Tensor-Network Study of Confining Flux Tubes and Effective Strings in $(2+1)$D Compact $U(1)$ Gauge Theory

Flux profiles, spectra and entanglement

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Why?

- Understanding Hamiltonian EST physics.
- Finer couplings without autocorrelation or statistical fluctuations.
- Entanglement measures and real-time evolution.

Variational uncertainties, truncation and finite-volume systematics.

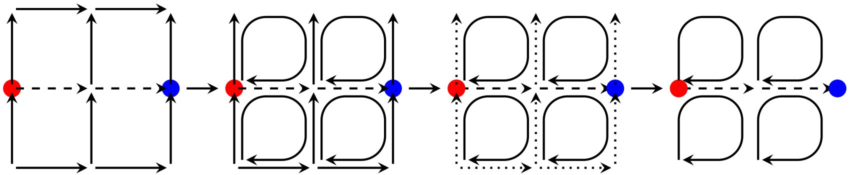
Hamiltonian and operators

Dual Hamiltonian

$$\hat{H}_{\text{dual}} = \frac{g^2}{2} \sum_{\mathbf{x},i} (\hat{L}_{\mathbf{x}} - \hat{L}_{\mathbf{x}+a\mathbf{i}} + n_{ij}^{(\gamma)})^2 + \frac{1}{2g^2} \sum_{\mathbf{x}} (2 - \hat{P}_{\mathbf{x}} - \hat{P}_{\mathbf{x}}^{\dagger})$$

Gauss's law

Forbids the dynamics of the remaining gauge links.



Plaquette basis [2510.27611] [2510.18594]

- Solve the one-plaquette problem:

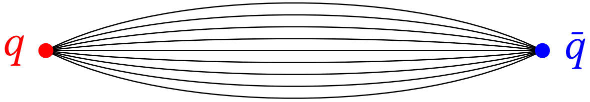
$$\hat{H}_{\text{dual}} = -2g^2 \frac{\mathrm{d}^2}{\mathrm{d}\varphi_P^2} + \frac{1}{g^2}(1 - \cos\varphi_P), \qquad \hat{H}_{\text{dual}} \psi_n = E_n \psi_n.$$

- Digitise by retaining the lowest Λ eigenstates.
- Project each local operator into this basis:

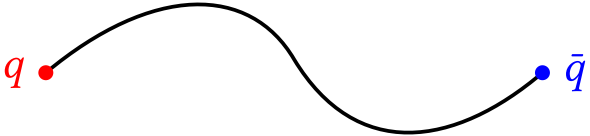
$$(\mathcal{O}_\Lambda)_{mn} = \langle \psi_m | \hat{\mathcal{O}} | \psi_n \rangle , \quad m, n = 0, \ldots, \Lambda - 1.$$

Flux tube as a string

- Chromo-electric and chromo-magnetic fields are confined inside a thin flux tube.



- The fluctuations of the string generate a finite width of the flux tube.



- **Effective string theory** (EST) describes long confining strings.

Hamiltonian EST

- The **Nambu–Gotō** EST predicts

$$V_{\mathrm{NG}}(R) = V_0 + \sigma R \sqrt{1 - \frac{\pi v_s}{12\sigma R^2}}\,.$$

- **Lüscher** term potential:

$$V_{\text{Lüscher}}(R) = V_0 + \sigma R - \frac{\pi v_s}{24R}\,.$$

v_s: worldsheet velocity, c_b: bulk light speed, v_s/c_b: $\widehat{H}$-rescaling invariant.

Measurement of $\Delta E^2_{\mathrm{ws}}(\hat{k}) = m^2_{\mathrm{ws}} + c_b^2 \hat{k}^2$ is still needed.

Effective strings in compact U(1)

- For compact $U(1)$ [2412.01313], where $m_{0^{--}}/\sqrt{\sigma} \to 0$ as $g^2 \to 0$:

$$E_0^{\text{closed}}(R) = \sigma R - \frac{\pi \nu_s}{6R} + \frac{c_b}{\pi R}\,\mathrm{Li}_2(e^{-\mu R}) + \cdots\,,$$
$$V_{\text{open}}(R) = V_0 + \sigma R - \frac{\pi \nu_s}{24R} + \frac{c_b}{4\pi R}\,\mathrm{Li}_2(e^{-\mu R}) + \cdots\,.$$

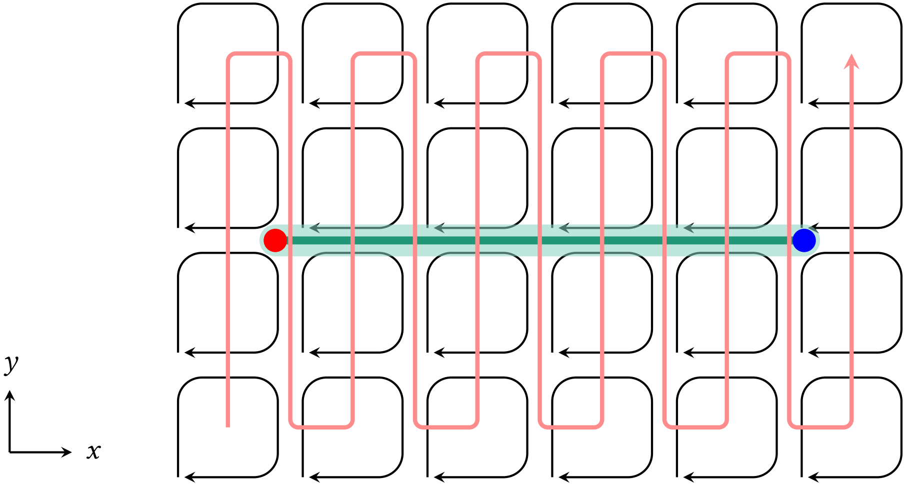
  The massive correction is suppressed for $\mu R \gg 1$.

- Profile of the flux tube [2601.19520]:

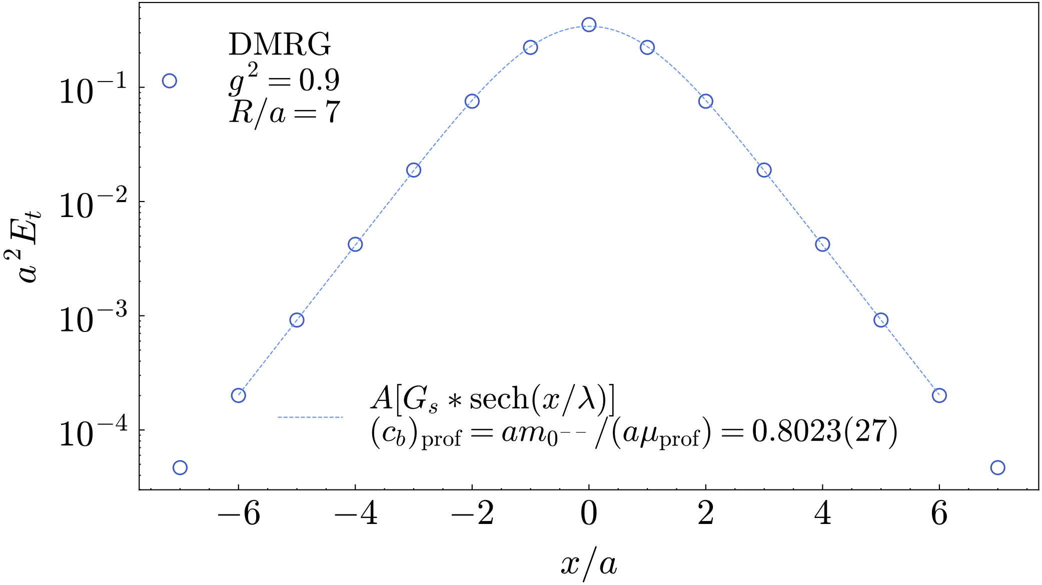
$$\langle F_{01}(y)\rangle = \frac{e^2}{\sqrt{2\pi^3}\,\lambda s}\int_{-\infty}^{\infty} dt\, e^{-t^2/(2s^2)}\,\mathrm{sech}\left(\frac{y-t}{\lambda}\right),$$
$$\lambda = \mu_{\text{prof}}^{-1}, \qquad s^2 \propto \log(MR).$$

Matrix product state

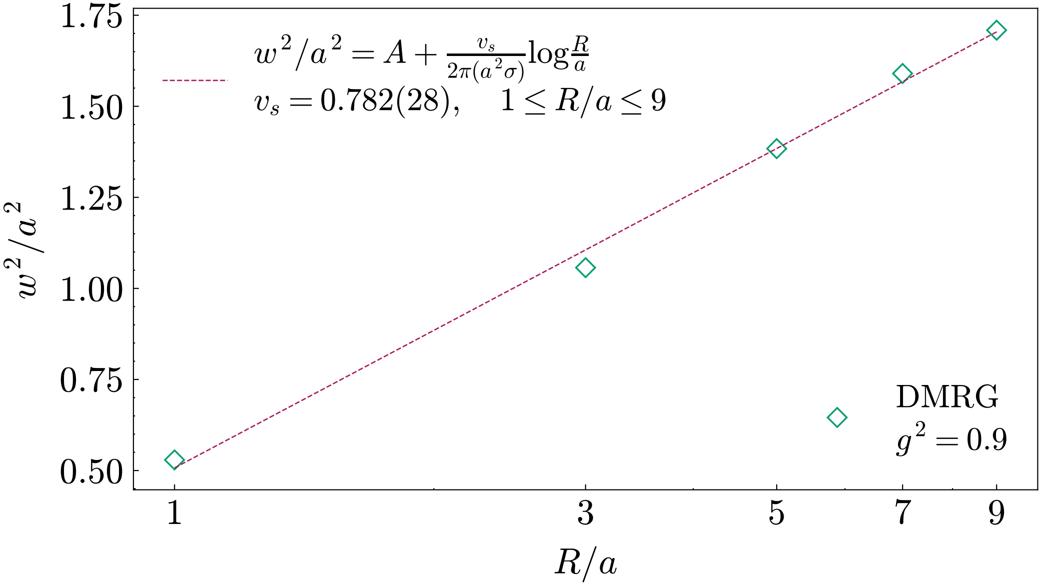
$$i_{\mathrm{MPS}} = x(N_y - 1) + y$$



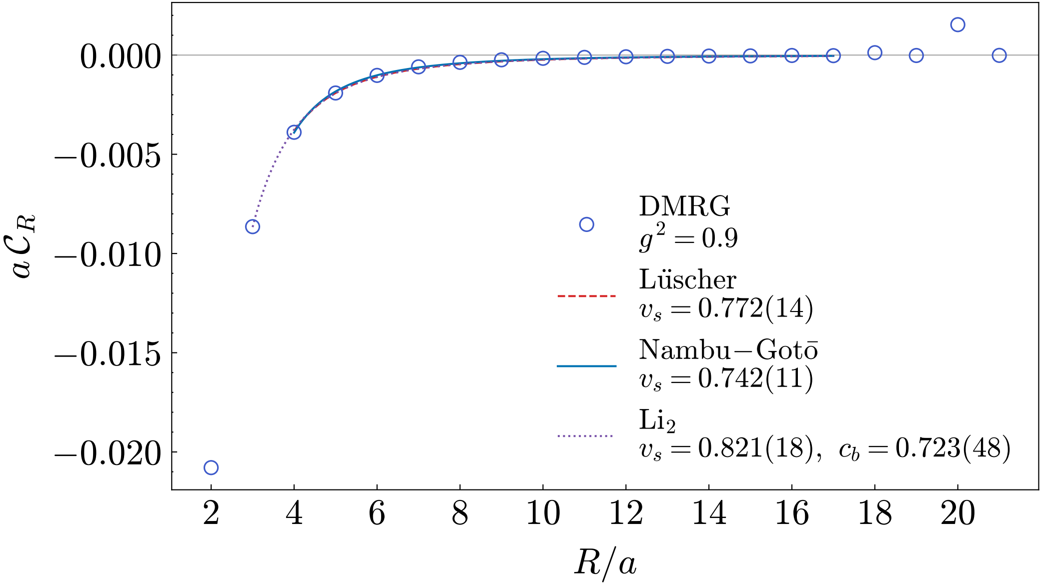
Transverse flux tube profile



Logarithmic broadening and roughening [2601.19520]

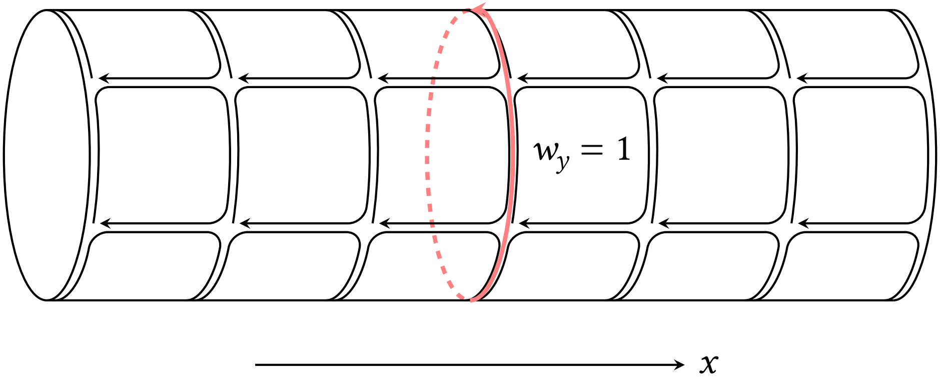


Open string potential

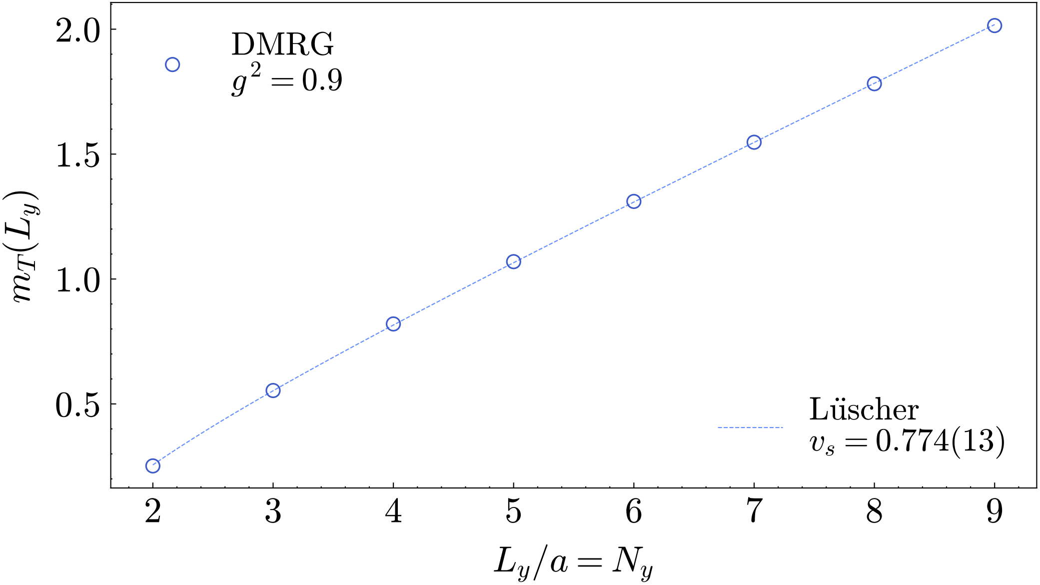


Torelon winding sector

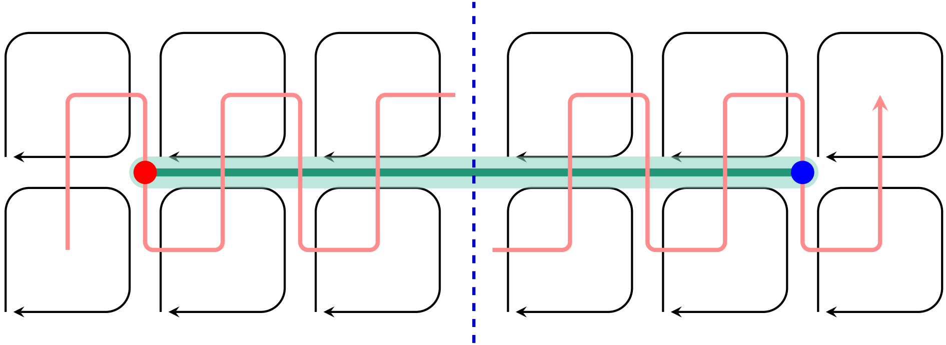
$$m_T = E_0(w_y = 1) - E_0(w_y = 0) = \sigma L_y - \pi v_s/(6L_y) + \cdots$$



Torelon potential

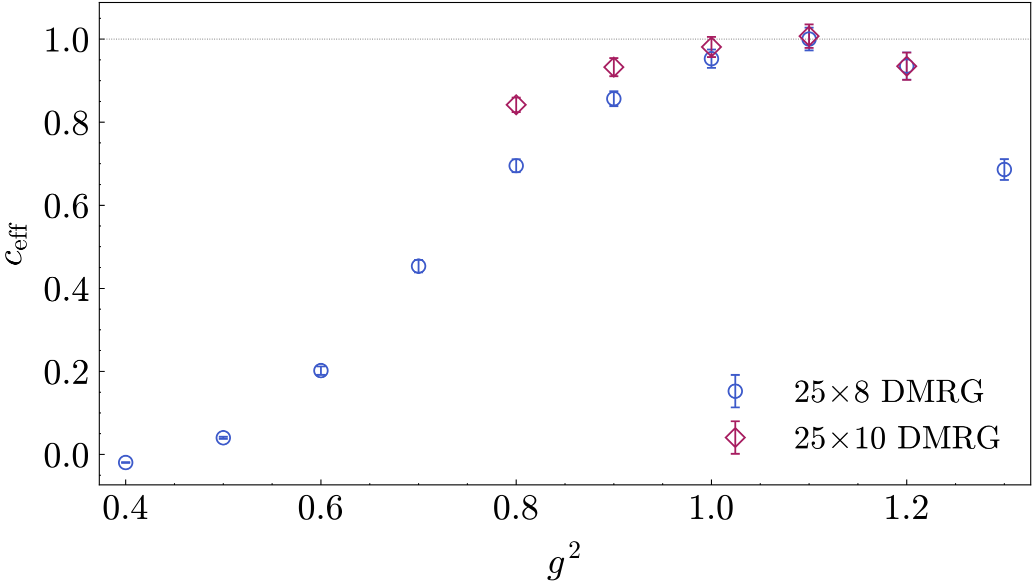


Flux tube entanglement entropy



- Long strings: one massless transverse mode $\Longrightarrow c = 1$ $(1+1)$D CFT.
- Measure at the mid-string cut: $\Delta S_{q\bar{q}}(R) = S_{q\bar{q}}(R) - S_{\mathrm{vac}}$.
- For an open string, fit $\Delta S_{q\bar{q}}(R) = A + \dfrac{c_{\mathrm{eff}}}{6} \ln(R/a)$ to obtain c_{eff}.

Entanglement entropy coefficient



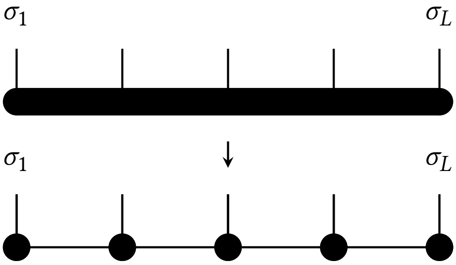
Conclusions and outlook

- **Conclusions:** Hamiltonian tensor networks access flux-tube profiles, string spectra and entanglement.
- **Outlook:** Real-time evolution and further entanglement measures.

Thank you

Questions and discussion

Matrix product states



$$= \sum_{\sigma_1\ldots\sigma_L} \underbrace{c_{\sigma_1\ldots\sigma_L}}_{\mathcal{O}(d^L)} |\sigma_1 \ldots \sigma_L\rangle$$

$$= \sum_{\sigma_1\ldots\sigma_L} \underbrace{A^{\sigma_1} \ldots A^{\sigma_L}}_{\mathcal{O}(\chi^3 L d^3)} |\sigma_1 \ldots \sigma_L\rangle$$

- Introduce a Lagrange multiplier λ.
- Extremize

$$\langle\psi| \hat{H} |\psi\rangle - \lambda \langle\psi|\psi\rangle$$

- At the minimum, $|\psi\rangle$ is the ground state and λ is its energy.

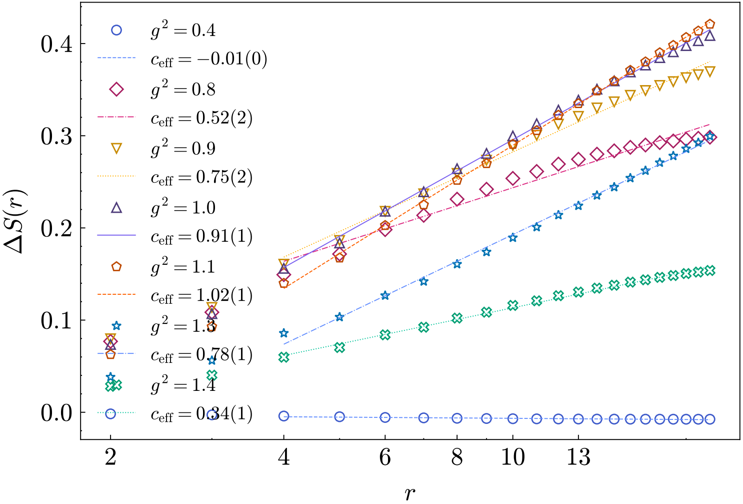
Current picture

- Profile broadening: $v_s = 0.782(28)$ at $g^2 = 0.9$.
- Open curvature: negative through $R = 17$; $v_s = O(1)$, not plateaued.
- Torelon coefficient rising; production scans still incomplete.
- Weak-coupling m_A: finite-box photon; $c_{\text{eff}} \simeq 1$.

Work in progress and future directions

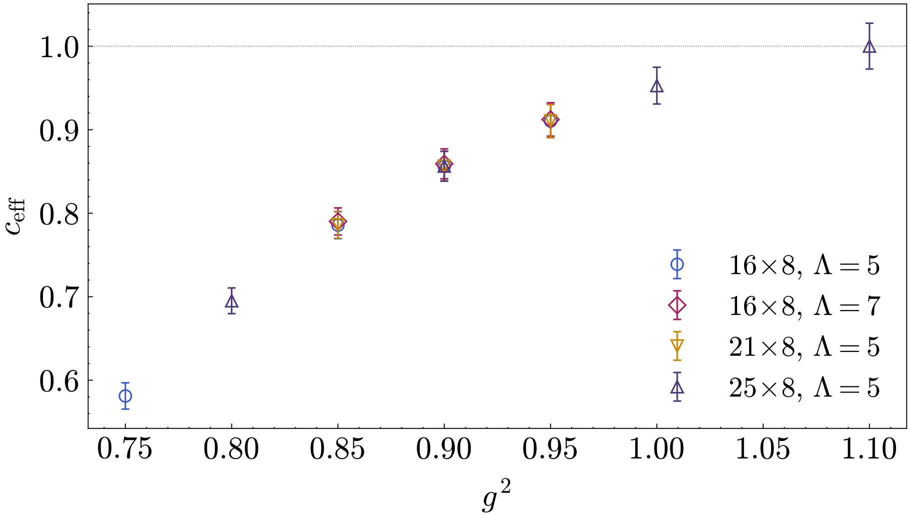
- Finer couplings and longer strings: larger volumes, controlled bond-dimension and Λ extrapolations, and direct v_s/c_b tests.
- Real-time evolution: quenches, flux-tube formation and breaking, string–string scattering, and entanglement growth.
- Dynamical matter: fermionic and scalar charges, screening, string breaking, bound states, and finite density.
- Entanglement: von Neumann and Rényi entropies, entanglement spectra, and separating worldsheet from bulk contributions.
- Spectrum and algorithms: excited strings, torelons, glueballs and massive worldsheet modes; optimized winding-sector DMRG and excited-state targeting.
- Longer term: finite-temperature states and extensions beyond compact U(1).

Open-string entanglement



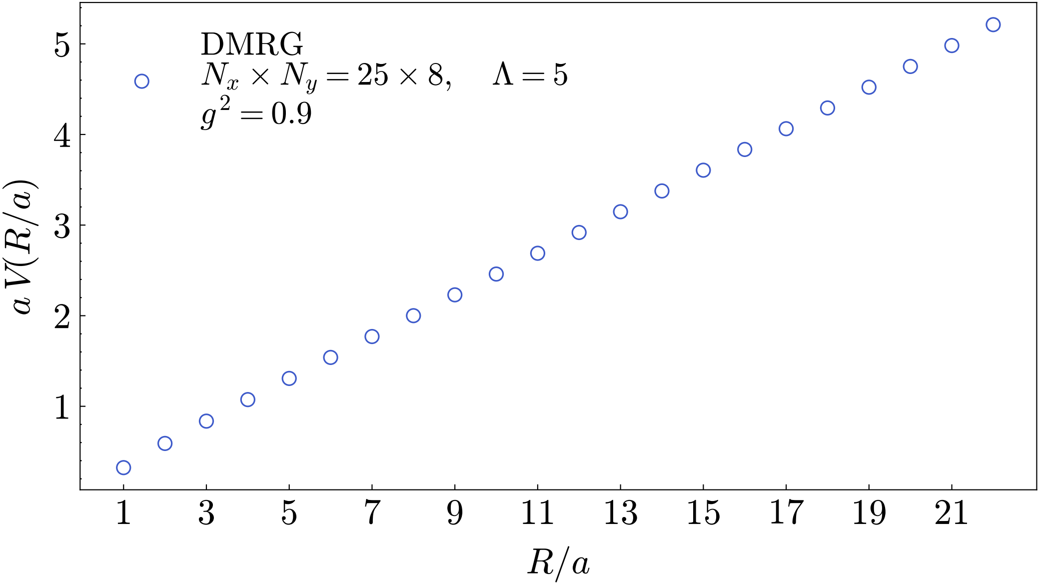
$\Delta S(r) = S_{Q\bar{Q}}(r) - S_0 = A + (c_{\mathrm{eff}}/6)\log r$; full-range fits bend near the walls.

Effective entanglement coefficient

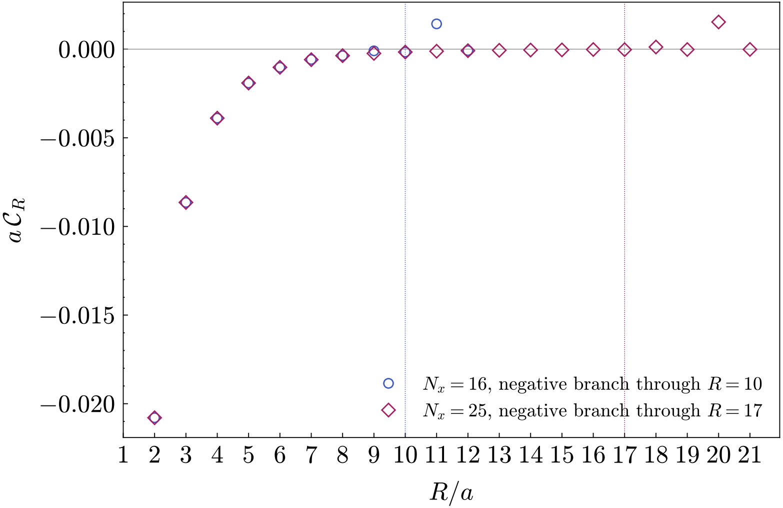


On $4 \leq r \leq 10$: $c_{\text{eff}} = 0.856(18)$ at $g^2 = 0.9$ and $1.000(27)$ at $g^2 = 1.1$.

Open-string potential

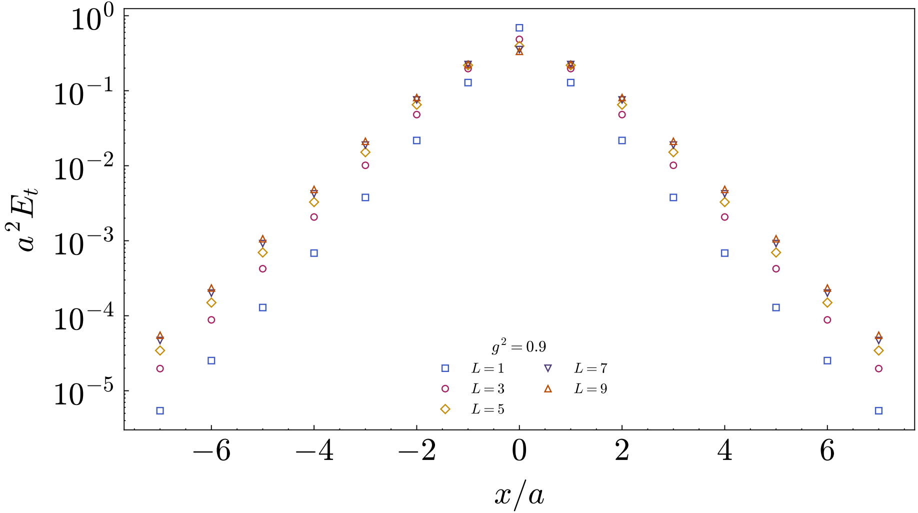


Finite-length effect in the curvature



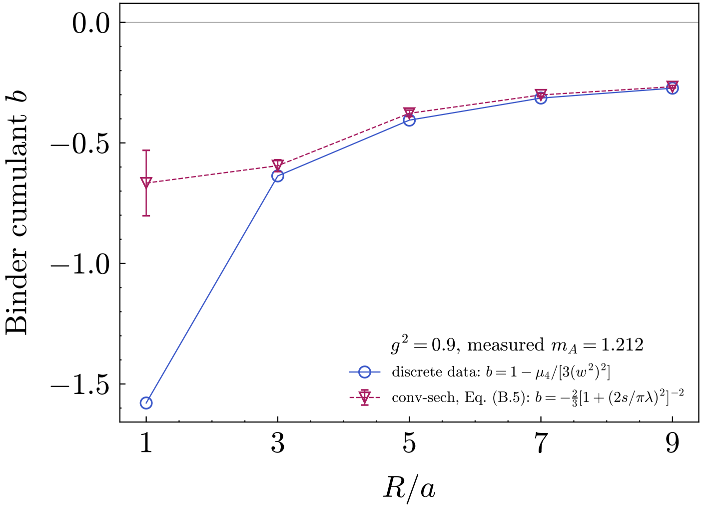
$N_x = 16 \to 25$ extends the initial negative branch from $R = 10$ to $R = 17$.

Flux-tube profiles



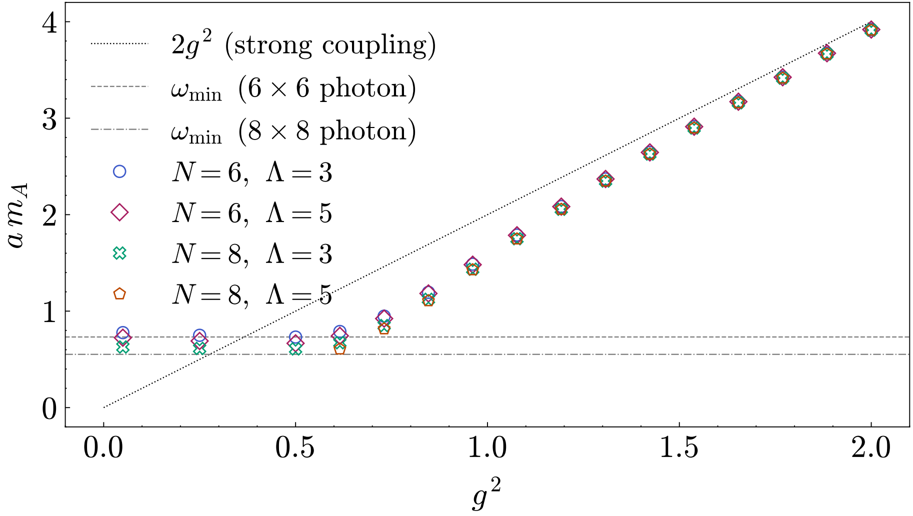
15×10, $\Lambda = 5$, open y, $g^2 = 0.9$; $R/a = 1, 3, 5, 7, 9$.

Binder cumulant



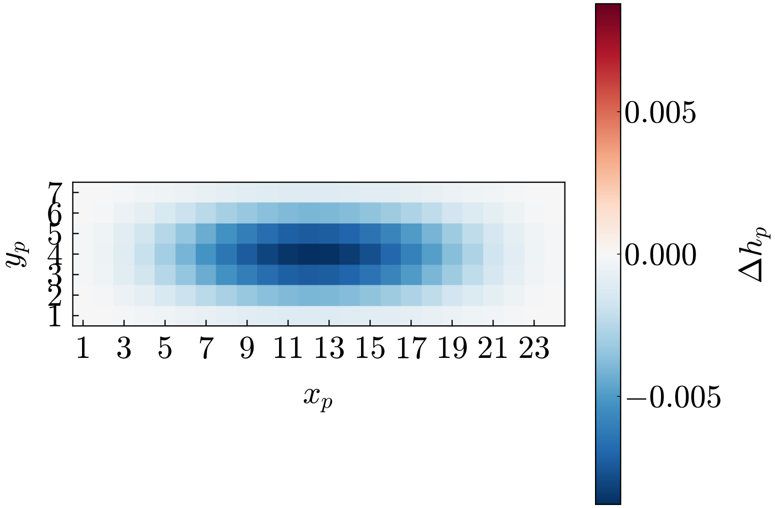
The ABS moment prediction agrees for $R/a \geq 3$; $R/a = 1$ is not a developed string.

Antisymmetric glueball gap



Weak coupling approaches the finite-box photon; strong coupling gives $am_A \to 2g^2$.

25×8 companion glueball



Open y, $\Lambda = 5$: $am_A = 1.791486 \to 2.563610$ for $g^2 = 1.1 \to 1.4$.

Hamiltonian continuum limits

$$g^2 = ae^2, \qquad \widehat{m} = am_{0^{--}}, \qquad \hat{\sigma} = a^2\sigma, \qquad g^2 \to 0.$$

$$\widehat{m} \sim \sqrt{\frac{1}{g^2}}\, e^{-c/g^2},$$
$$\hat{\sigma} \sim \sqrt{g^2}\, e^{-c/g^2}, \qquad c > 0.$$

$$\frac{\widehat{m}}{g^2} \to 0, \qquad \frac{\sqrt{\hat{\sigma}}}{g^2} \to 0, \qquad \frac{\widehat{m}}{\sqrt{\hat{\sigma}}} \to 0.$$
$$\widehat{m} \ll \sqrt{\hat{\sigma}} \ll g^2$$

Three possible continuum limits

Hold fixed e^2, $\sqrt{\sigma}$, or $m_{0^{--}}$.

Continuum limit I: e^2 fixed

$$\boxed{e^2 = \frac{g^2}{a} = \text{fixed}}, \qquad a = \frac{g^2}{e^2} \to 0.$$

$$\frac{m_{0^{--}}}{e^2} = \frac{\widehat{m}}{g^2} \to 0,$$

$$\frac{\sqrt{\sigma}}{e^2} = \frac{\sqrt{\hat{\sigma}}}{g^2} \to 0.$$

Continuum theory

Free massless photon.

Continuum limit II: $\sqrt{\sigma}$ fixed

$$\boxed{\sqrt{\sigma} = \text{fixed}}, \qquad a = \frac{\sqrt{\hat{\sigma}}}{\sqrt{\sigma}} \to 0.$$

$$\frac{m_{0^{--}}}{\sqrt{\sigma}} = \frac{\widehat{m}}{\sqrt{\hat{\sigma}}} \to 0,$$

$$\frac{e^2}{\sqrt{\sigma}} = \frac{g^2}{\sqrt{\hat{\sigma}}} \to \infty.$$

Neutral sector

Free massless photon; the linear regime moves to infinite distance.

Continuum limit III: $m_{0^{--}}$ fixed

$$\boxed{m_{0^{--}} = \text{fixed}}, \qquad a = \frac{\widehat{m}}{m_{0^{--}}} \to 0.$$

$$\frac{\sqrt{\sigma}}{m_{0^{--}}} = \frac{\sqrt{\hat{\sigma}}}{\widehat{m}} \to \infty,$$

$$\frac{e^2}{m_{0^{--}}} = \frac{g^2}{\widehat{m}} \to \infty.$$

Continuum theory

Free massive scalar; strings and charged states decouple.

Backup: fit forms used for the open potential

Model	Potential
Lüscher	$V_0 + \sigma R - \frac{\pi v_s}{24R}$
Nambu–Gotō	$V_0 + \sigma R\sqrt{1 - \frac{\pi v_s}{12\sigma R^2}}$
ABS analogue	$V_0 + \sigma R - \frac{\pi v_s}{24R} + \frac{1}{4\pi R}\,\mathrm{Li}_2(e^{-mR})$
Massive correction	$V_0 + \sigma R - \frac{\pi v_s}{24R} + cK_0(mR)$

- Every potential fit uses the unsubtracted $V(R)$.
- Curvatures use the exact lattice second difference of each model.
- Fixed curvature fits end one point before the first $\mathscr{C}_R \geq 0$.
- A fit window must contain at least one more point than fit parameters.

Backup: continuum normalization and the Hamiltonian limit

In the Hamiltonian convention $\beta_H = 1/g^2$, Loan et al. fit

$$am_A = c_1\sqrt{8\pi^2\beta_H}e^{-\pi^2 v_H(0)\beta_H}, \qquad v_H(0) = 0.3214,$$
$$a^2\sigma = \beta_H^{-1/2}e^{a_0-a_1\beta_H}, \qquad a_0 = 2.359, \quad a_1 = 3.159.$$

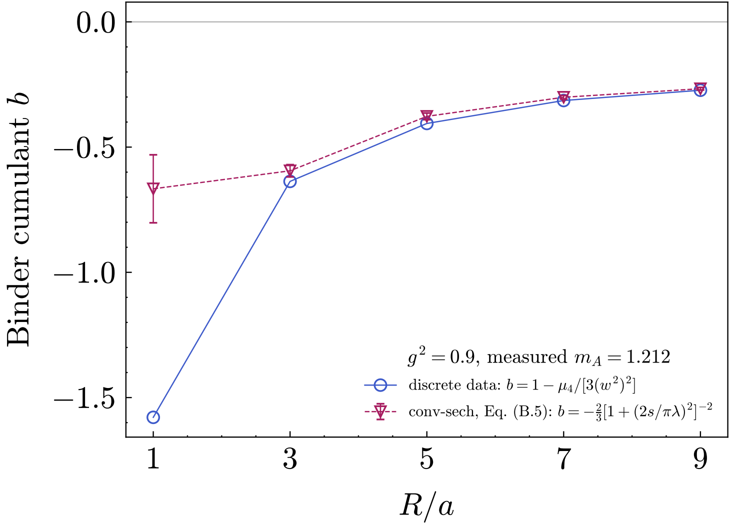
Thus

$$\frac{m_A}{\sqrt{\sigma}} \propto \beta_H^{3/4}e^{-1.59\beta_H} \longrightarrow 0.$$

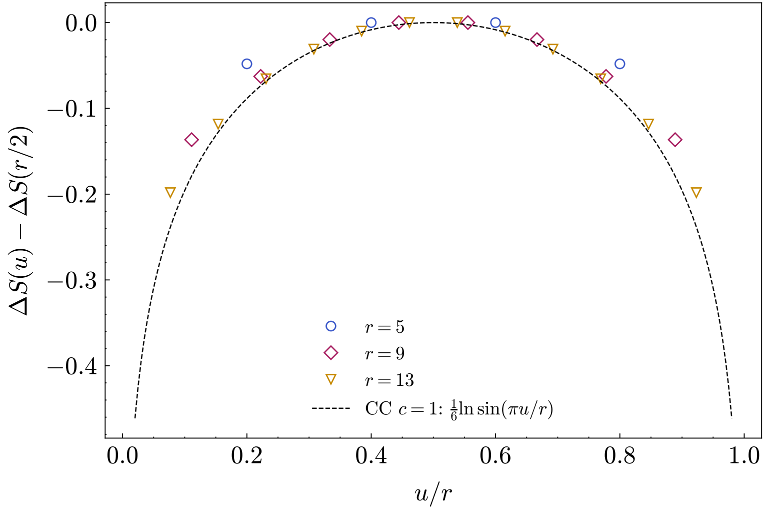
- The fitted normalization of m_A is about five times the leading semiclassical value.
- The scaling window quoted in that study is approximately $1.35 \lesssim \beta_H \lesssim 2.25$.
- Our small open boxes hit the photon momentum floor before the non-perturbative mass becomes visible.

Loan–Brunner–Sloggett–Hamer, [hep-lat/0209159].

Backup: Binder cumulant at $g^2 = 0.9$



Backup: entanglement arc along the string



$\Delta S(u) - \Delta S(r/2) = \frac{1}{6}\log\sin(\pi u/r)$ for the $c = 1$ reference.

Backup: selected references

- Polyakov, compact-QED confinement: Nucl. Phys. B120 (1977) 429; Phys. Lett. B72 (1978) 477.
- Göpfert–Mack, continuum limit: Commun. Math. Phys. 82 (1982) 545.
- Loan et al., Hamiltonian-limit spectrum and tension: [hep-lat/0209159].
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- Caselle et al., intrinsic-width moments: [2601.19520].
- Calabrese–Cardy, entanglement entropy: [hep-th/0405152].
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