) \frac{|\hat{\kappa}_s|}{|\Gamma(1/2+is)|} \, ds = \Delta\tilde{\varrho}_\kappa(s_{\max})$$

Bound on $[\varrho_\kappa - \varrho_\kappa(s_{\max})]$: derivation

1. We consider the following ratio:

$$\left|\frac{\bar{\mathcal{C}}_s}{\Gamma(1/2+is)}\right| \leq \int_0^{+\infty} |\varrho(\omega)||\hat{v}_s(\omega,t_0,a)|\,d\omega$$

Using the definition of $\hat{v}_s(n,a,t_0)$, we get:

$$\left|\frac{\bar{\mathcal{C}}_s}{\Gamma(1/2+is)}\right| \leq \frac{\sqrt{2s\sinh(\pi s)}}{\pi} \max_{\omega\in\mathbb{R}^+} |\mathcal{K}_{is}(\omega,t_0)| \int_0^{+\infty} \frac{a|\varrho(\omega)|}{\sqrt{1-e^{-a\omega}}}\,d\omega$$

where:

$$\mathcal{K}_{is}(\omega,t_0) = \sqrt{\frac{\pi}{a}}\left|\Gamma\left(2t_0/a-1/2+is\right)\right|\sqrt{\frac{e^{-a\omega}}{1-e^{-a\omega}}}\times$$
$$\times P^{1-2t_0/a}_{-\frac{1}{2}+is}\left(\coth\left(\frac{a\omega}{2}\right)\right)$$

Bound on $[\varrho_\kappa - \varrho_\kappa(s_{\max})]$: derivation

2. Using the fact that:

$$\frac{1}{\sqrt{1-e^{-a\omega}}} = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} e^{-a\omega n}$$

and assuming $\varrho(\omega) \geq 0$, we write:

$$\begin{aligned}
\mathfrak{r} &= \int_0^{+\infty} \frac{a|\varrho(\omega)|}{\sqrt{1-e^{-a\omega}}}\, d\omega = \\
&= a \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \int_0^{+\infty} \varrho(\omega) e^{-a\omega n}\, d\omega = \\
&= a \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \mathcal{C}(an)
\end{aligned}$$

Bound on $[\varrho_\kappa - \varrho_\kappa(s_{\max})]$: derivation

3. For $t > t_{\max}$, we use

$$|\mathcal{C}(an)| \leq |\mathcal{C}(an_{\max})|\, e^{-a\omega_0(n-n_{\max})}, \quad n_{\max} = t_{\max}/a$$

so we obtain:

$$\mathfrak{r} \leq a \sum_{n=0}^{n_{\max}} (-1)^n \binom{-\frac{1}{2}}{n} \mathcal{C}(an) +$$
$$+ |\mathcal{C}(an_{\max})| a \sum_{n=n_{\max}+1}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} e^{-a\omega_0(n-n_{\max})}$$

Bound on $[\varrho_\kappa - \varrho_\kappa(s_{\max})]$: derivation

4. To conclude, using the definitions:

$$\varrho_\kappa = \int_0^{+\infty} \frac{\bar{\mathcal{C}}_s}{|\Gamma(1/2+is)|}\hat{\kappa}_s\, ds$$

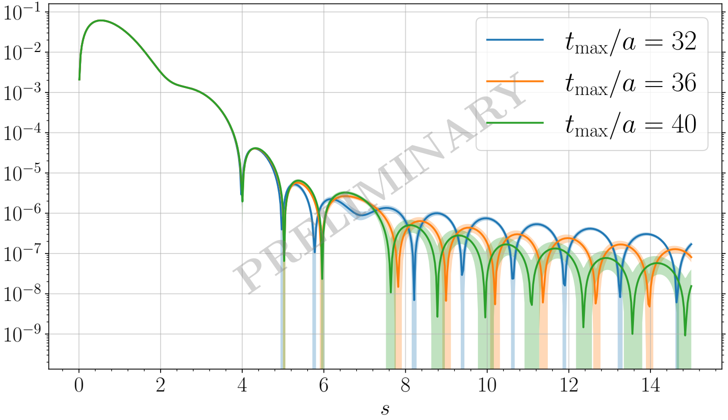
$$\varrho_\kappa(s_{\max}) = \int_0^{s_{\max}} \frac{\bar{\mathcal{C}}_s}{|\Gamma(1/2+is)|}\hat{\kappa}_s\, ds$$

we have:

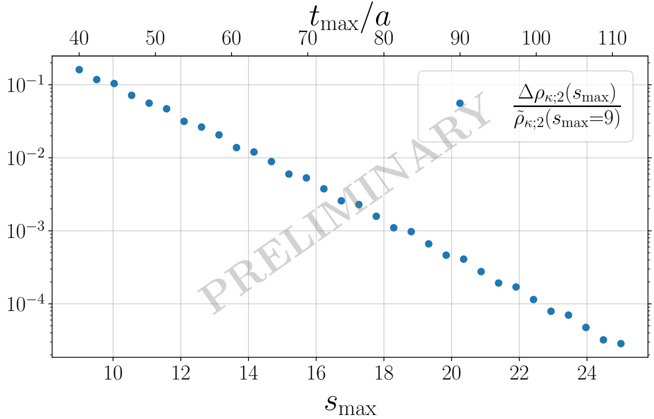
$$|\varrho_\kappa - \varrho_\kappa(s_{\max})| \leq \int_{s_{\max}}^{+\infty} \left|\frac{\bar{\mathcal{C}}_s}{\Gamma(1/2+is)}\right| |\hat{\kappa}_s|\, ds \leq$$

$$\leq \mathfrak{r} \int_{s_{\max}}^{\infty} \frac{\sqrt{2s\sinh(\pi s)}}{\pi} \max_{\omega\in\mathbb{R}^+} |\mathcal{K}_{is}(\omega, t_0)|\, |\hat{\kappa}_s|\, ds = \Delta\varrho_\kappa(s_{\max})$$

Discrete Mehler-Fock transform of the vector-vector correlator



Dependence of $\Delta\rho_{\kappa;m}(s_{\max})$ with $s_{\max} \propto t_{\max}$



Gaussian kernel, $\omega_* = 3M_\pi$, $\sigma = M_\pi$