Beyond the leading-order HVP contribution to the muon $g-2$

Arnau Beltran Martínez for the Mainz lattice group

Johannes Gutenberg-Universität Mainz

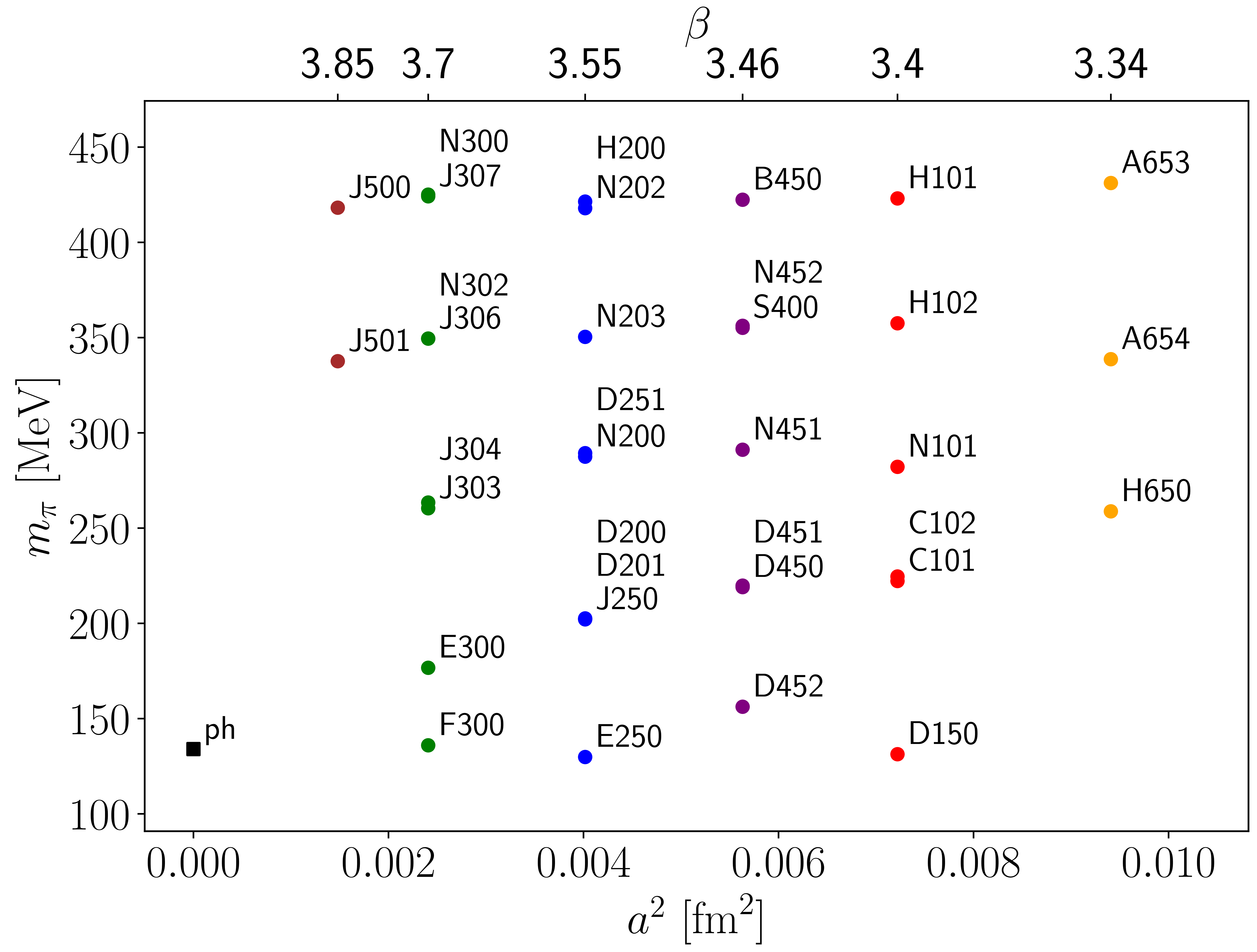
- Introduction and setup
- Time Momentum Representation
- Higher-order HVP
- The electron $g-2$

See A. Conigli's talk

- Large **data set** already created (CLS)

$$G^{(\gamma,\gamma)}(t) = -\frac{1}{3}\int d^3\mathbf{x}\, \langle\, j_k^{\gamma}(t,\mathbf{x})\, j_k^{\gamma}(0)\, \rangle$$

- Deep and robust **methodologies** have also been derived
  - Isospin basis decomposition
  - Window splitting
  - Volume corrections
  - …



See A. Conigli's talk

$$\mathcal{O}^{\mathrm{hvp}}(Q) \sim \int dt\, w(t;Q)\, G^{\gamma,\gamma}(t)$$

- We probe **observables sensitive to the HVP**, each weighting a different region of the hadronic spectrum / Euclidean momentum
  - **Long distance** - $a_\mu^{\mathrm{hvp,\,lo}}$, $a_e^{\mathrm{hvp,\,lo}}$, $\Pi'(0)$
  - **Short distance** - $\Delta\alpha_{\mathrm{had}}\left(M_Z^2\right)$, $\Delta\left(\sin^2\theta_W\right)_{\mathrm{had}}$, $a_\mu^{\mathrm{hvp,\,nlo}}$, $a_\tau^{\mathrm{hvp,\,lo}}$ (*can be computed to high precision*)

- Compare **data- and lattice-based determinations** across these observables to test their (in)consistency

- Introduction and setup
- **Time Momentum Representation**
- Higher-order HVP
- The electron $g-2$

Representations of the *master formula*

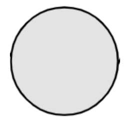
TMR

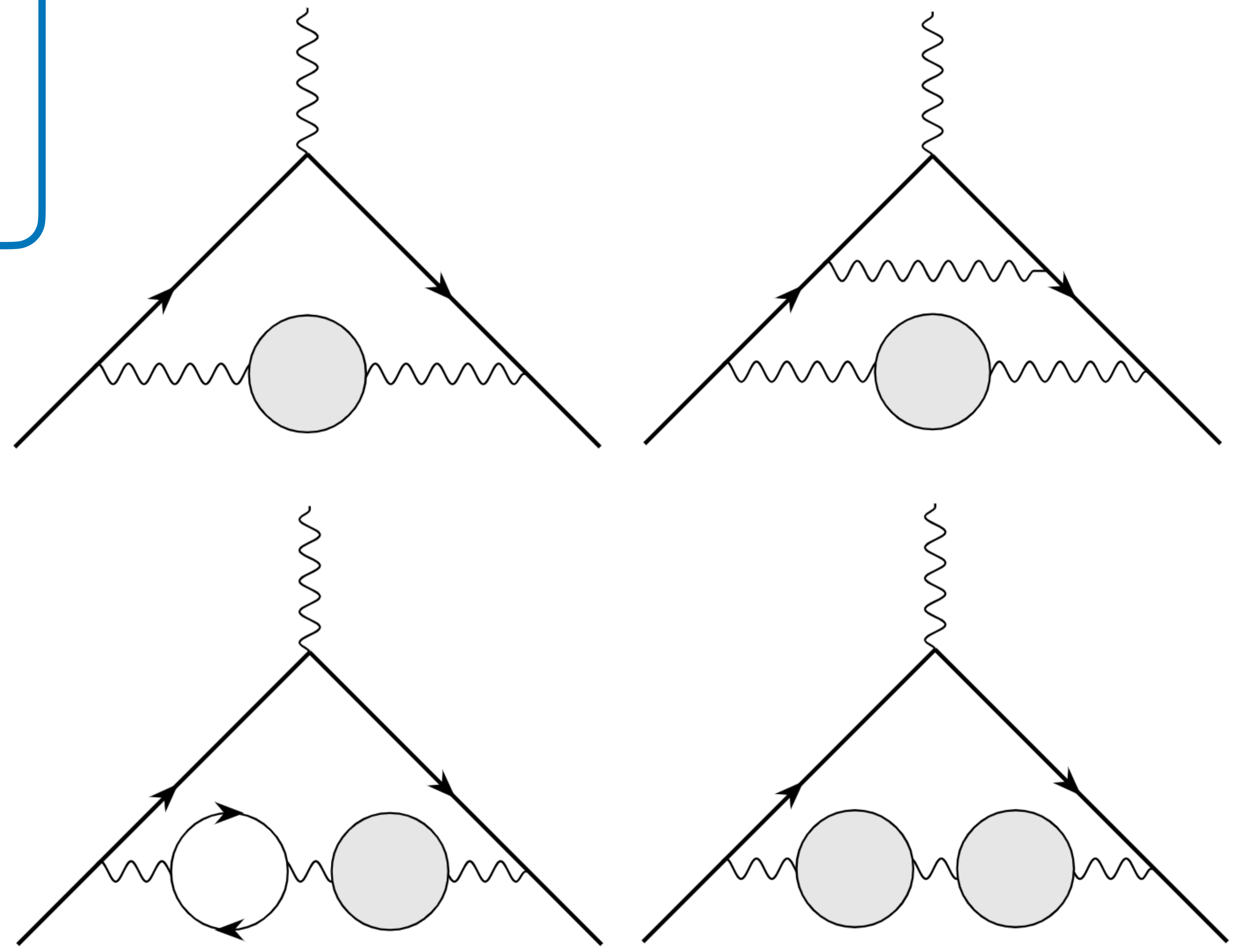
[D. Bernecker and H. B. Meyer 2011]

$$a_\ell^{\mathrm{hvp,(i)}} = \left(\frac{\alpha}{\pi}\right)^{2+N_i} \int dt_1 \ldots dt_{m_i}\, \tilde{f}^{(i)}(\hat{t}_1, \ldots, \hat{t}_{m_i})\, G(t_1) \times \ldots \times G(t_{m_i})$$

$$\tilde{f}^{(i)}(\hat{t}_1, \ldots, \hat{t}_{m_i}) = \int_0^\infty d\hat{\omega}^2\, \hat{f}^{(i)}(\hat{\omega}^2) \prod_{j=1,\ldots,m_i} \frac{4\pi^2}{m_\ell^2 \hat{\omega}^2} \left[\hat{\omega}^2 \hat{t}_j^2 - 4\sin^2\frac{\hat{\omega}\hat{t}_j}{2}\right]$$

$$N_i = \begin{cases} 0 & (i) = (2) \\ 1 & (i) = (4a), (4b), (4c) \\ \ldots & \end{cases}$$

$m_i = \#$ 



We want analytical control over $\tilde{f}$ for all NLO diagrams

Representations of the *master formula*

[D. Bernecker and H. B. Meyer 2011]

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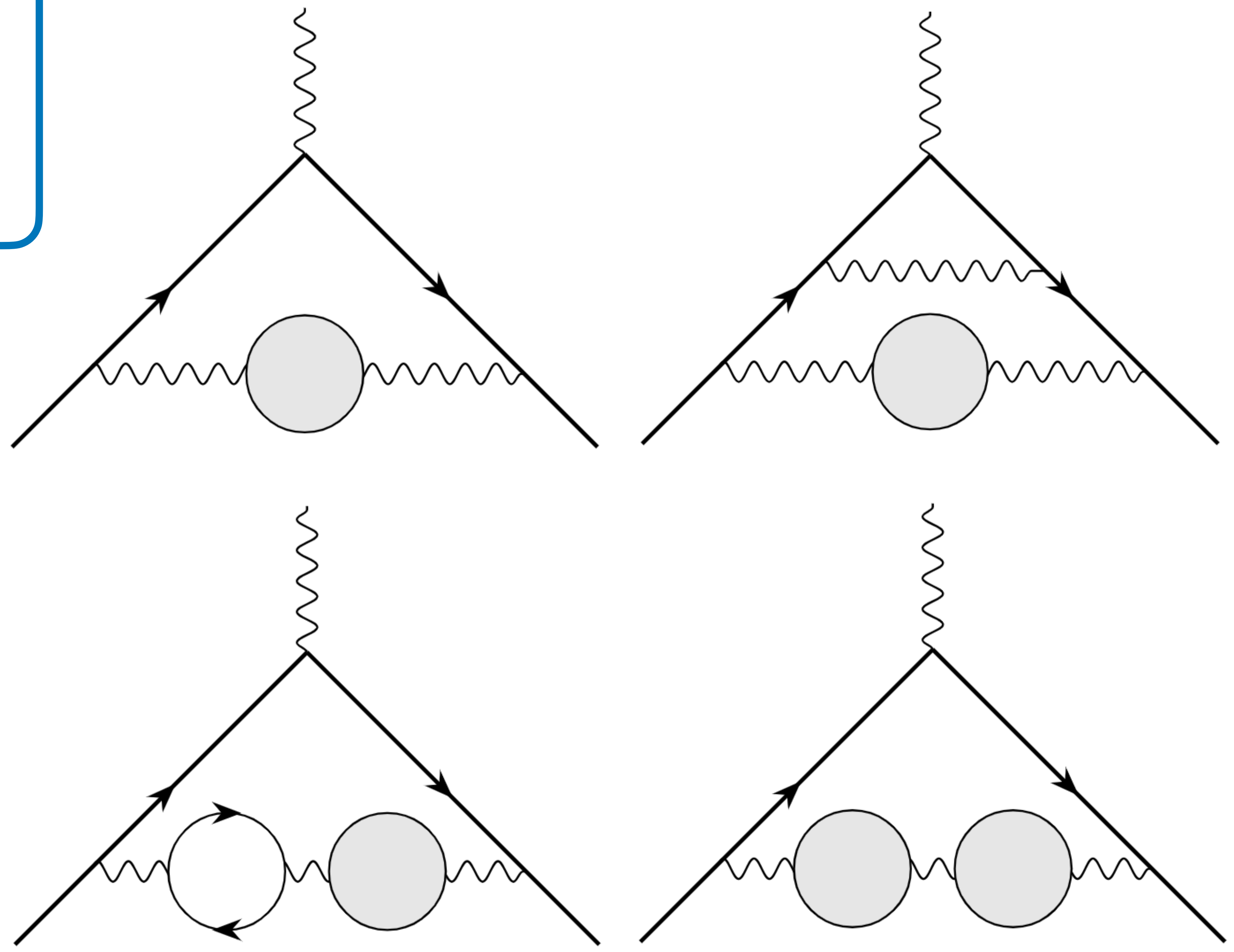

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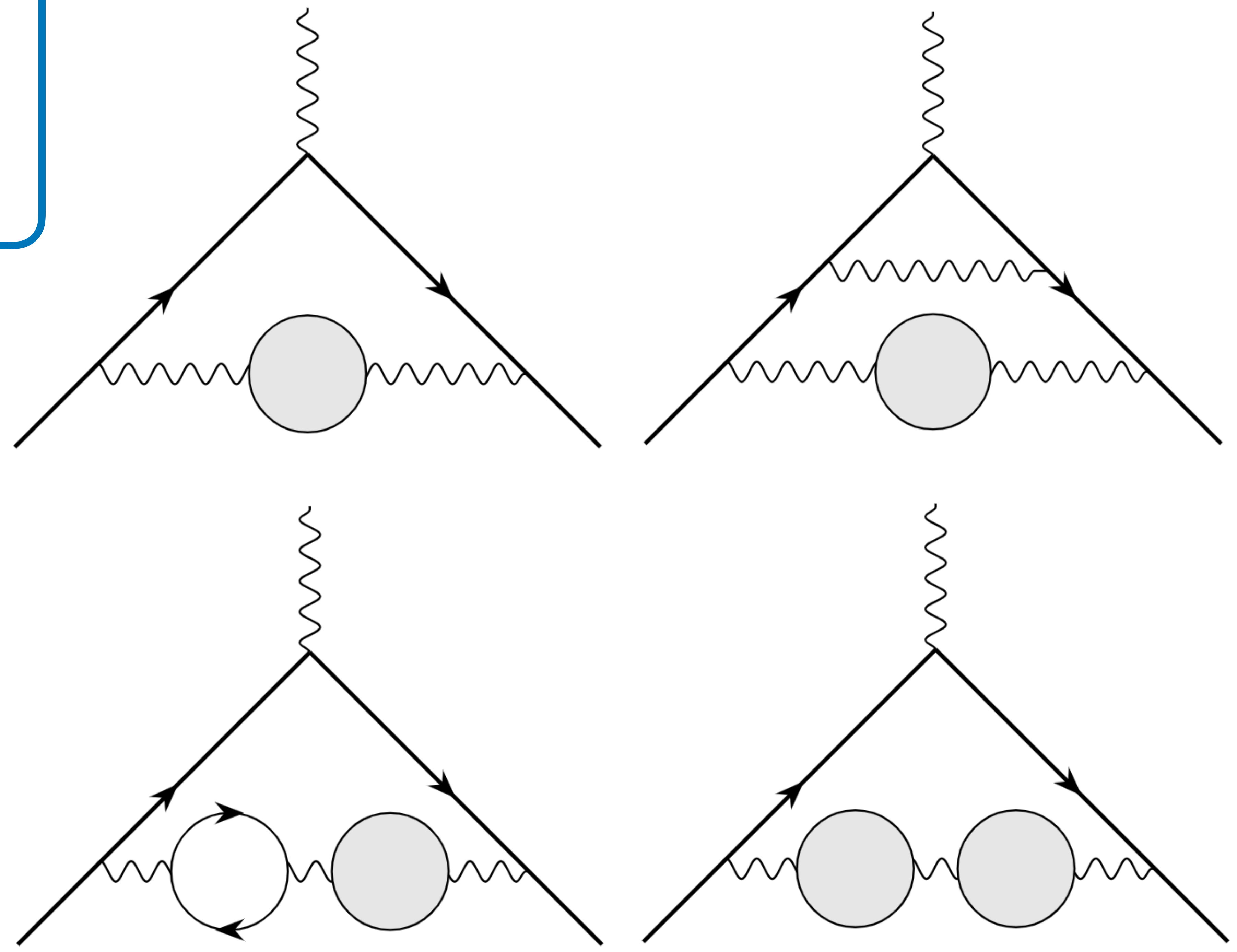
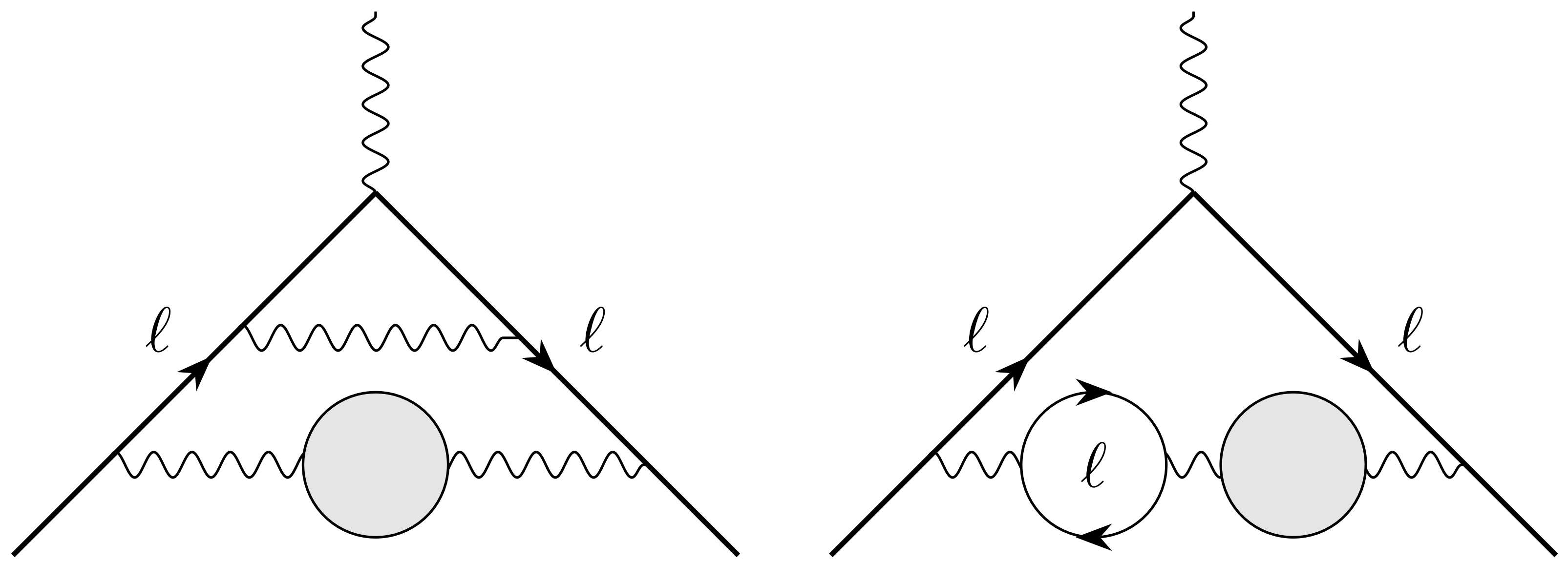


We want analytical control over $\tilde{f}$ for all NLO diagrams

Time-kernel for the NLOa

TMR



$$\frac{m_\ell^2}{8\pi^2}\tilde{f}^{(4a)}(\hat{t}) \;=\; \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\; \hat{f}^{(4a)}(\hat{\omega}^2)\; \left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]$$

[S. Laporta et al. 2024]

$$\frac{m_\ell^2}{16\pi^2}\tilde{f}^{(4a)}(\hat{t}) \overset{\hat{t}\ll 1}{=} \sum_{\substack{n\geq 4\\ n\ \text{even}}} \frac{\hat{t}^n}{n!}\left(a_n + b_n\pi^2 + c_n\left(\gamma_E + \ln\hat{t}\right) + d_n\left(\gamma_E + \ln\hat{t}\right)^2\right)$$

Dominant piece: $\mathscr{D} \;\sim\; \hat{t}^2$

Suppressed piece: $\mathscr{S}$

Can be used to interpolate between $\hat{t} \ll 1$ and $\hat{t} \gg 1$

$$\frac{m_\ell^2}{16\pi^2}\tilde{f}^{(4a)}(\hat{t}) \overset{\hat{t}\gg 1}{=} \mathscr{D}(\hat{t}) \;+\; \mathscr{S}(\hat{t})$$

Time-kernel for the NLOa

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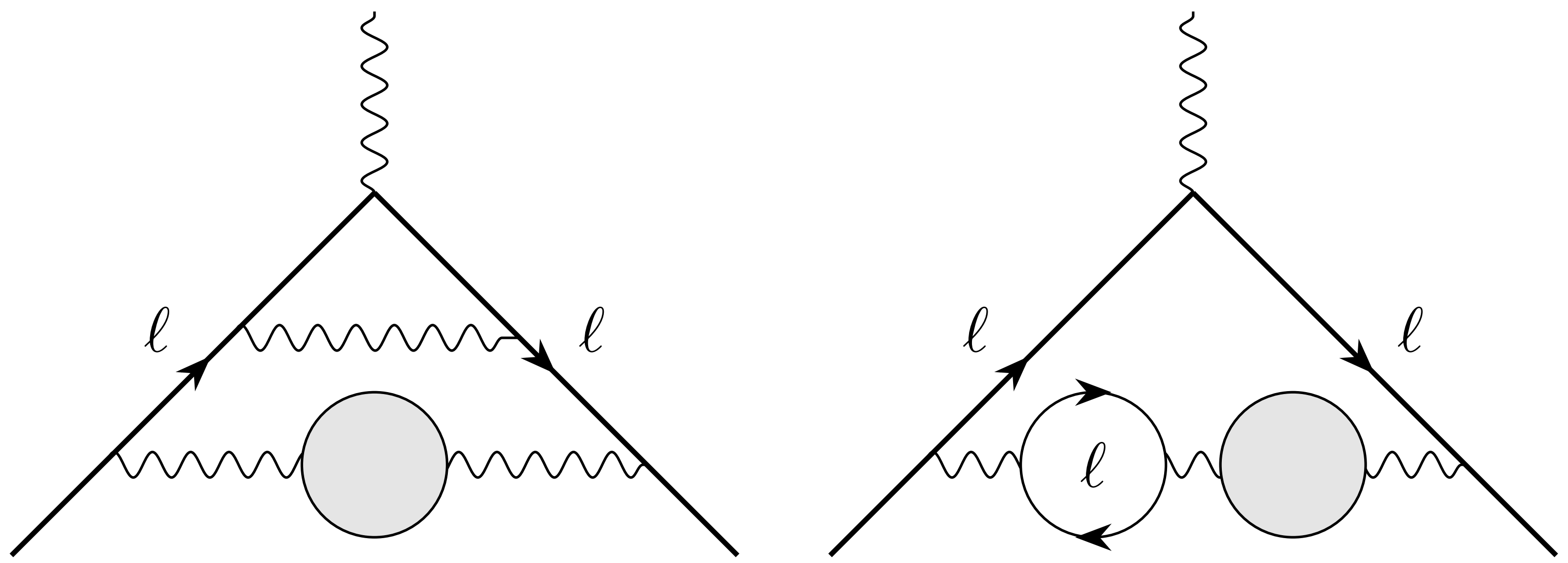

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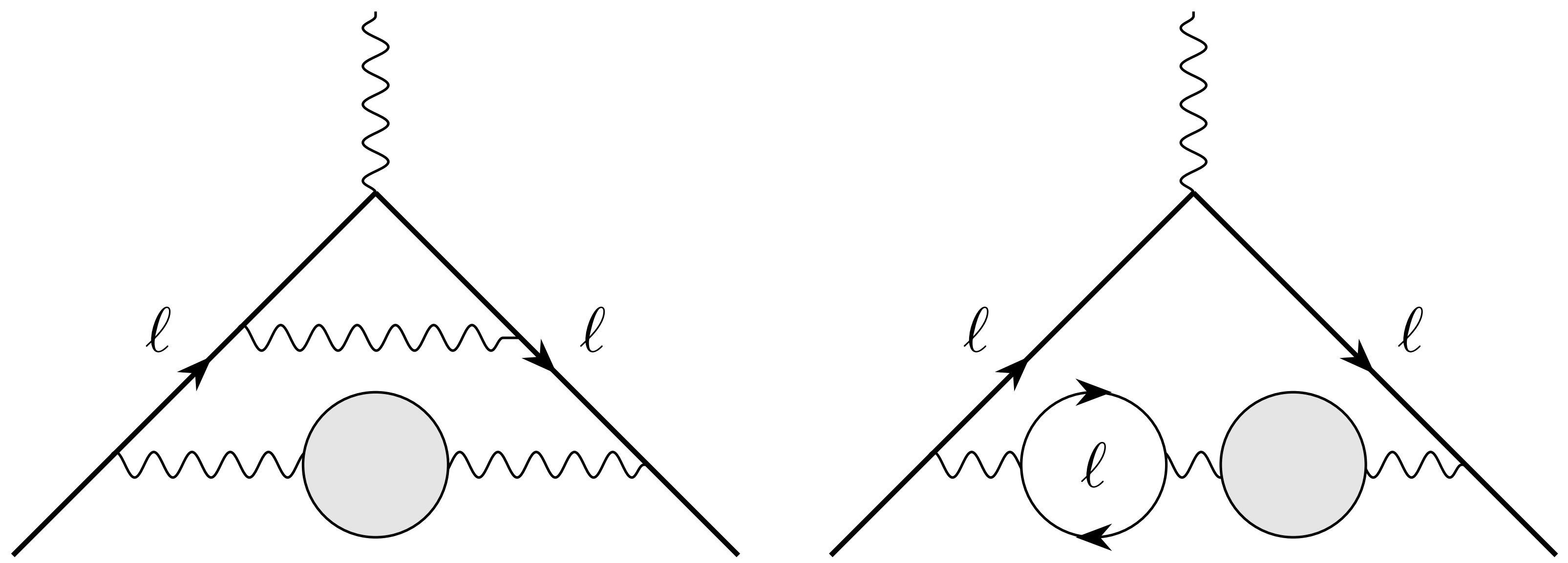

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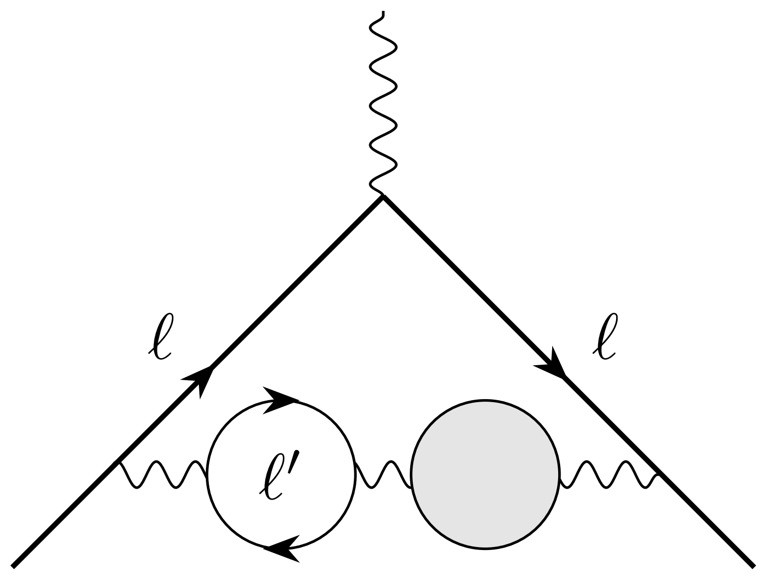
Time-kernel for the NLOb

TMR



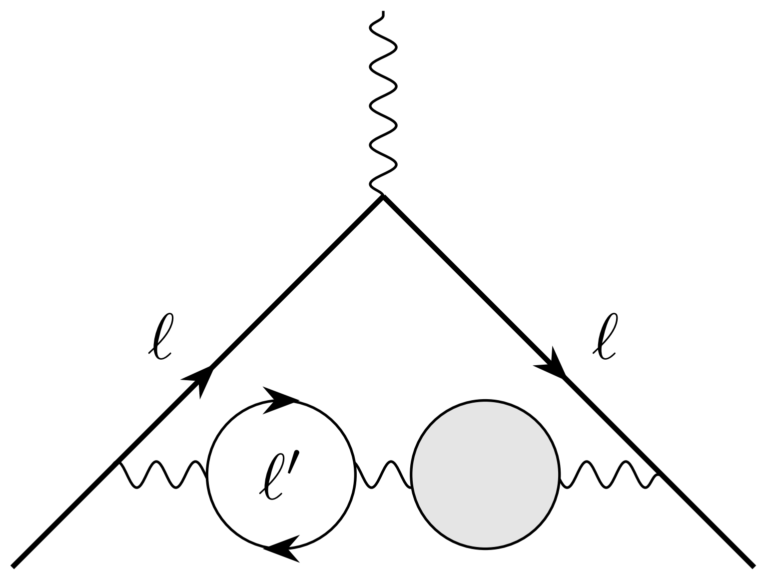
$$\frac{m_\ell^2}{8\pi^2}\tilde{f}^{(4b)}(\hat{t}) \;=\; 2\int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\;\hat{f}^{(2)}(\hat{\omega}^2)\;F_l(\omega; m_{\ell'})\;\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]$$

$$M \equiv \frac{m_{\ell'}}{m_\ell} \ll 1$$

$$\frac{m_\ell^2}{16\pi^2}\tilde{f}^{(4b)}_{\ell'}(\hat{t}) \overset{\hat{t}\ll 1}{=} \sum_{n\geq 4}\sum_{m\geq 0}\frac{\hat{t}^n}{n!}\left(\frac{m_{\ell'}}{m_\ell}\right)^m\left(e_{nm} + f_{nm}\pi^2 + g_{nm}\left(\gamma_E + \ln\hat{t}\right) + h_{nm}\left(\gamma_E + \ln\hat{t}\right)^2\right)$$

$$\frac{m_\ell^2}{16\pi^2}\tilde{f}^{(4b)}_{\ell'}(\hat{t}) \overset{\hat{t}\to\infty}{=} \mathscr{D}\left(\hat{t}, \frac{m_{\ell'}}{m_\ell}\right) \;+\; \mathcal{S}(m_\ell\, t) \;+\; \mathcal{S}'(m_{\ell'}\, t)$$

THREE REGIONS : $\hat{t} \ll 1 \ll 1/M$, $1 \ll \hat{t} \ll 1/M$ **AND** $1 \ll 1/M \ll \hat{t}$



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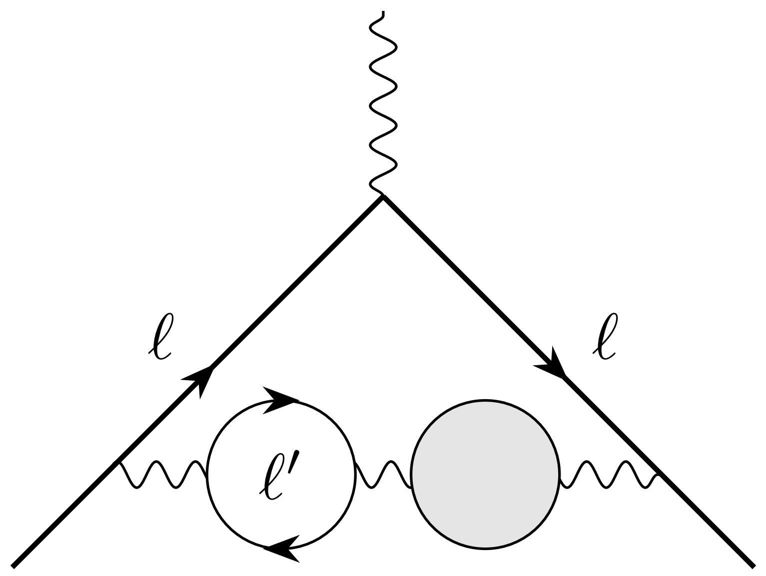

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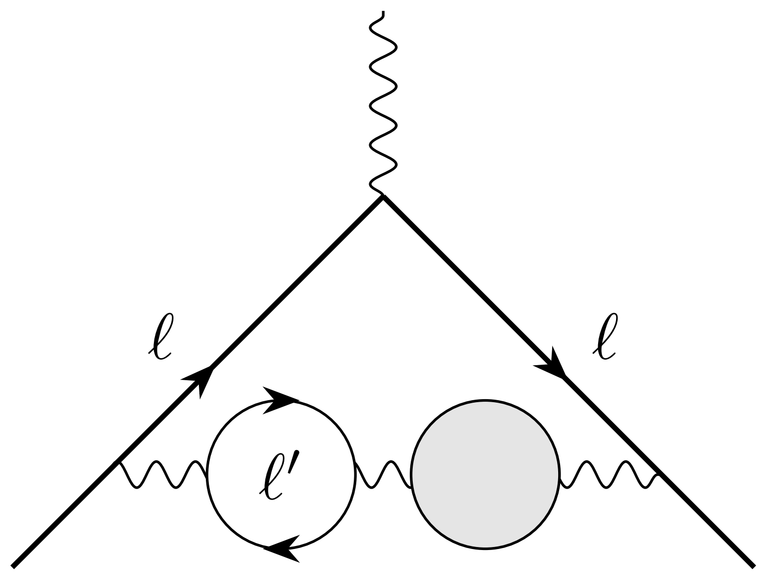

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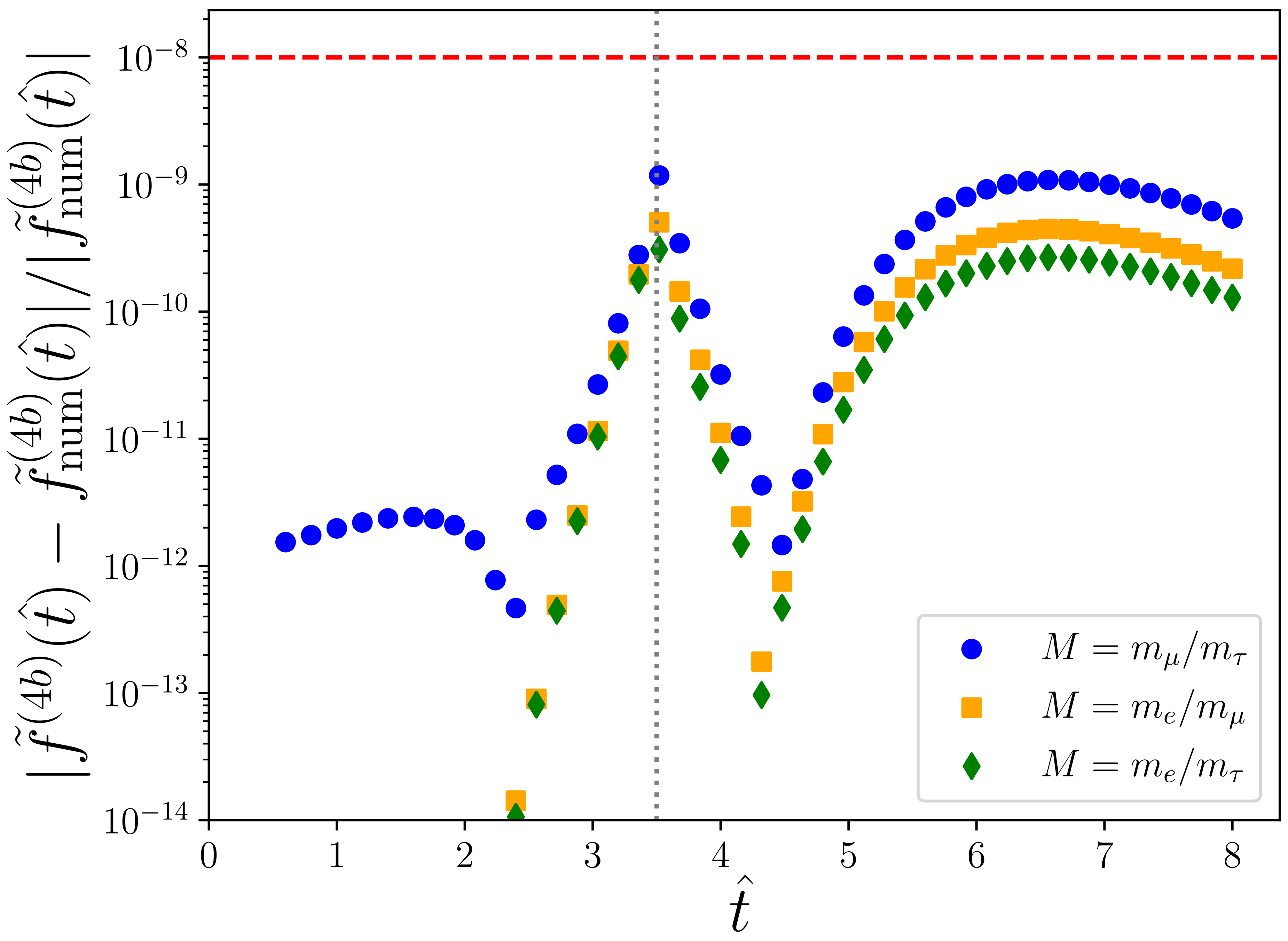
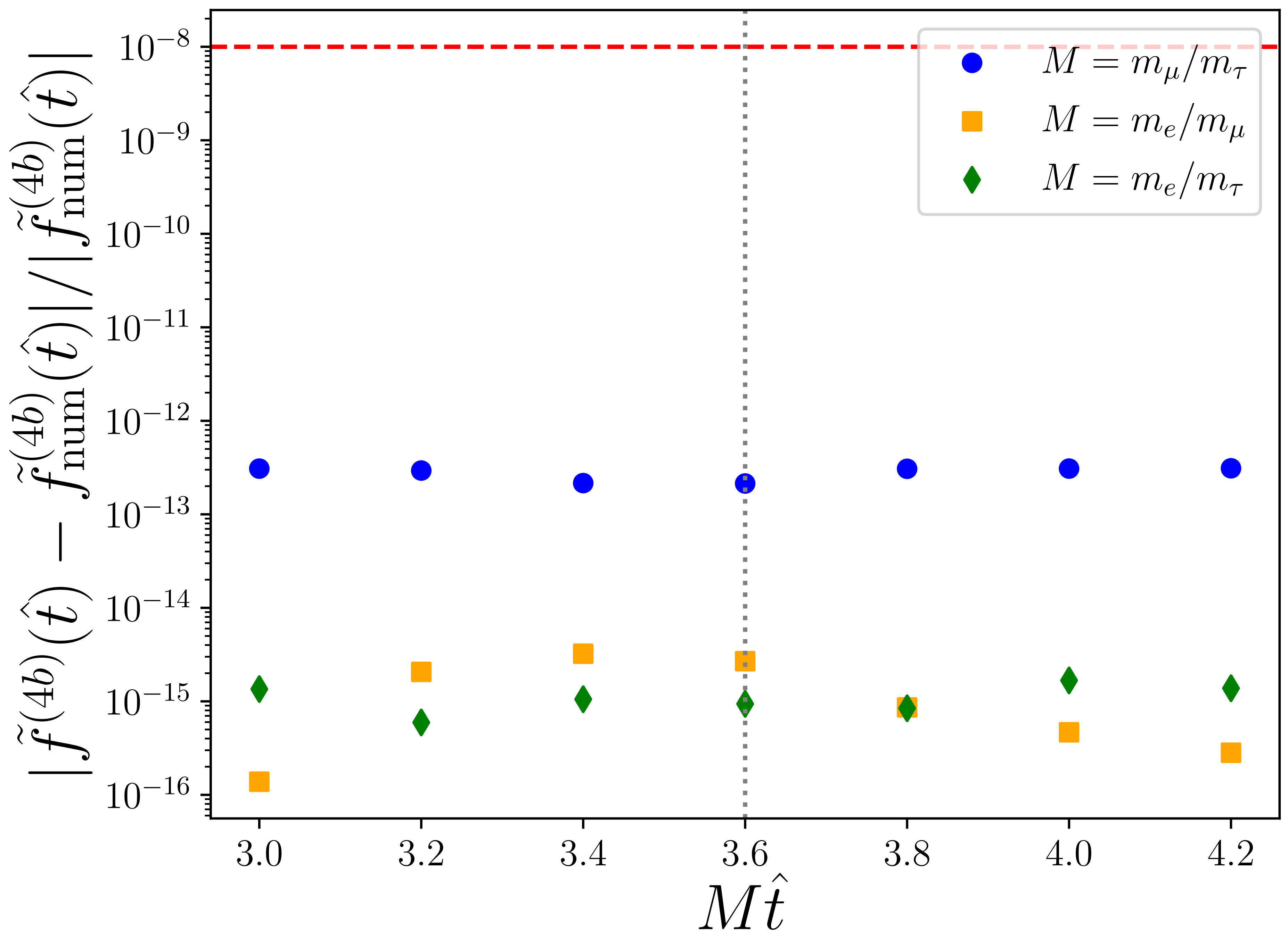
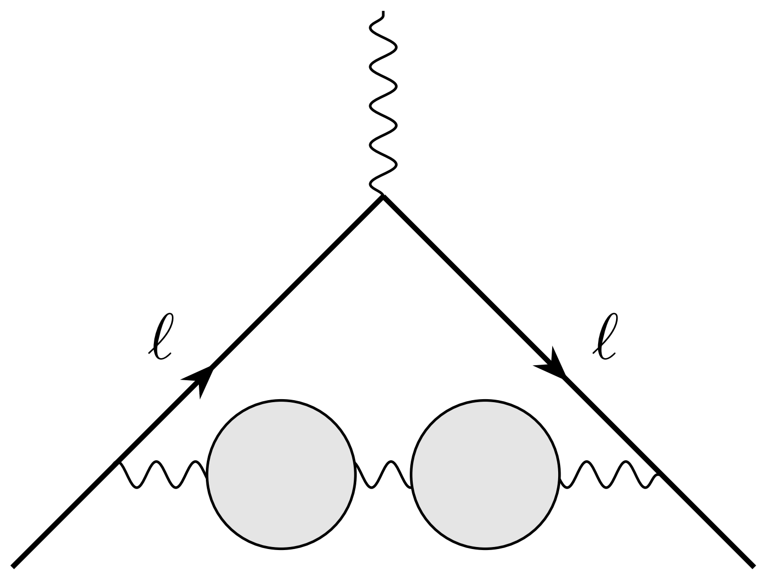




Time-kernel for the NLOc

TMR



$$\frac{m_\ell^4}{32\pi^4}\tilde{f}^{(4c)}(\hat{t},\hat{\tau}) = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}^3}\,\hat{f}^{(2)}(\hat{\omega}^2)\left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]\left[\hat{\omega}^2\hat{\tau}^2 - 4\sin^2\frac{\hat{\omega}\hat{\tau}}{2}\right]$$

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$$\mathscr{A}(\theta) = \frac{1}{\theta^2} - \frac{1}{2} + 2\left(\ln\theta + \gamma_E\right) - \frac{2}{\theta}K_1(2\theta) + \frac{1}{4}G^{2,1}_{1,3}\left(\theta^2\,\middle|\,\begin{matrix}\frac{3}{2}\\0,1,\frac{1}{2}\end{matrix}\right)$$

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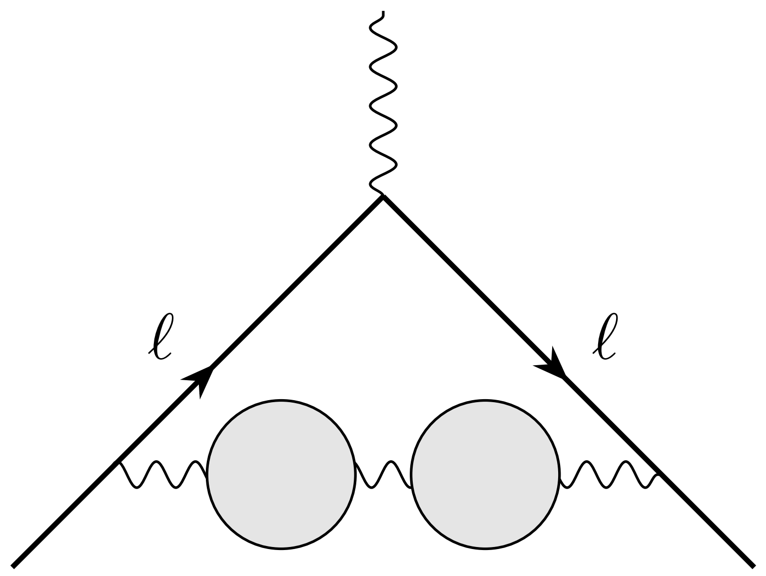
Time-kernel for the NLOc

TMR



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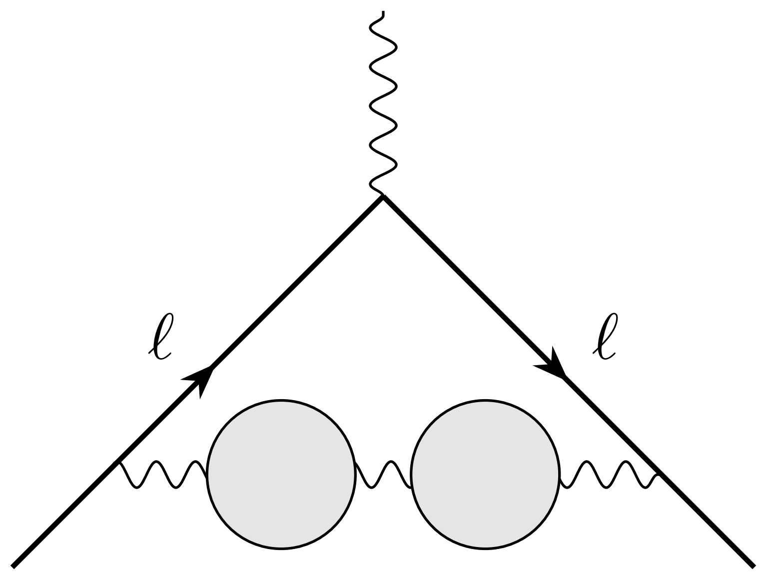
Time-kernel for the NLOc

TMR



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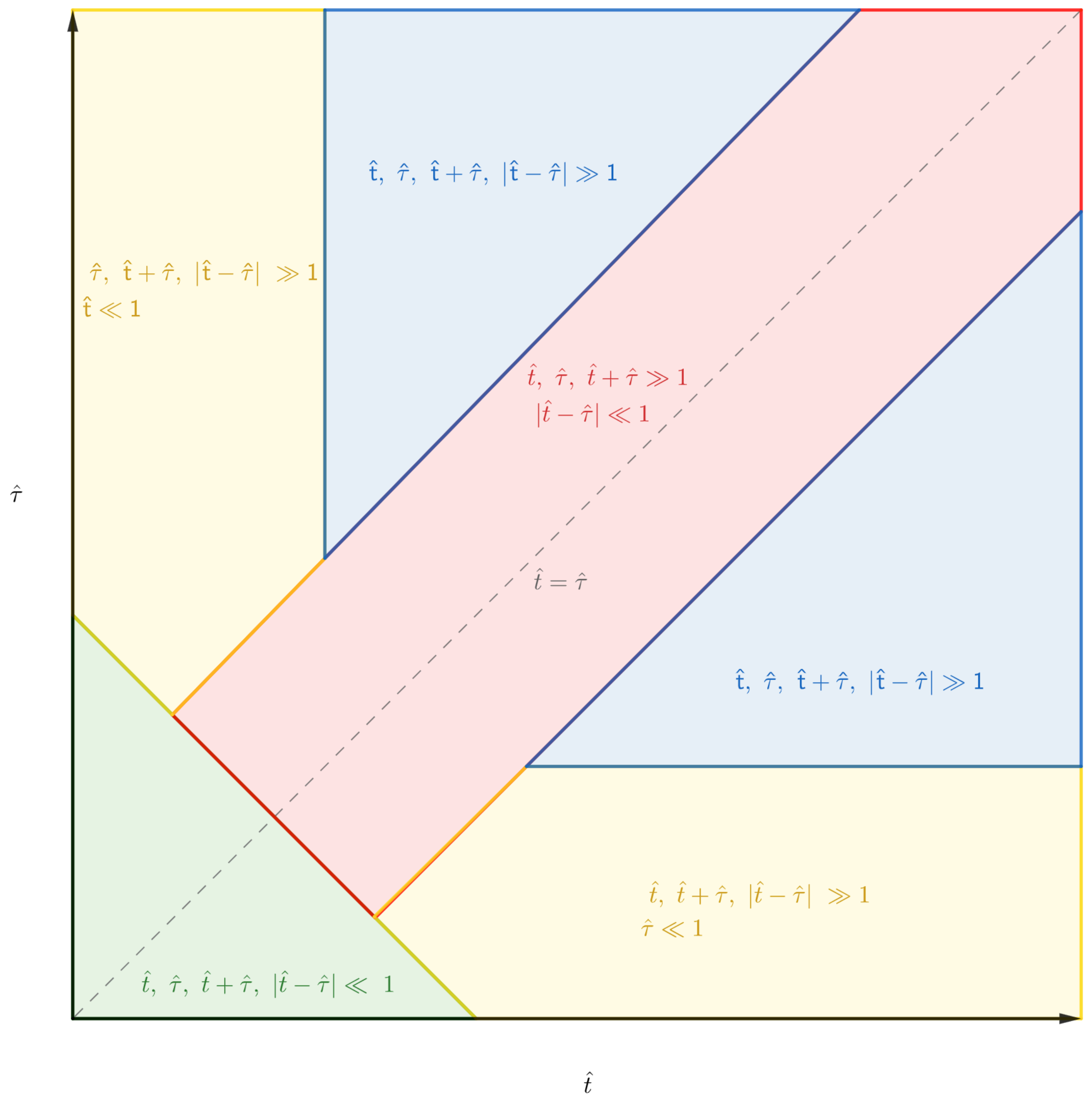
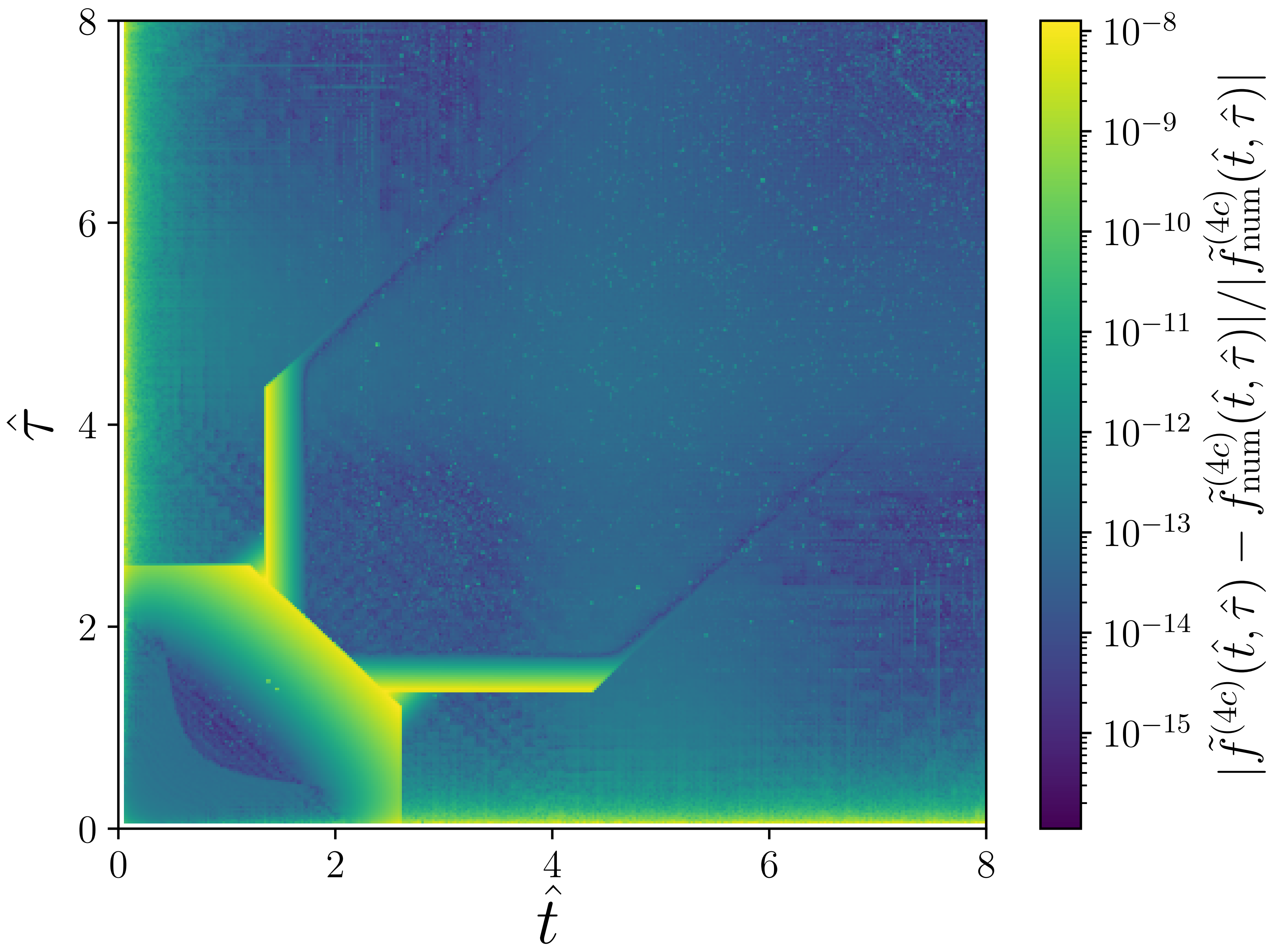




Accumulated error

TMR

$$a_\ell^{\mathrm{hvp},(i)} = \left(\frac{\alpha}{\pi}\right)^{2+N_i} \int_0^\infty dq^2\, f^{(i)}(\hat{q}^2)\, \left(\overline{\Pi}(q^2)\right)^{m_i}$$

vs

$$a_\ell^{\mathrm{hvp,(i)}} = \left(\frac{\alpha}{\pi}\right)^{2+N_i} \int dt_1 \ldots dt_{m_i}\, \tilde{f}^{(i)}(\hat{t}_1, \ldots, \hat{t}_{m_i})\, G(t_1) \times \ldots \times G(t_{m_i})$$

We observe a **relative error** $\lesssim 10^{-10}$ for all observables

On the time-momentum representation of the hadronic vacuum polarization contribution to $g-2$ at next-to-leading order. **Manuscript in preparation**, 2026

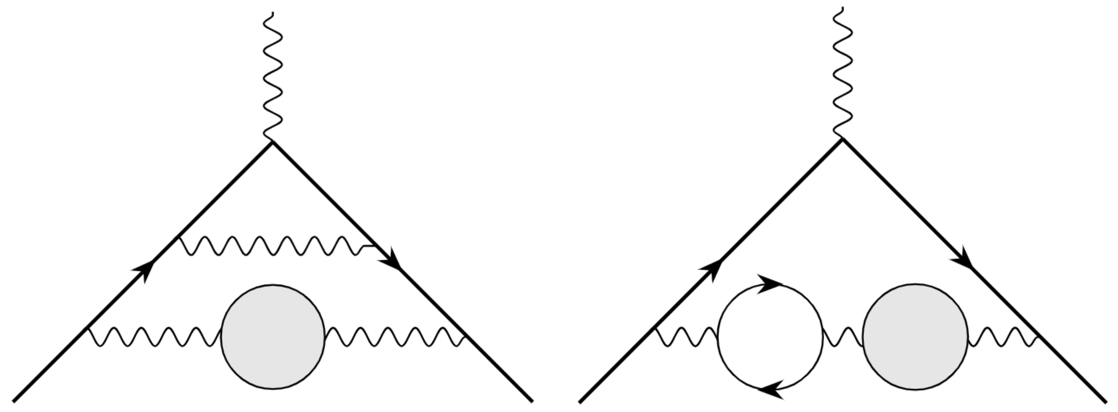
hvpKernels.jl

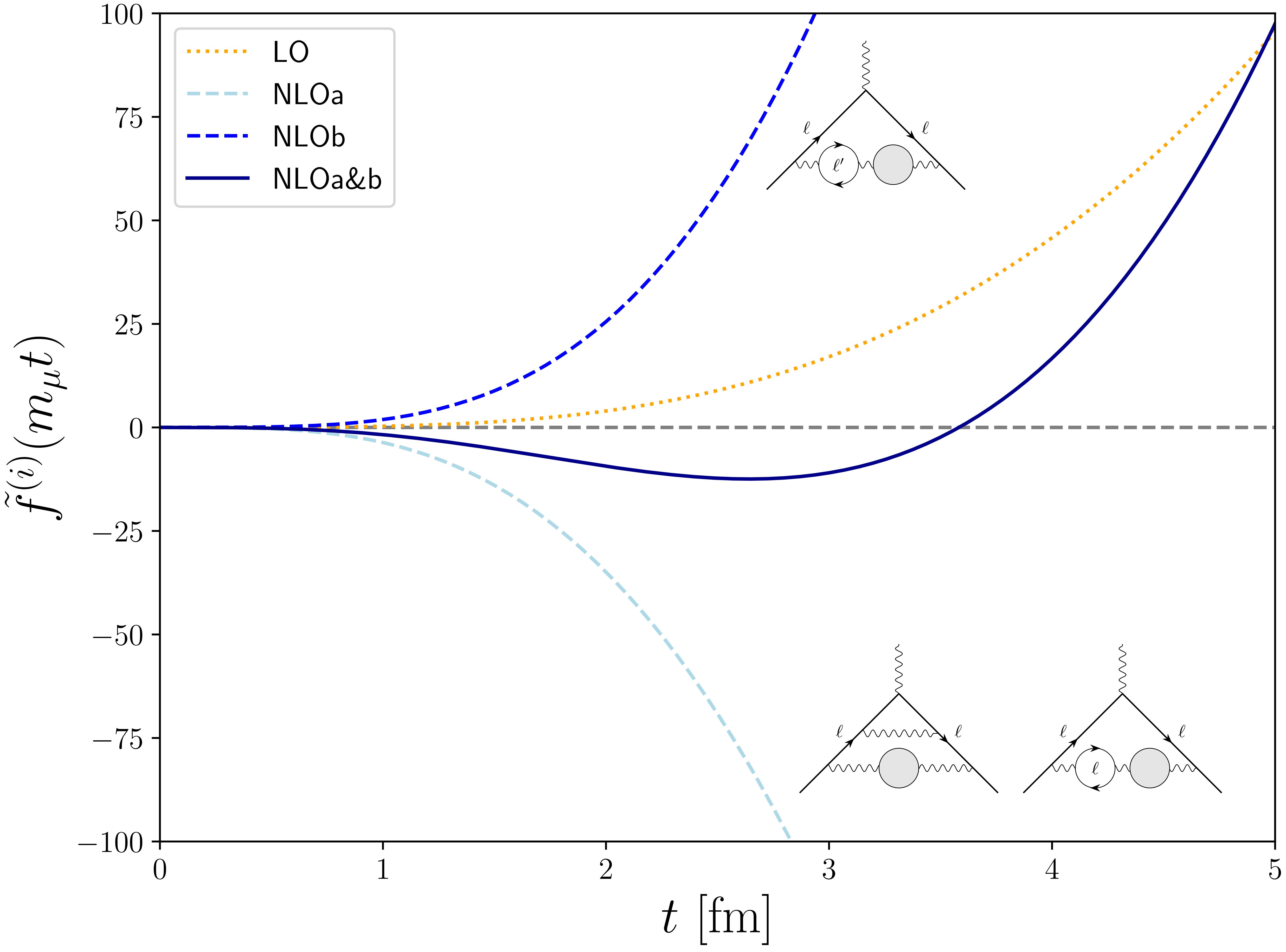
- Introduction and setup
- Time Momentum Representation
- Higher-order HVP
- The electron $g-2$

Long distance cancellation

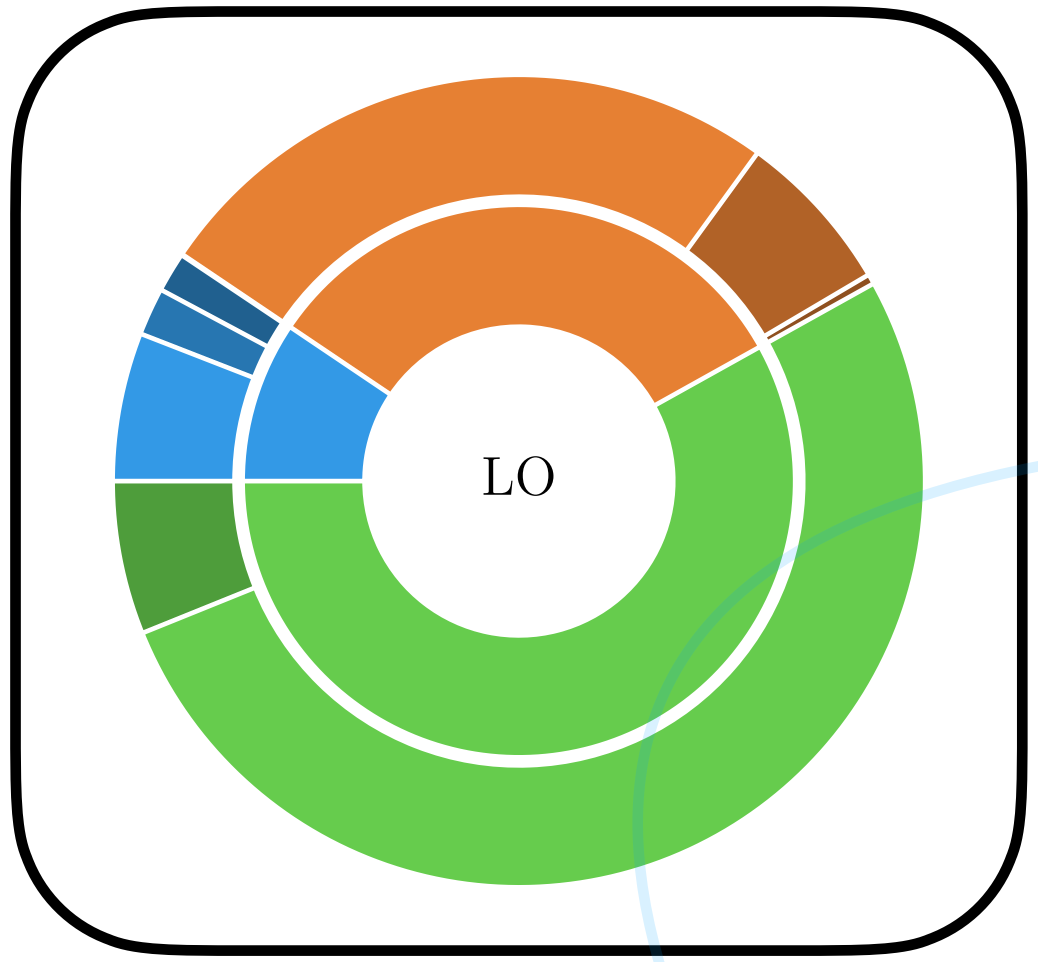
We set $m_\ell = m_\mu$

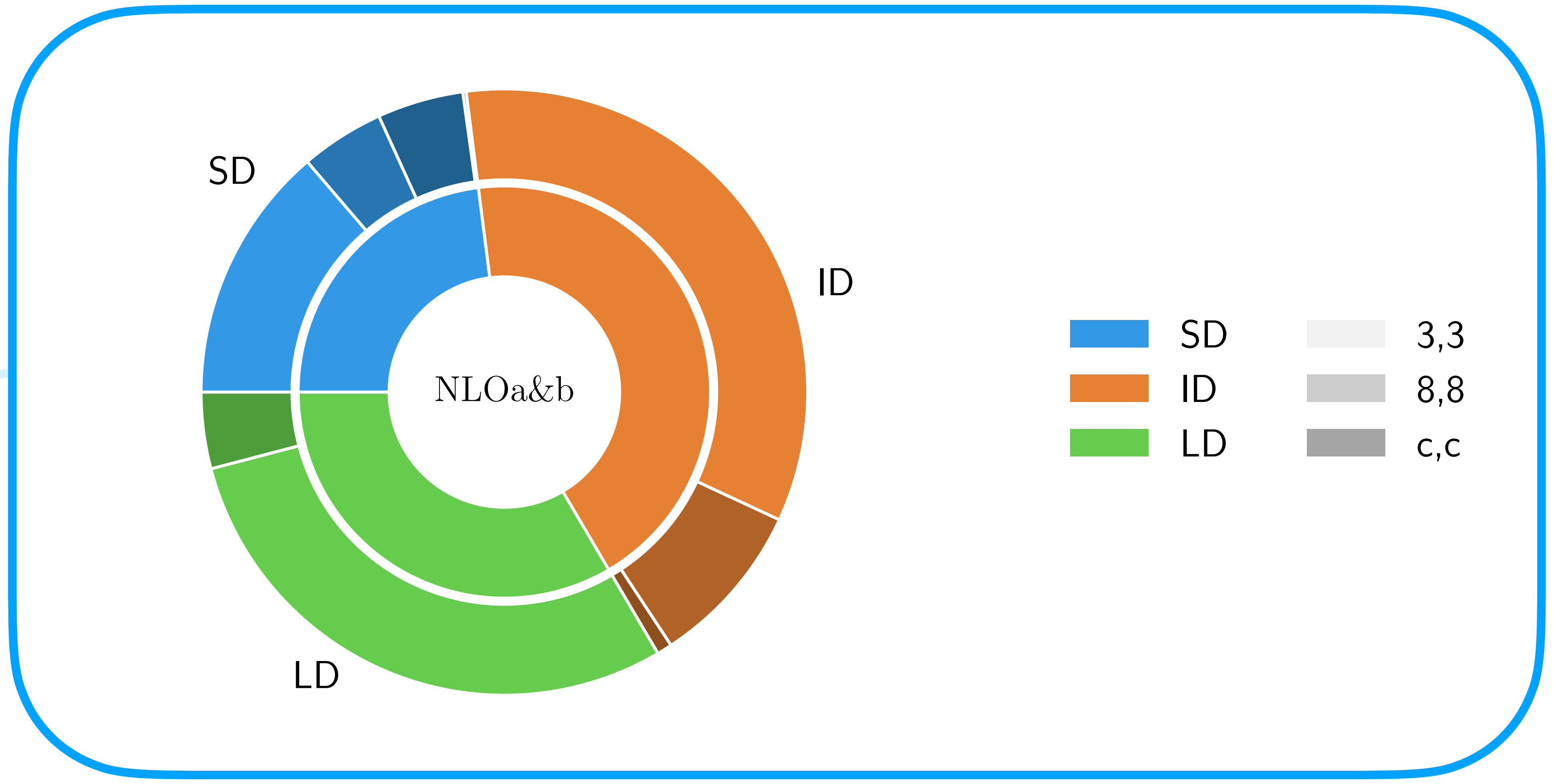
- NLOa and NLOb have opposite signs
- Their addition exactly cancels around $t \sim 3.6$ fm
- **NLOa&b $\equiv$ NLOa + NLOb** exploits this cancellation





Reduction of uncertainty

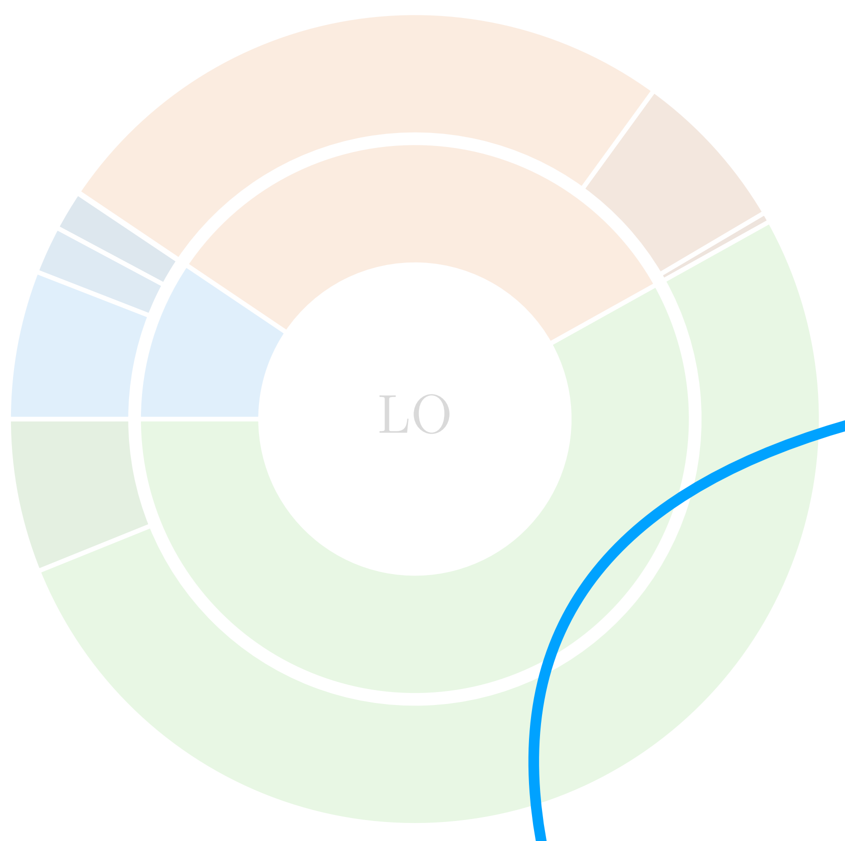


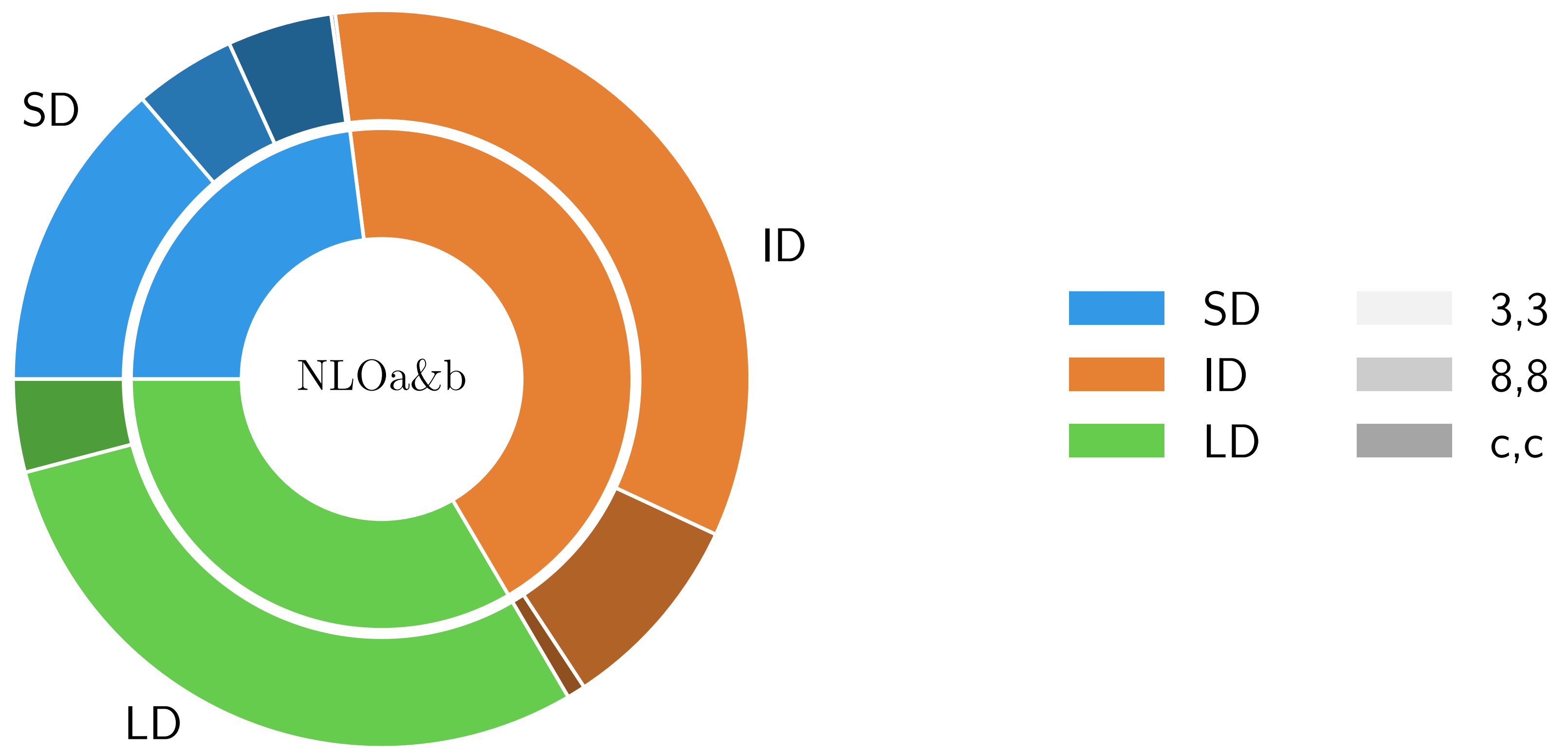


← Contribution to the Central Value

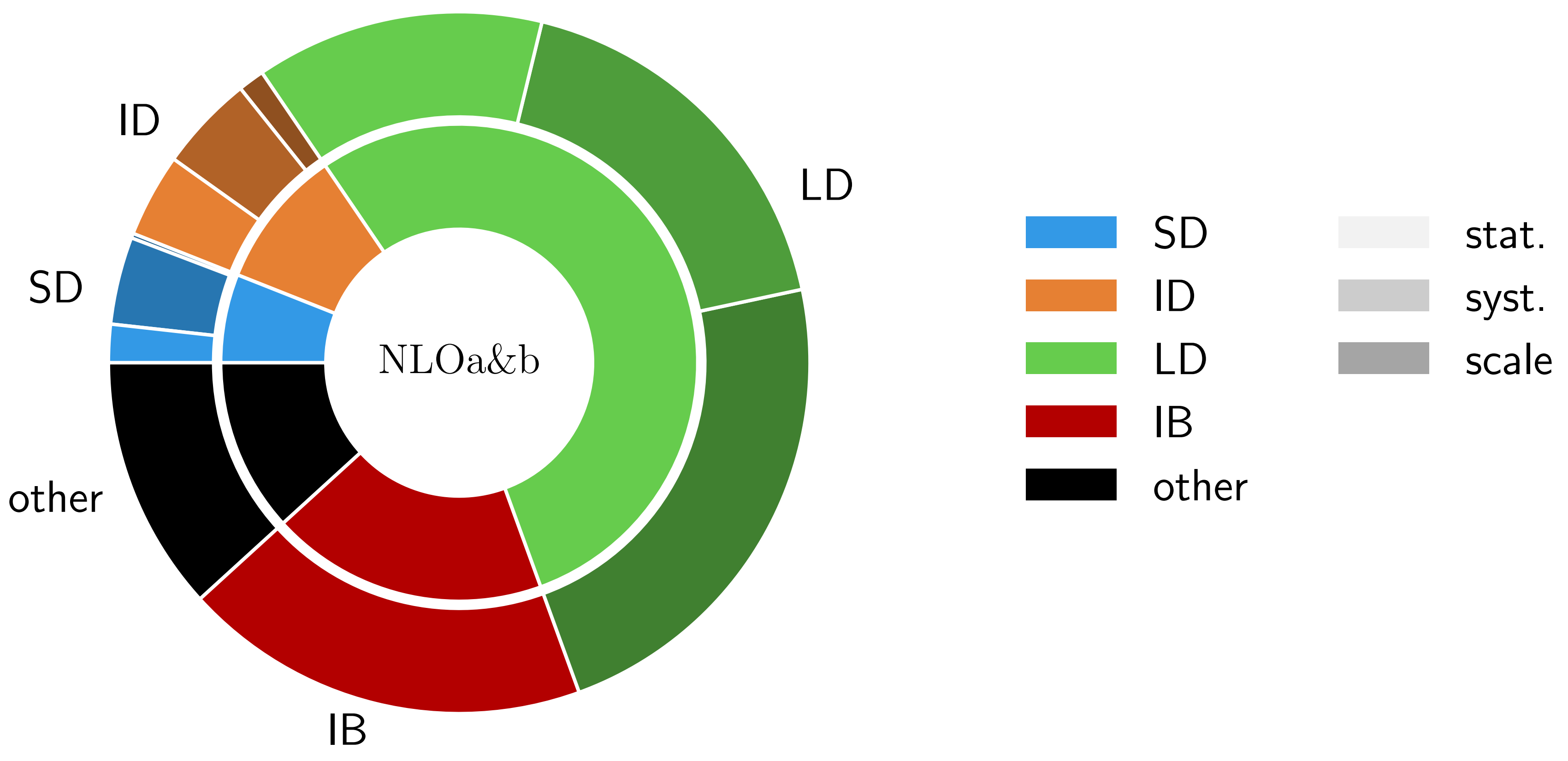
← Contribution to the Final Error

Reduction of uncertainty





← Contribution to the Central Value

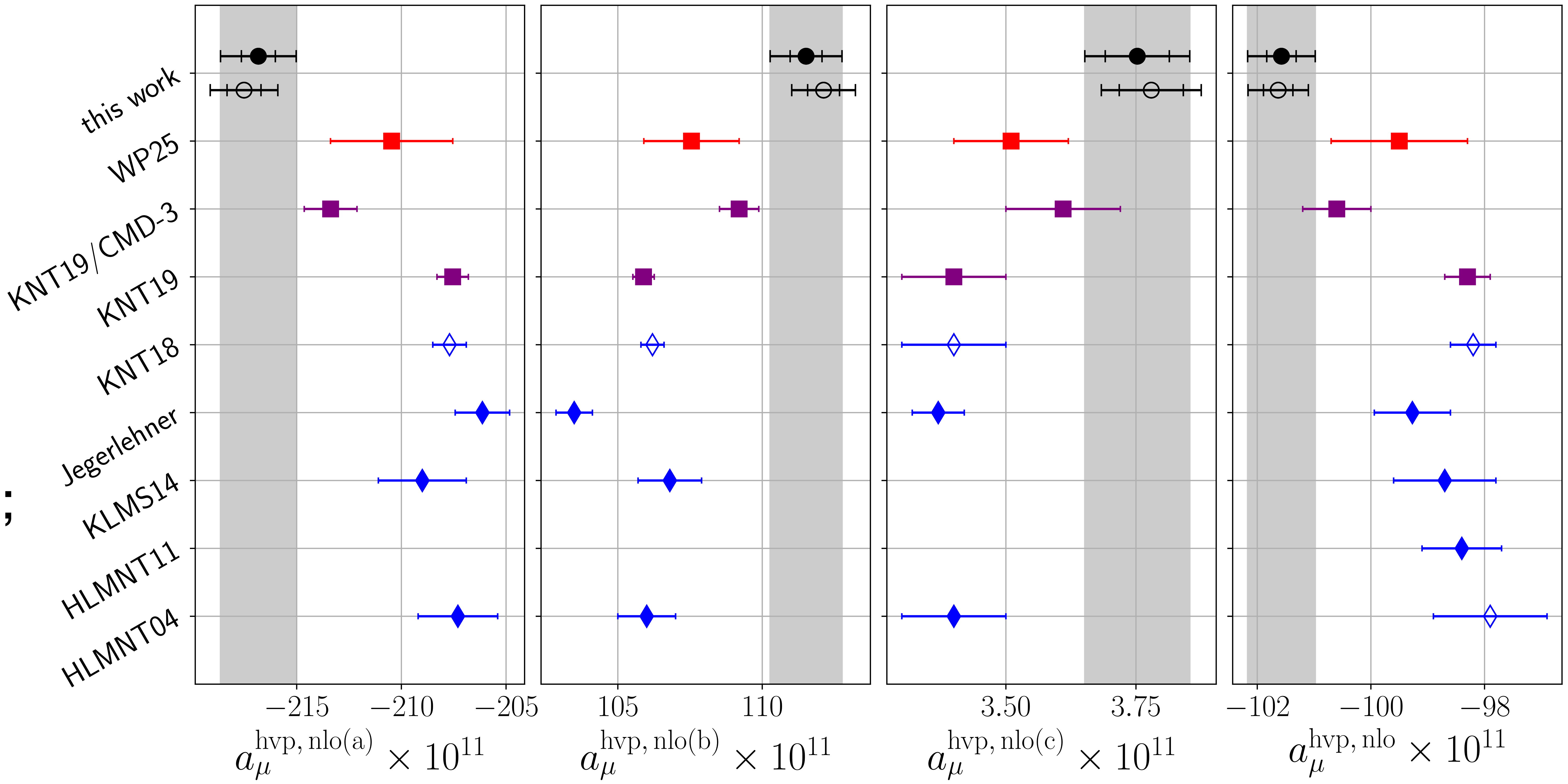


0.59 %

← Contribution to the Final Error

Reduction of uncertainty

- We have achieved a $0.6\,\%$ precision, improving on WP25
- The size of NLO HVP is similar to the HLbL but with the opposite sign
- Large cancellation in the Isospin Breaking corrections in NLOa+NLOb; *very reliable* QCD estimate

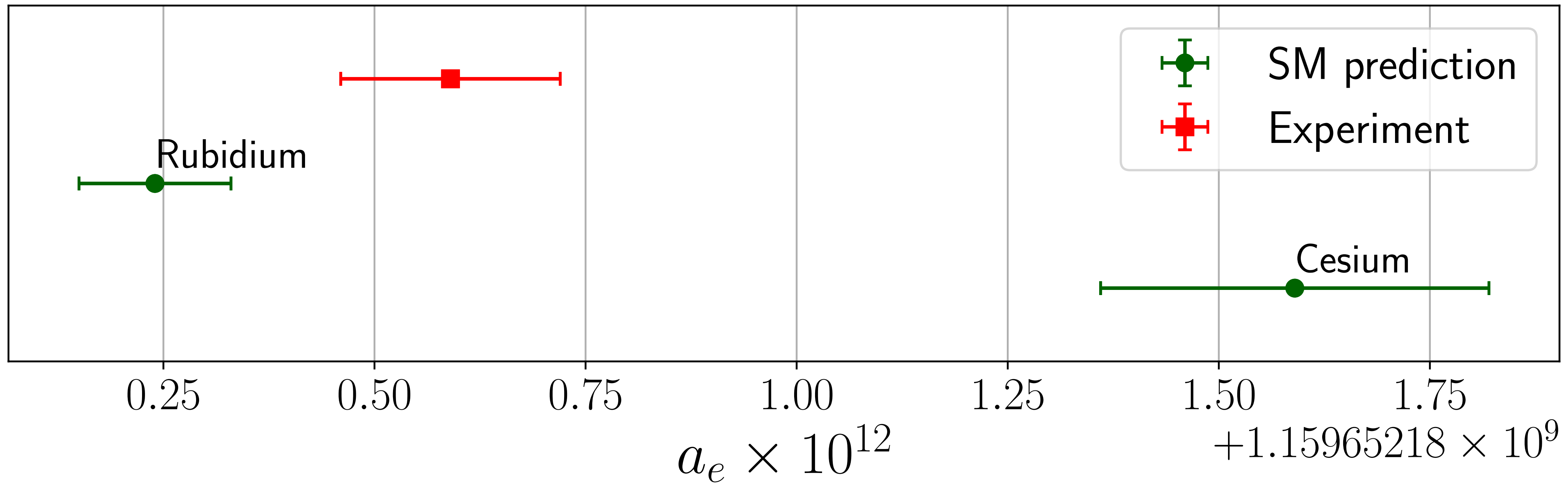


Higher-order hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD. 2026. A. Beltran et al.

$$a_\mu^{\mathrm{hvp,\,nlo}} = -\,101.57(26)_{\mathrm{stat}}(54)_{\mathrm{syst}}[60] \times 10^{-11}$$

- Introduction and setup
- Time Momentum Representation
- Higher-order HVP
- The electron $g-2$

- Current uncertainty on a_e is dominated by the **Cs/Rb discrepancy** in determinations of α (*atom interferometry*)



- Current error of a_e^{exp} already **sensitive to the size** of $a_e^{\mathrm{hvp,\,lo}}$ and with prospects of improving
- a_e^{HVP} suffers of the *usual* **discrepancies in the data-driven approach**
- Once α is resolved, **an unambiguous determination** of $a_e^{\mathrm{hvp,\,lo}}$ is as pressing as a complete 6^{th}-order QED computation

[H. Wittig 2025]

Relative contribution to $a_e \times 10^{12}$ of different sectors of the SM

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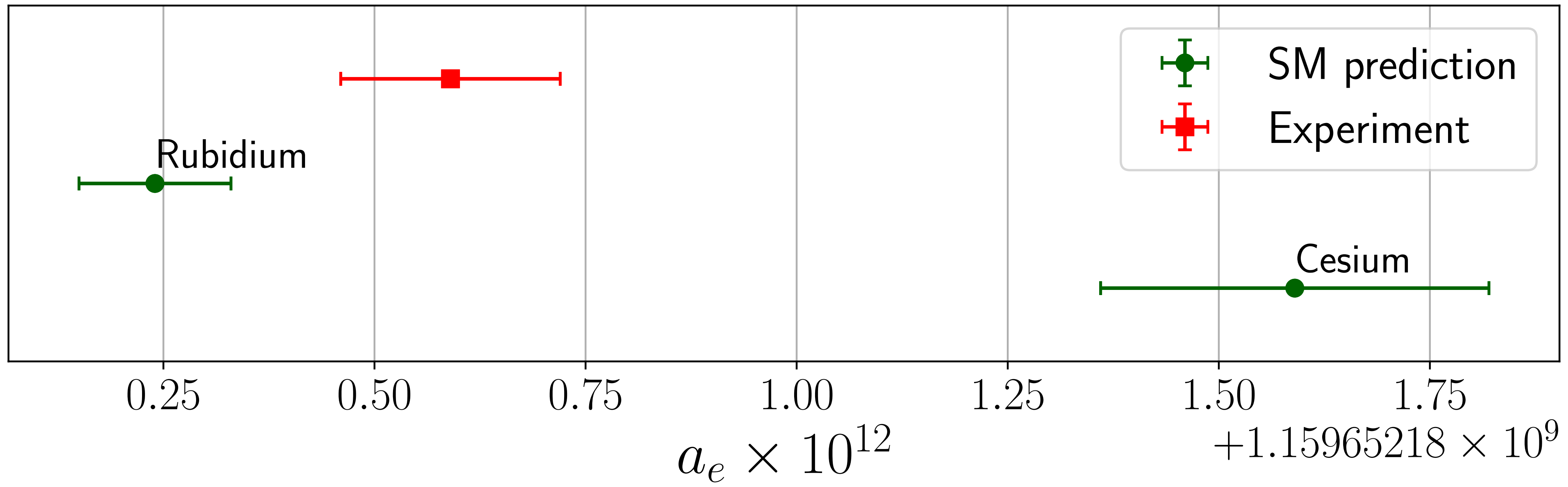

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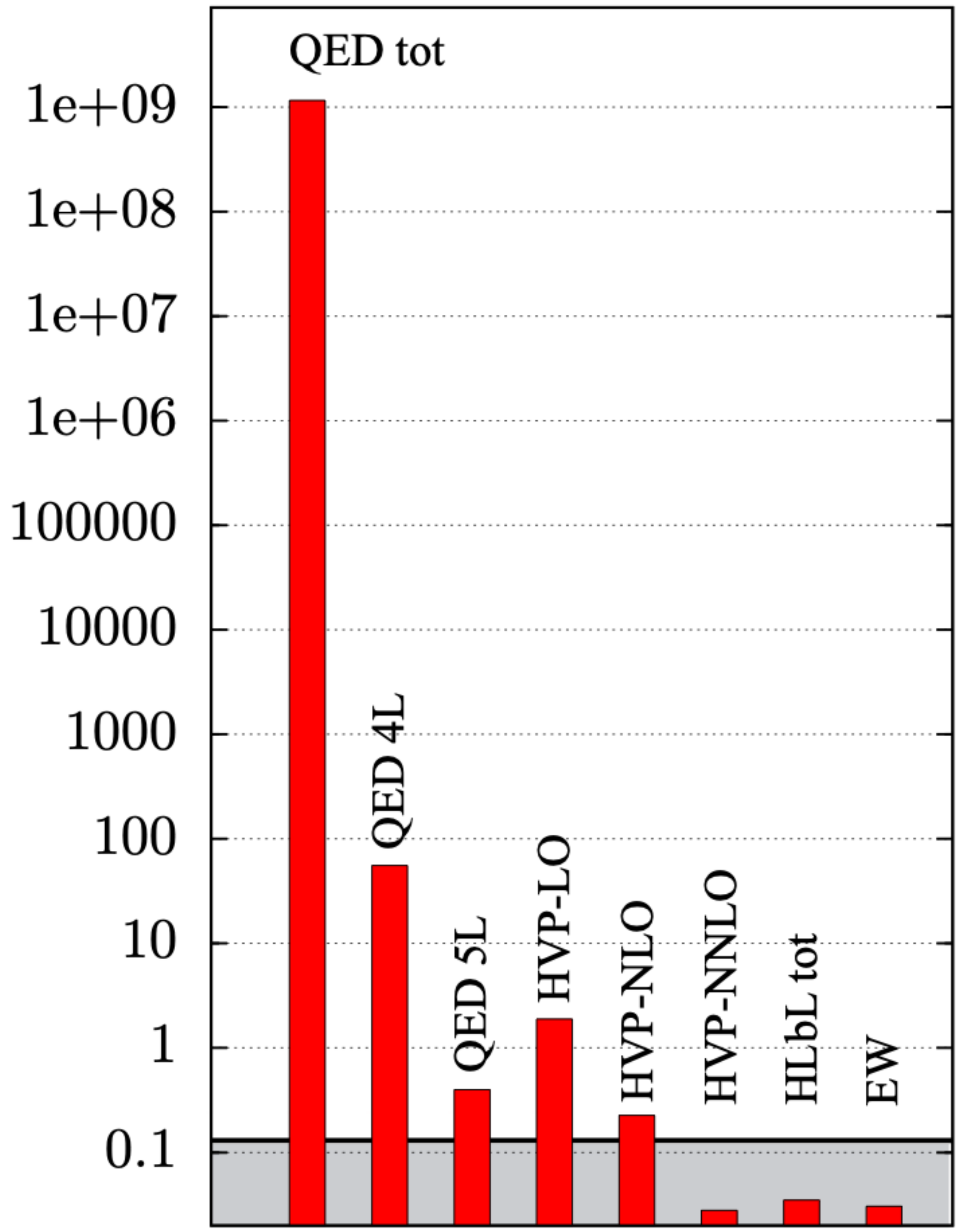
[H. Wittig 2025]



Relative contribution to $a_e \times 10^{12}$ of different sectors of the SM

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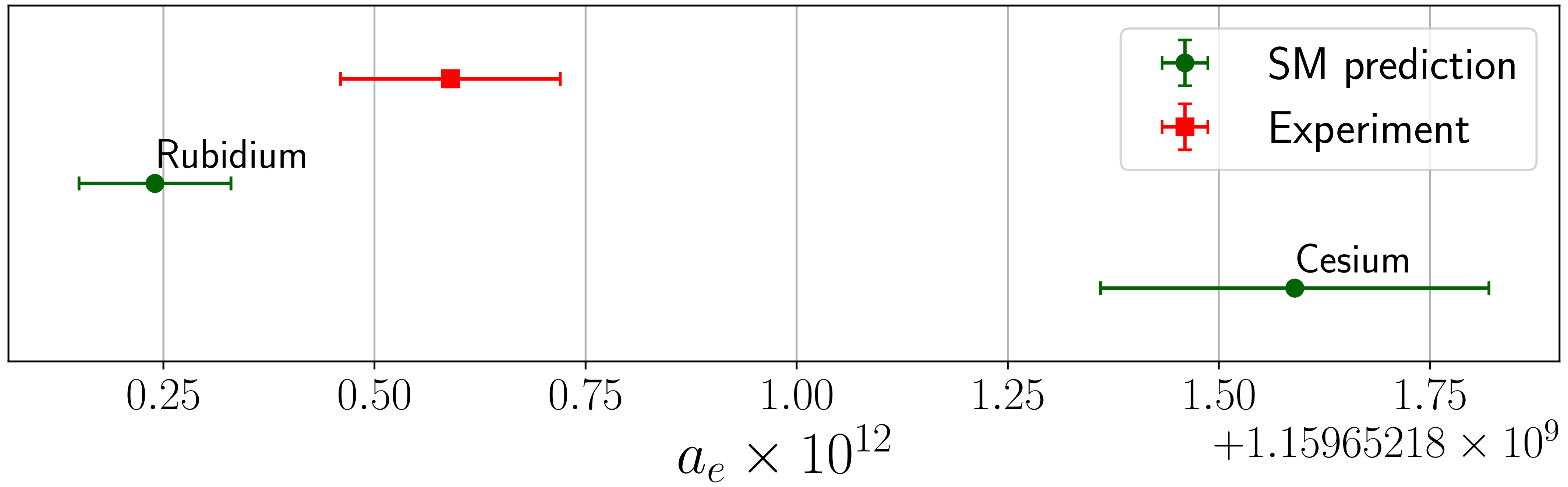

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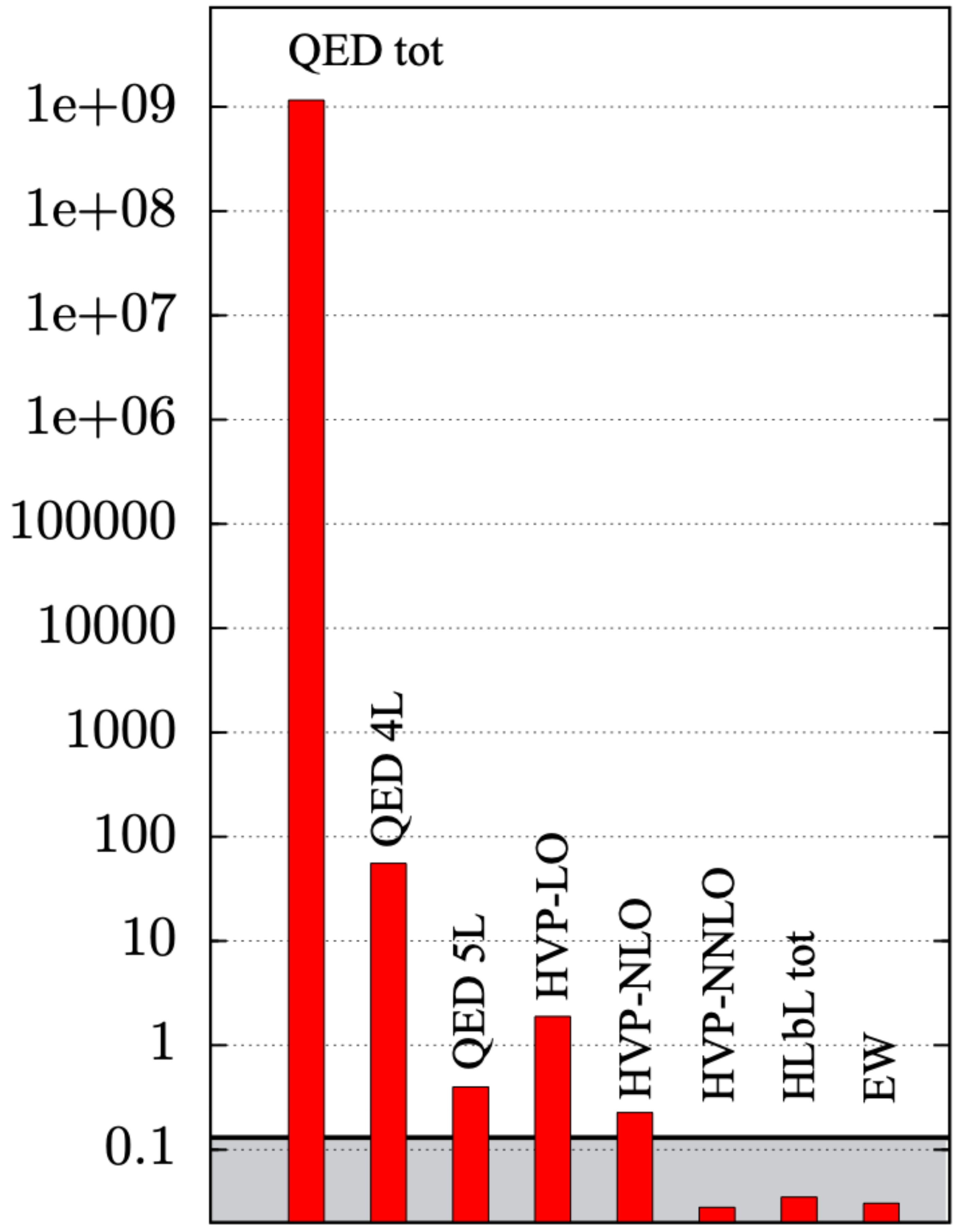
[H. Wittig 2025]



Relative contribution to $a_e \times 10^{12}$ of different sectors of the SM

Rescaling from $a_\mu^{\text{hvp, lo}}$

Electron $g-2$

$$a_{e-\mu} \equiv a_e - \left(\frac{m_e}{m_\mu}\right)^2 a_\mu$$

[S. Li et al. 2026]

- At the kernel level, there is an **exact cancellation** of the $\mathrm{O}(t^4)$ terms

$$\tilde{f}^{(2)}(\hat{t}) \overset{\hat{t}\ll 1}{\simeq} \frac{\pi^2 m_\ell^2}{9}\, t^4$$

$$a_e^{\text{hvp, lo}} - \left(\frac{m_e}{m_\mu}\right)^2 a_\mu^{\text{hvp, lo}} \longleftrightarrow \tilde{f}^{(2)}(m_e t) - \left(\frac{m_e}{m_\mu}\right)^2 \tilde{f}^{(2)}(m_\mu t) \overset{t\ll 1/m_\mu \ll 1/m_e}{=} \left(\mathrm{O}\left(m_\mu\right)^4 + \mathrm{O}\left(m_e\right)^4\right)\, t^6$$

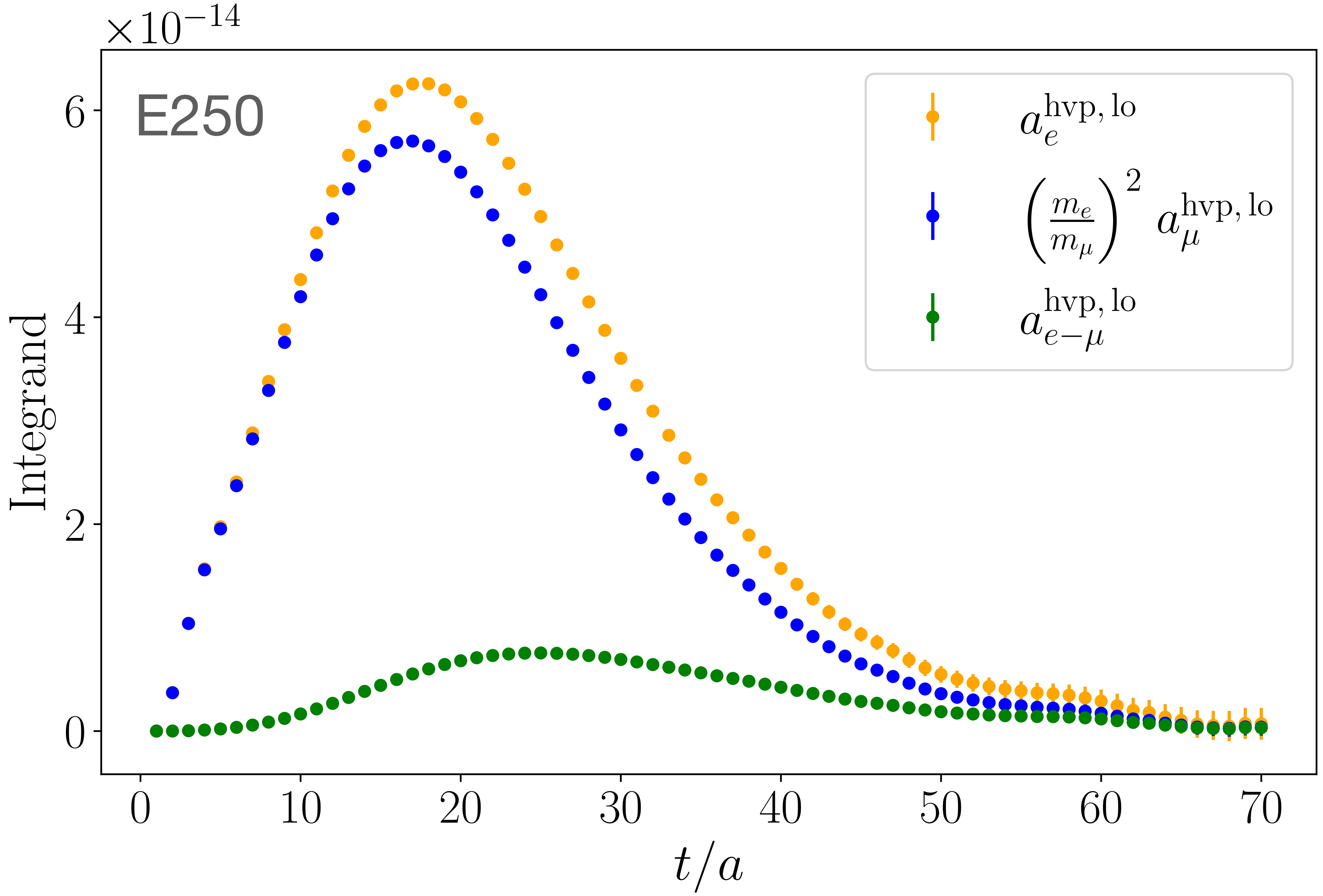
- $a_e^{\text{hvp, lo}}$ is computed as a **shift** from the rescaled $a_\mu^{\text{hvp, lo}}$

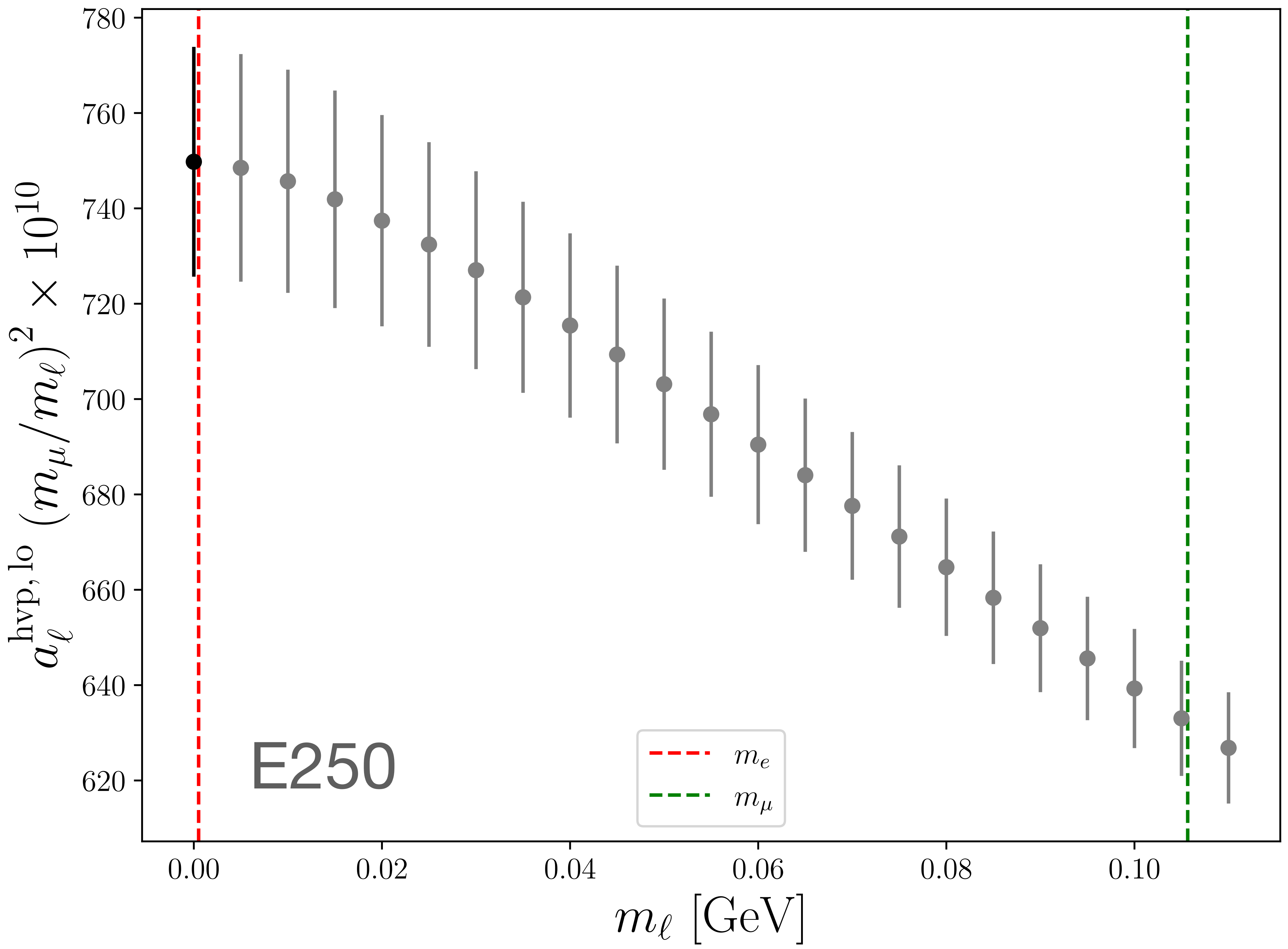
$$a_e^{\text{hvp, lo}} = \left(\frac{m_e}{m_\mu}\right)^2 a_\mu^{\text{hvp, lo}} + a_{e-\mu}^{\text{hvp, lo}}\,, \qquad \frac{a_{e-\mu}^{\text{hvp, lo}}}{a_e^{\text{hvp, lo}}} \sim 15\ \%$$

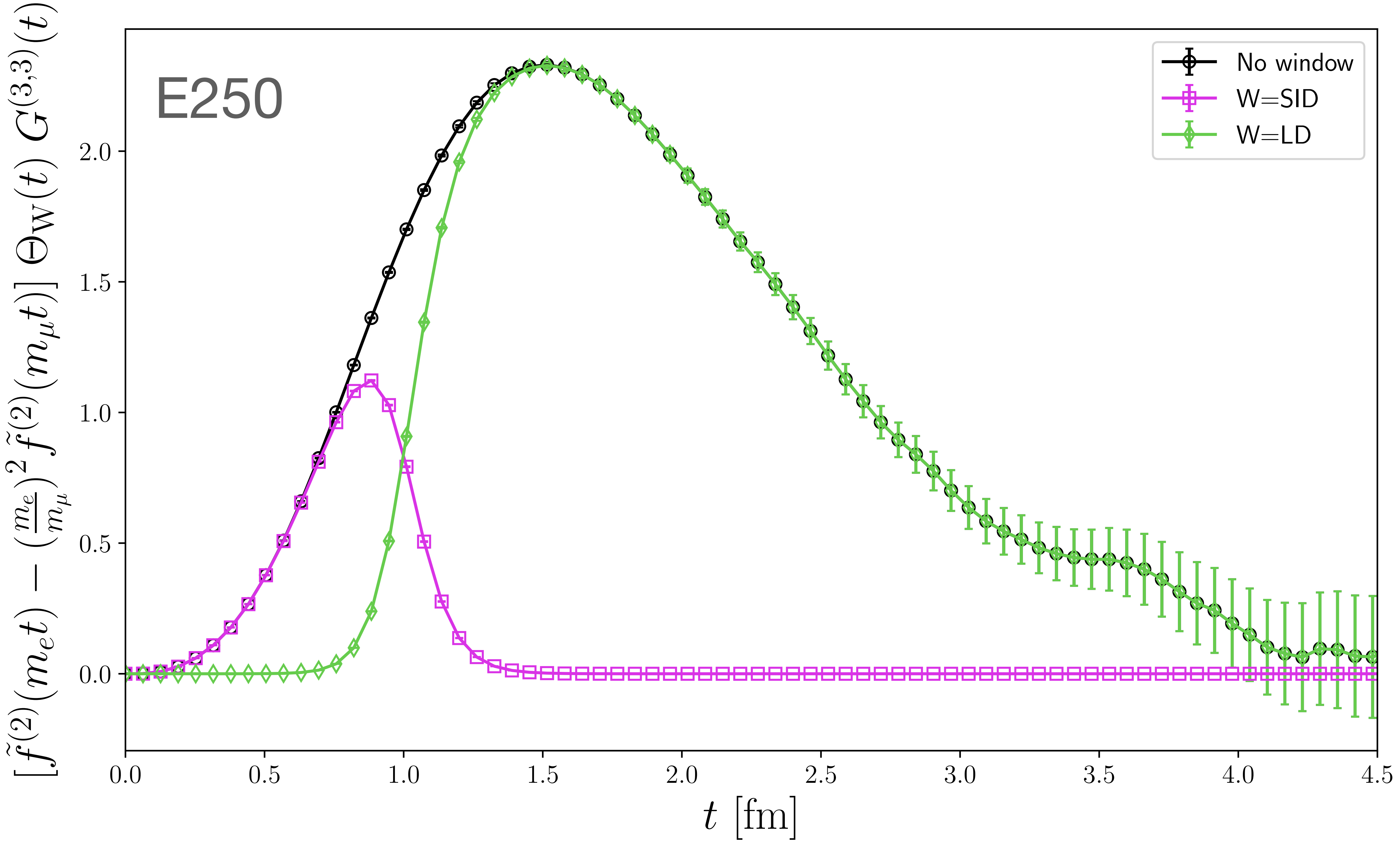
Rescaling from $a_\mu^{\text{hvp, lo}}$

Electron $g-2$

$$a_{e-\mu} \equiv a_e - \left(\frac{m_e}{m_\mu}\right)^2 a_\mu$$



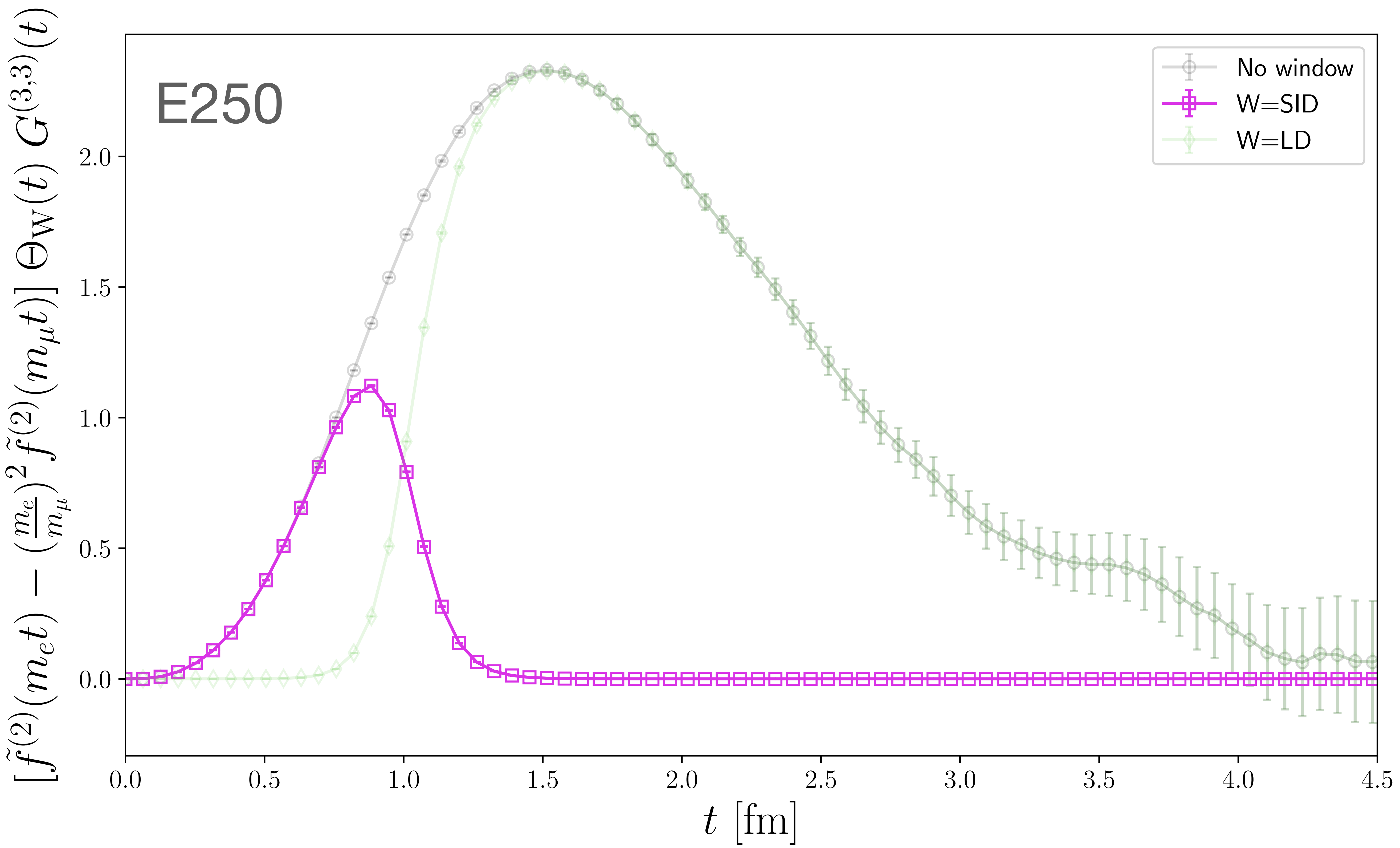


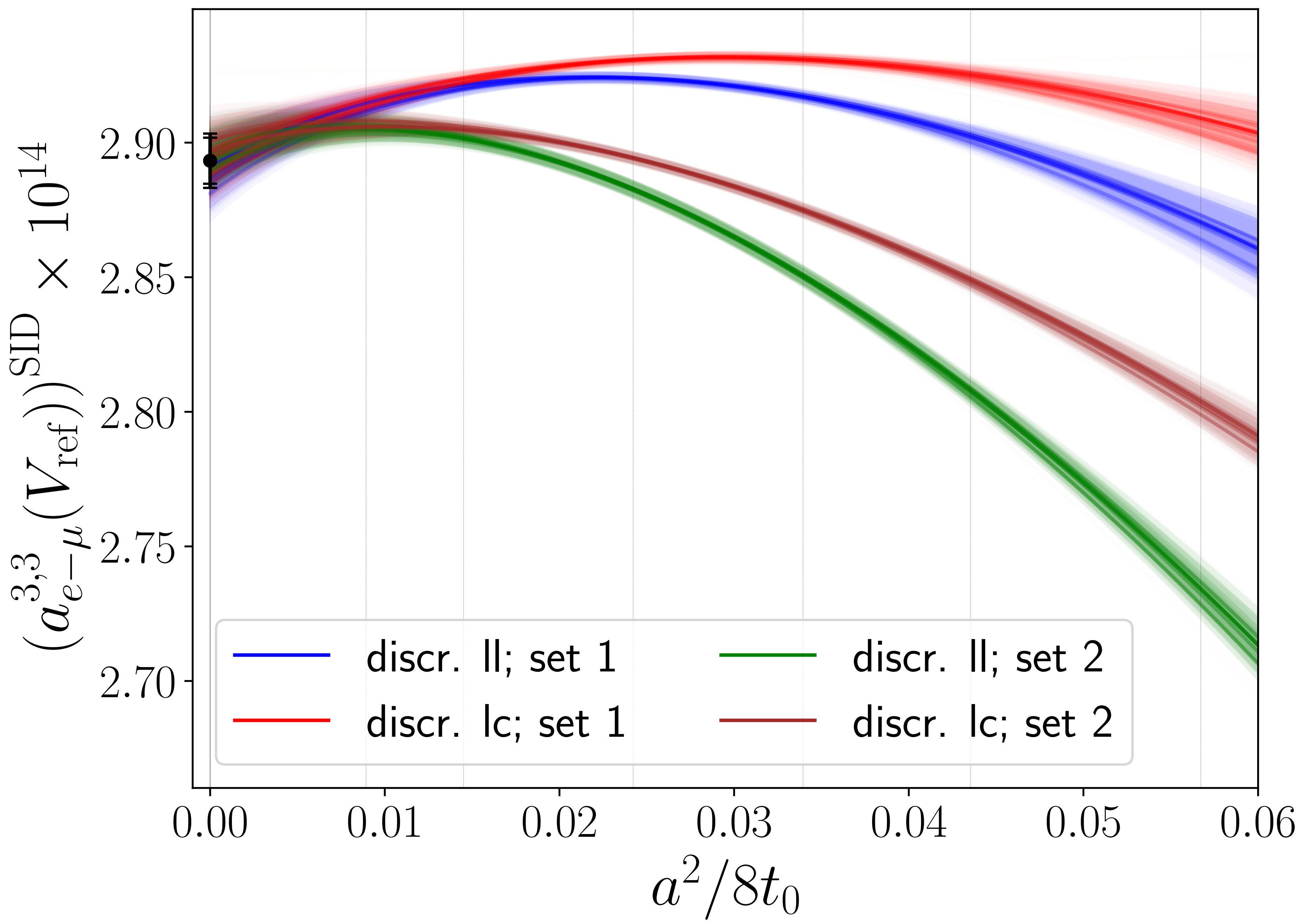


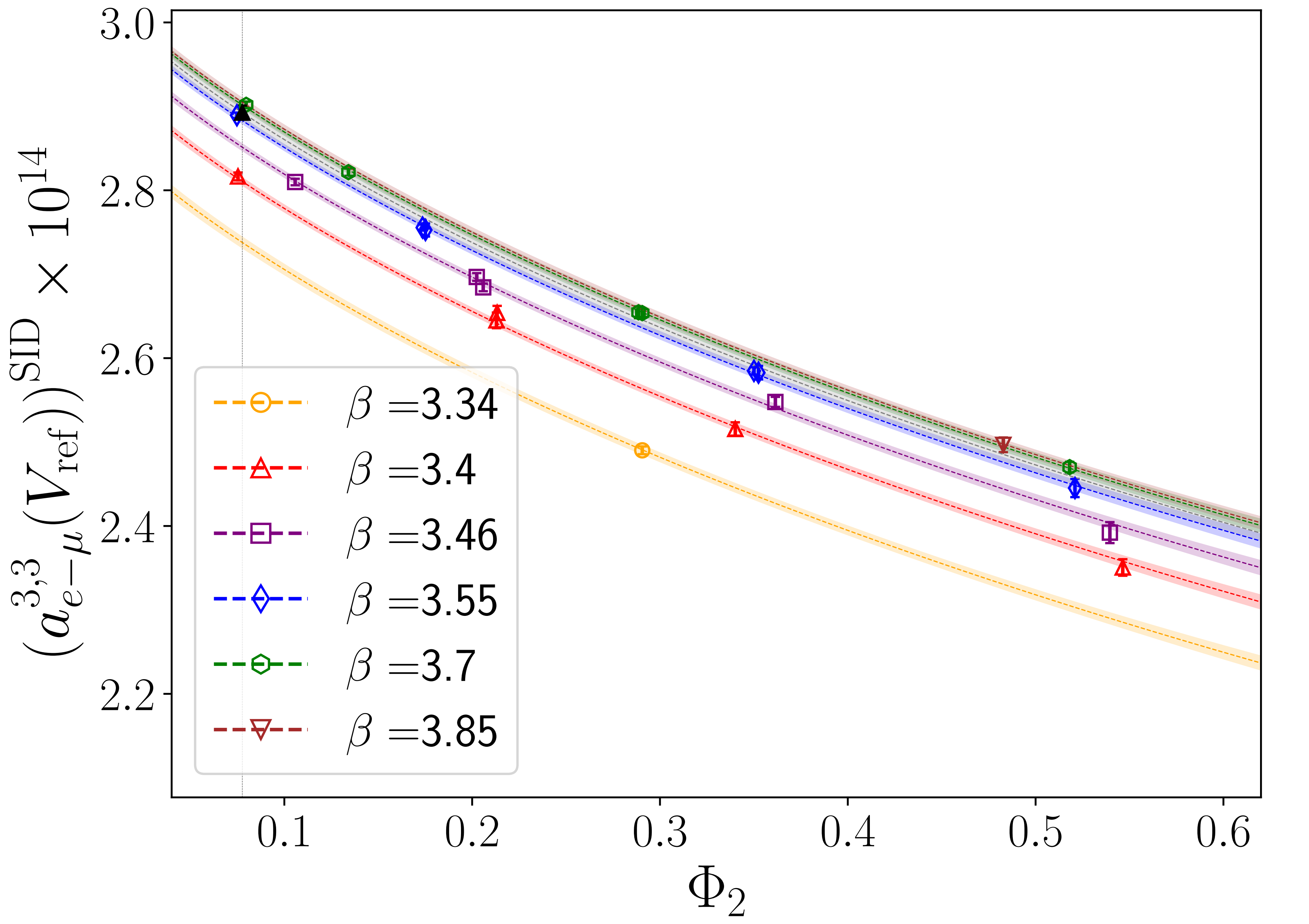
E250
No window
W=SID
W=LD
$[\tilde{f}^{(2)}(m_e t) - (\frac{m_e}{m_\mu})^2 \tilde{f}^{(2)}(m_\mu t)]\ \Theta_{\mathrm{W}}(t)\ G^{(3,3)}(t)$
t [fm]
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5

SID of $a_{e-\mu}^{\mathrm{hvp,\,lo}}$

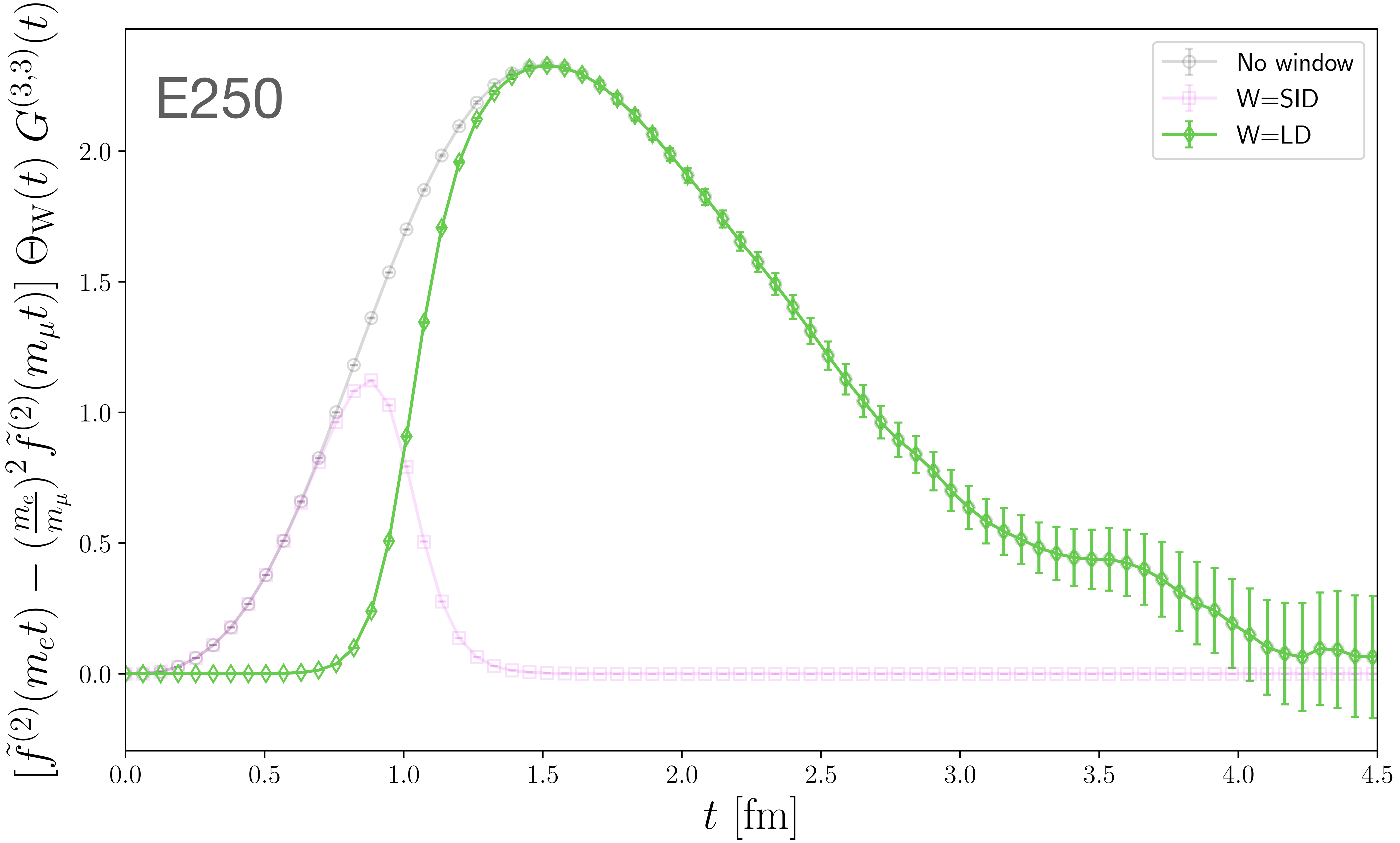
$g_e - 2$

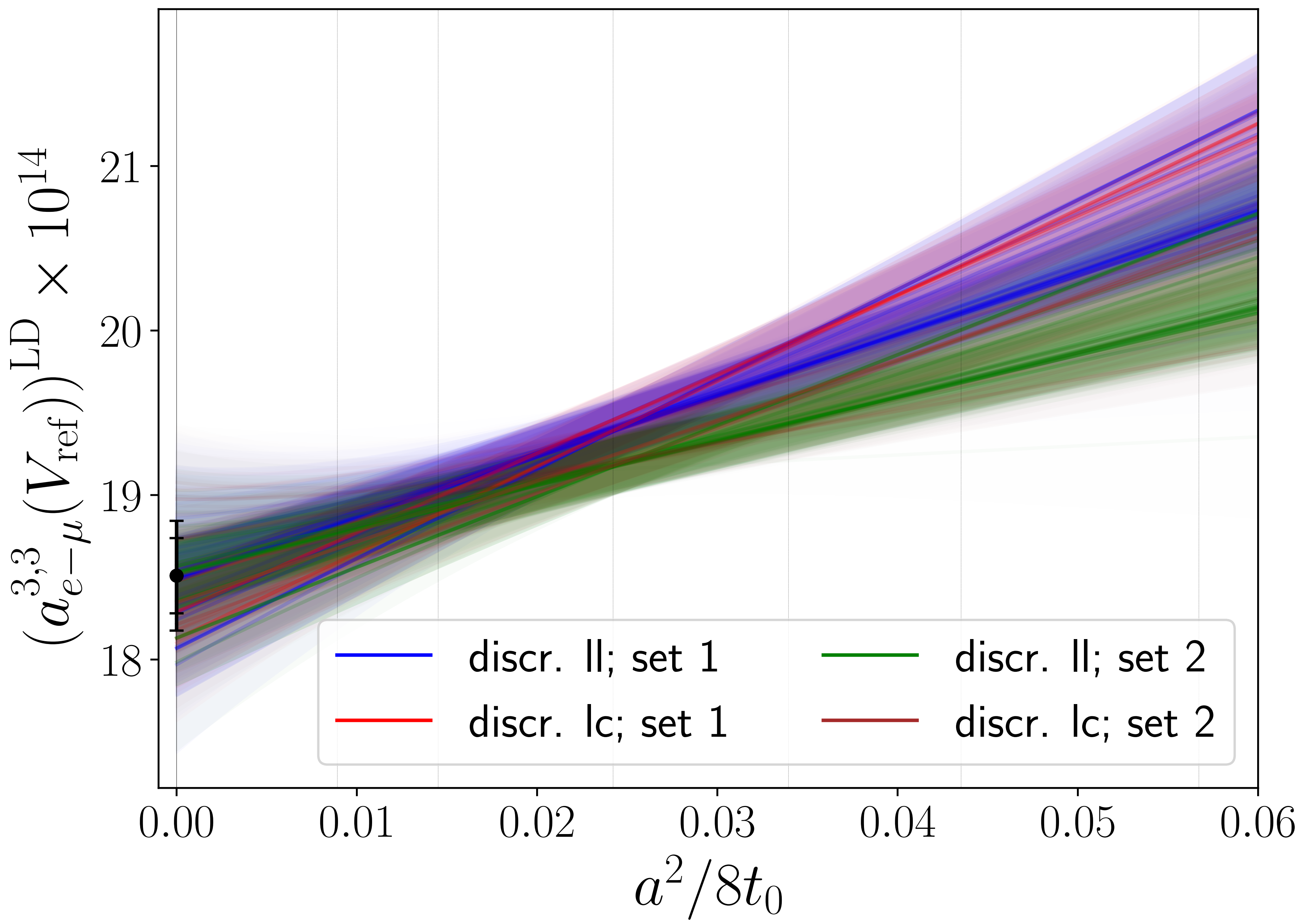


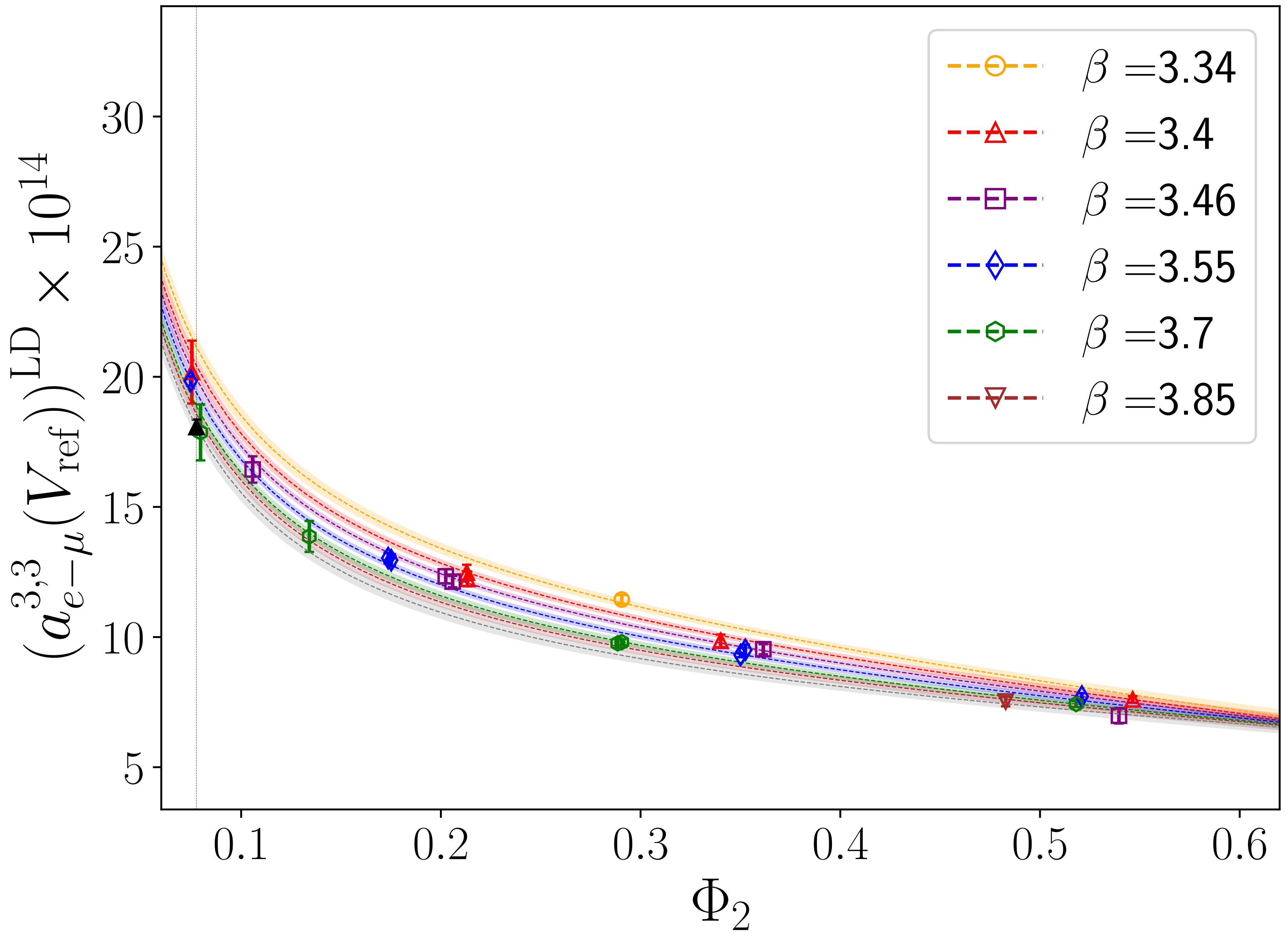




LD of $a_{e-\mu}^{\text{hvp, lo}}$ $g_e - 2$



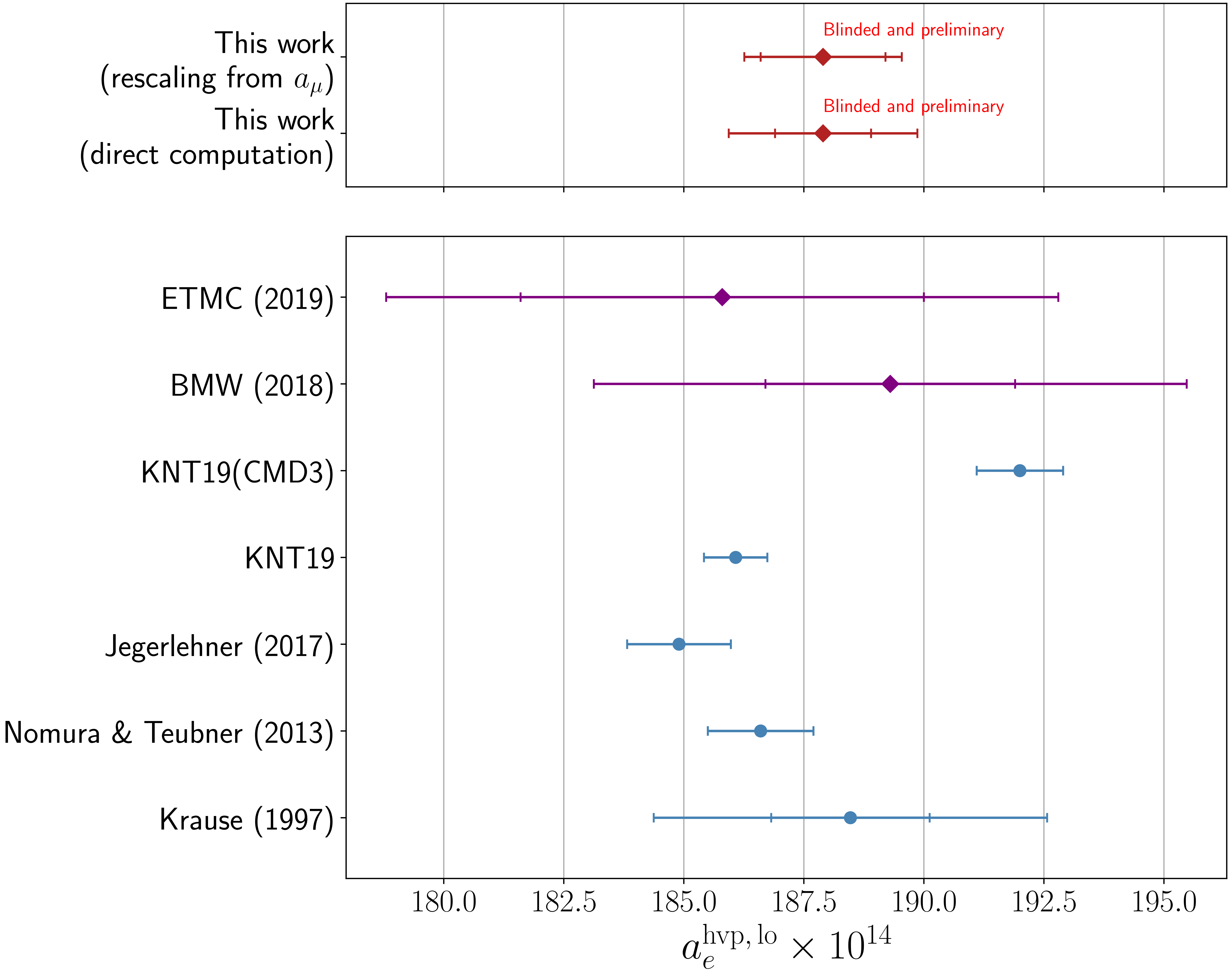




Comparison of results (isoQCD)

Electron $g-2$

- Calculation through $a_{e-\mu}^{\mathrm{HVP}}$ at $\sim 1\,\%$ precision
- The direct computation presents larger errors - dominated by scale setting (t_0^{ph})
- IB corrections are important for a_e^{hvp} and $a_{e-\mu}^{\mathrm{hvp}}$, still not included in this analysis



Summary

- We have total **analytical control on the time-kernels at NLO** for all Euclidean times
  - Manuscript in preparation
  - Working package already developed “hvpKernels.jl”
- Estimate for the **NLO HVP to the muon** $g-2$ is now complete and accepted for publication
- Progress on the determination of the **LO HVP to the electron** $g-2$
  - Rescaling from a_μ working at $\sim 1\,\%$ relative precision
  - Still room for improvement: **scale-setting dominates the error**

Related talks of the Mainz group:

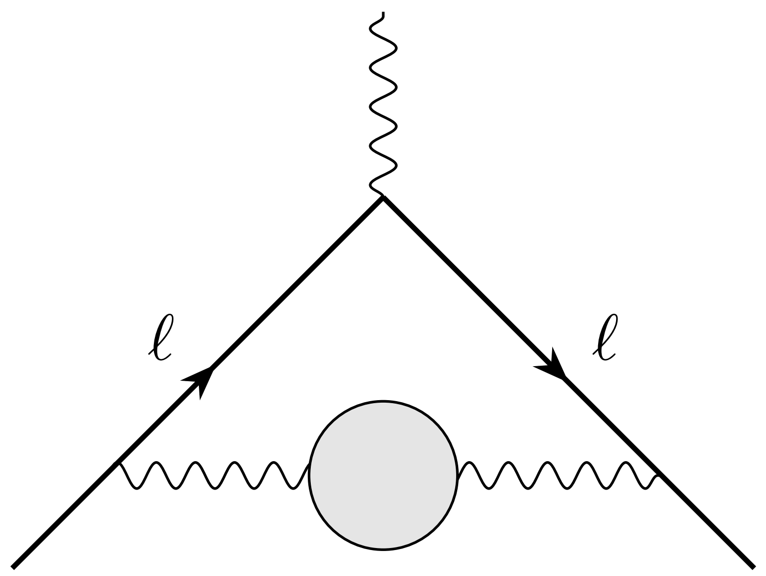
- Tue. 16:50 - H. B. Meyer
- Wed. 09:00 - A. Conigli
- Wed. 11:10 - D. Erb
- Wed. 11:30 - S. Lahrtz
- Wed. 11:50 - D. Albandea
- Wed. 09:00 - A. De Santis

Backup slides

- **Time Momentum Representation**
- Higher-order HVP
- The electron $g-2$

Time-kernel for the LO

TMR



$$\frac{m_\ell^2}{8\pi^2}\tilde{f}^{(2)}(\hat{t}) = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}^{(2)}(\hat{\omega}^2) \left[\hat{\omega}^2\hat{t}^2 - 4\sin^2\frac{\hat{\omega}\hat{t}}{2}\right]$$

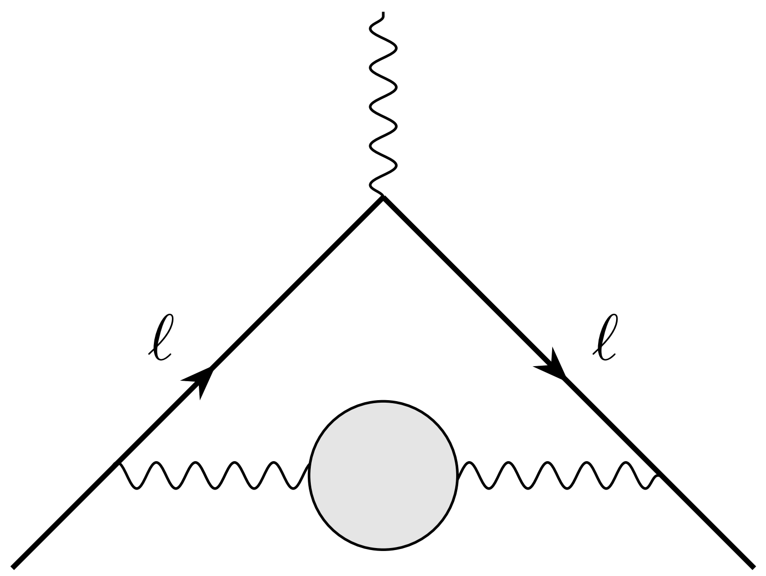
[M. Della Morte et al. 2017]

$$\frac{m_\ell^2}{16\pi^2}\tilde{f}^{(2)}(\hat{t}) = -\frac{1}{4} + \gamma_{\mathrm{E}} + \frac{1}{2\hat{t}^2} + \hat{t}^2 - \frac{1}{\hat{t}}K_1(2\hat{t}) + \ln(\hat{t}) + \frac{1}{8}G^{2,1}_{1,3}\left(\hat{t}^2 \,\middle|\, \begin{matrix} \frac{3}{2} \\ 0,1,\frac{1}{2} \end{matrix}\right)$$

$$\frac{m_\ell^2}{16\pi^2}\tilde{f}^{(2)}(\hat{t}) \overset{\hat{t}\ll 1}{=} \sum_{\substack{n\geq 4\\ n\ \mathrm{even}}} \frac{\hat{t}^n}{n!}\left(A_n + B_n\left(\gamma_E + \ln\hat{t}\right)\right)$$

$$\frac{m_\ell^2}{16\pi^2}\tilde{f}^{(2)}(\hat{t}) \overset{\hat{t}\gg 1}{=} \frac{1}{8}\hat{t}^2 - \frac{\pi}{4}\hat{t} + \gamma_E + \ln\hat{t} - \frac{1}{4} + \frac{1}{2\hat{t}^2} - e^{-2\hat{t}}\sqrt{\frac{\pi}{\hat{t}}}\left(\frac{1}{8} + O\left(\hat{t}^{-1}\right)\right)$$

Time-kernel for the LO



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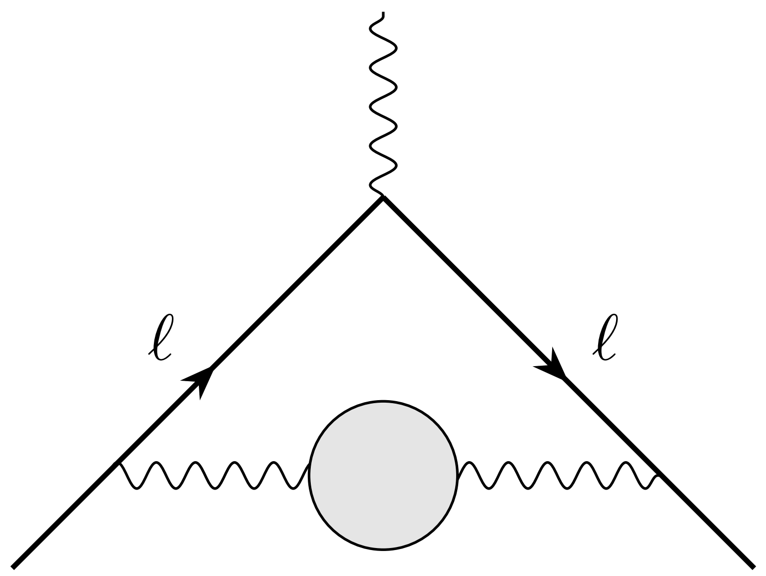
Time-kernel for the LO



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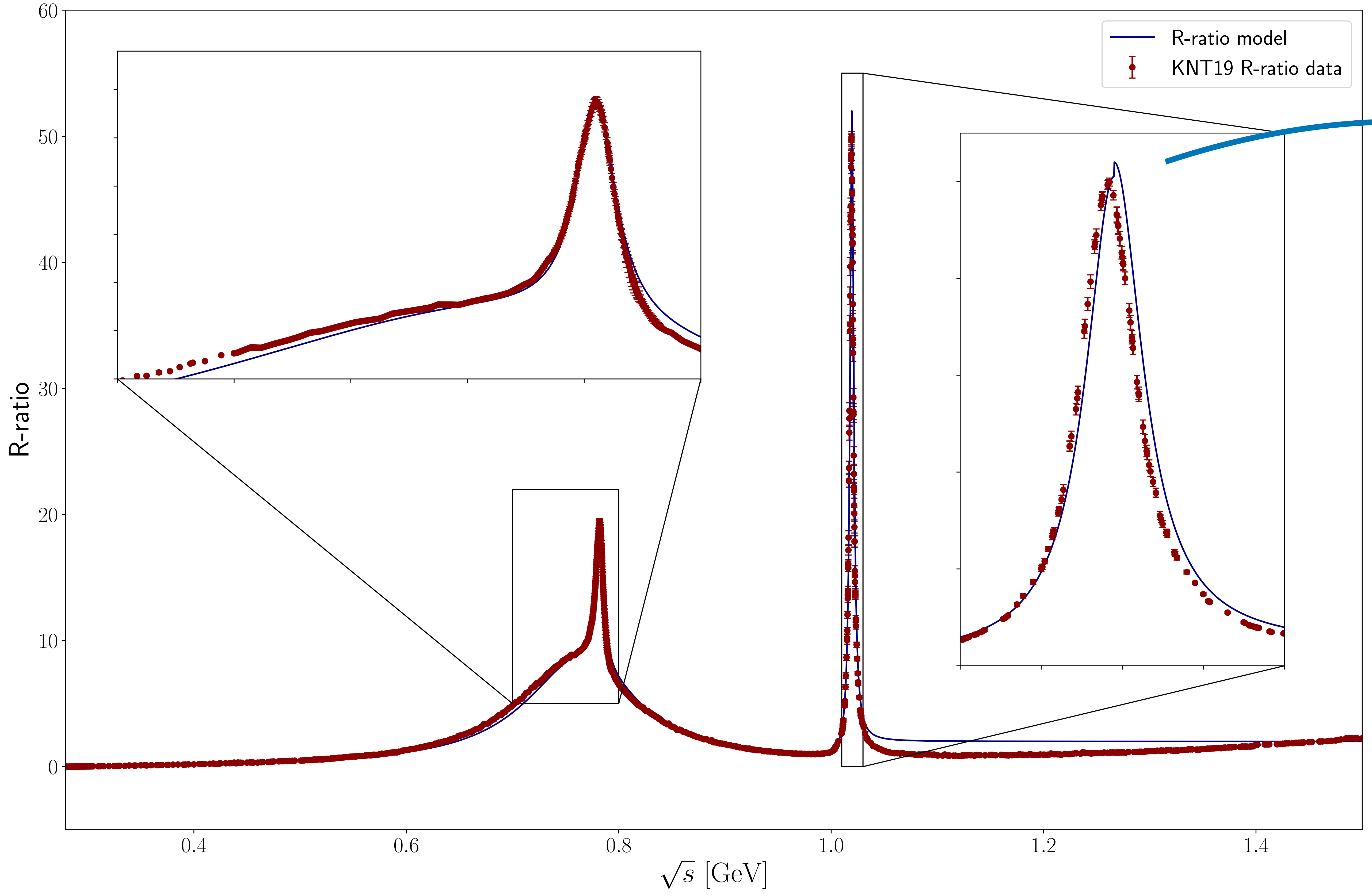
Accumulated error

TMR

$$a_\ell^{\mathrm{hvp},(i)} = \left(\frac{\alpha}{\pi}\right)^{2+N_i} \int_0^\infty dq^2\, f^{(i)}(\hat{q}^2)\, \left(\overline{\Pi}(q^2)\right)^{m_i}$$

vs

$$a_\ell^{\mathrm{hvp},(i)} = \left(\frac{\alpha}{\pi}\right)^{2+N_i} \int dt_1 \ldots dt_{m_i}\, \tilde{f}^{(i)}(\hat{t}_1, \ldots, \hat{t}_{m_i})\, G(t_1) \times \ldots \times G(t_{m_i})$$



We relate the R-ratio to

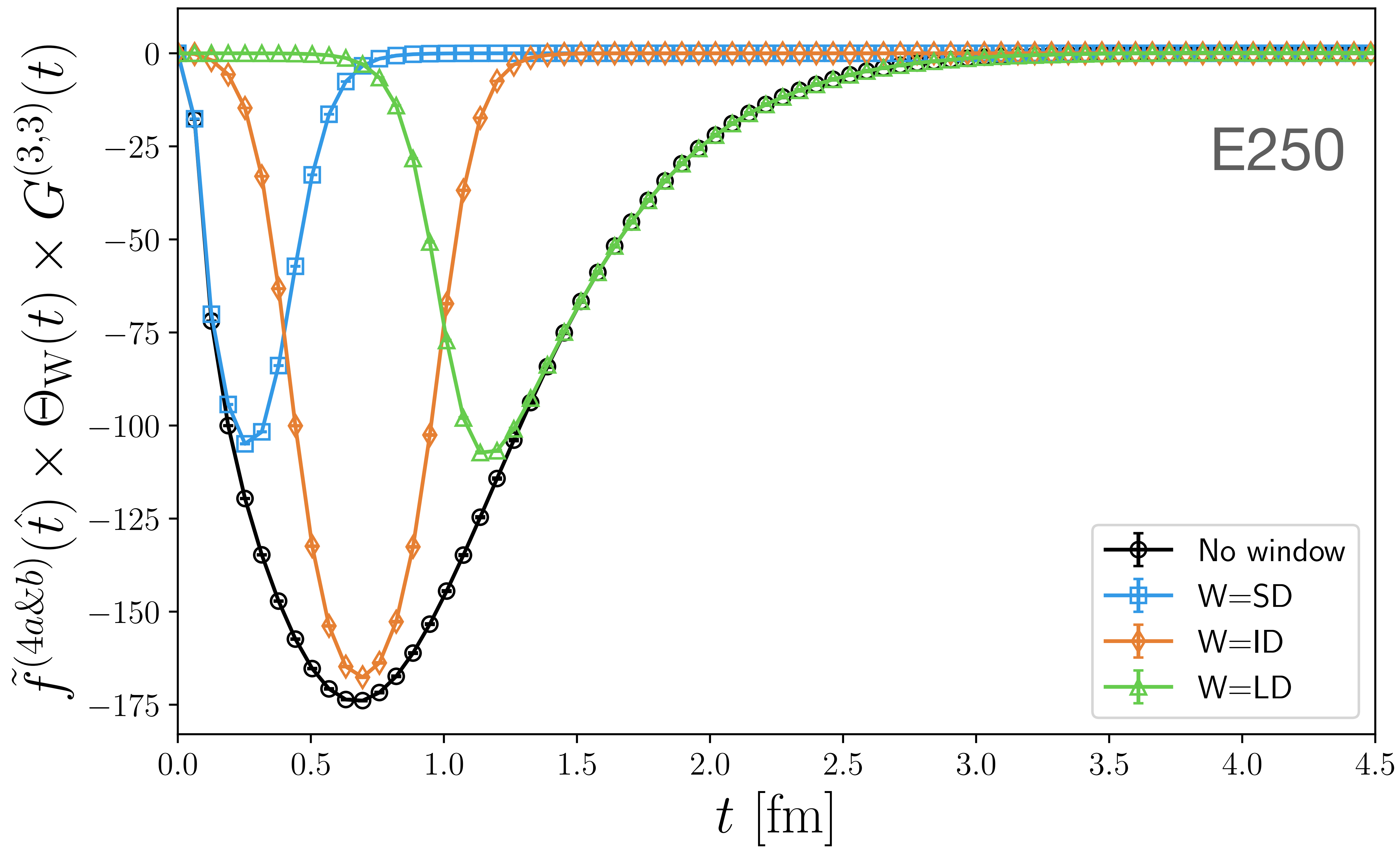
$\bar{\Pi}(\omega^2)$ and $G(t)$

- Time Momentum Representation

- Higher-order HVP

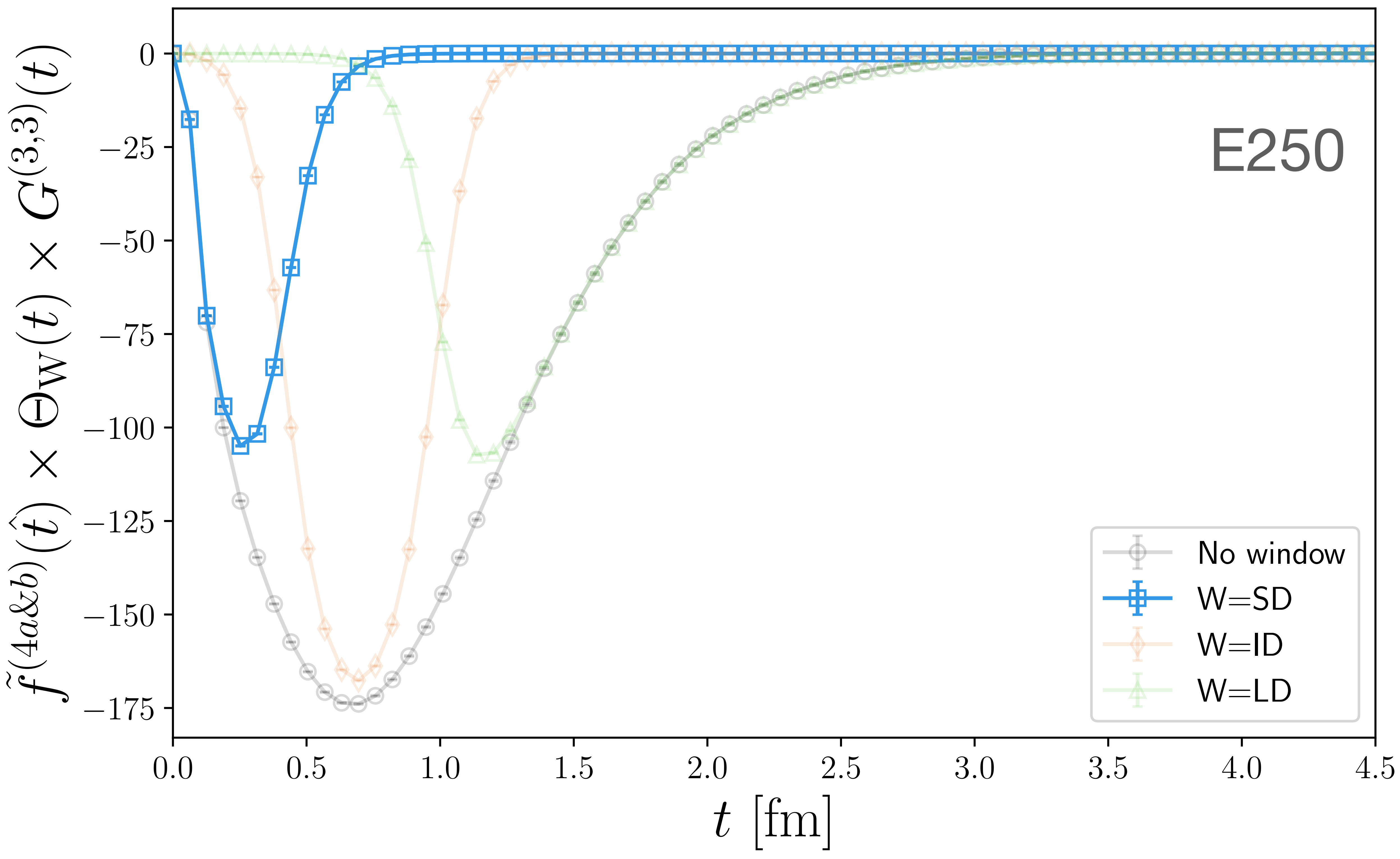
- The electron $g-2$

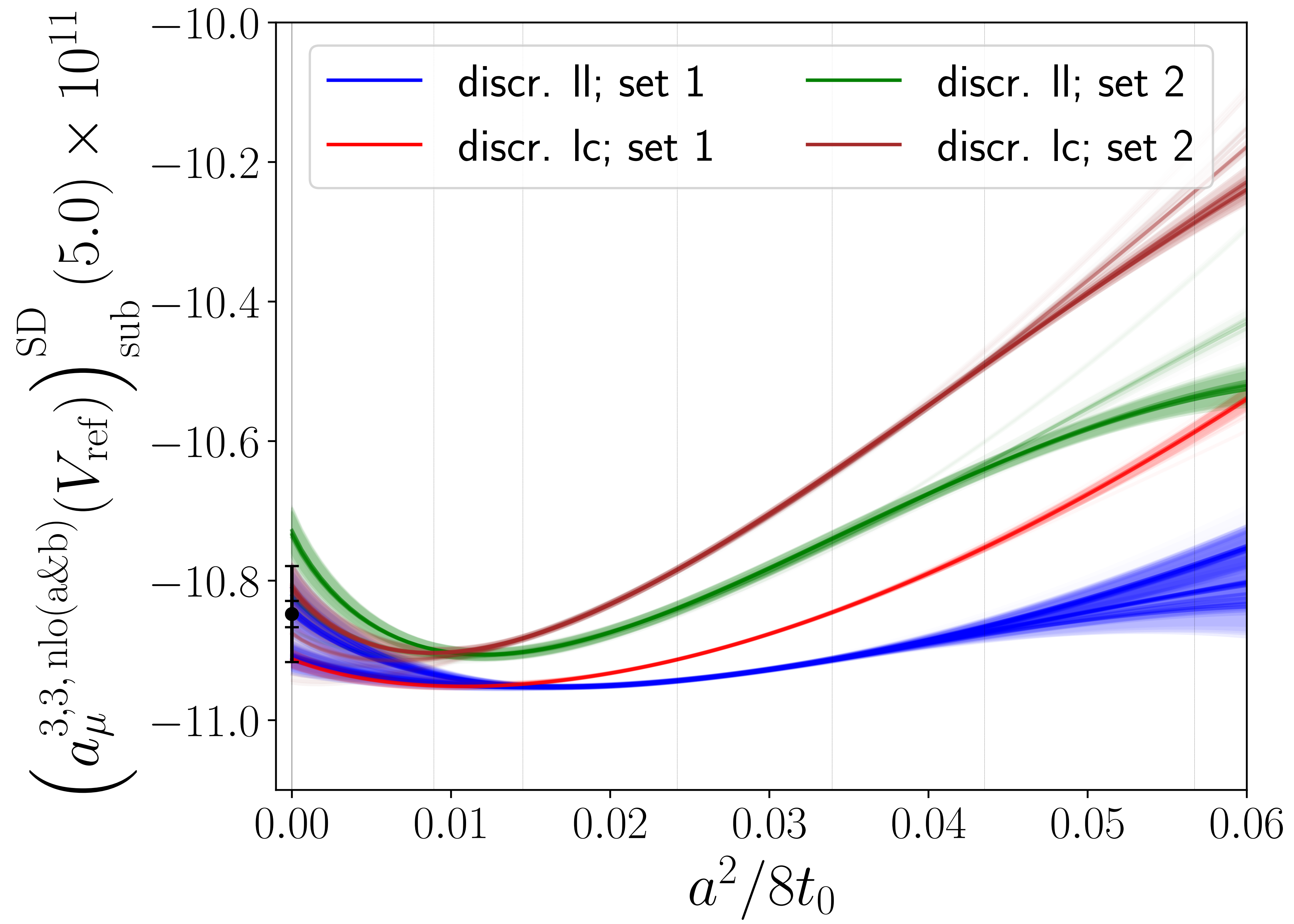
Windows NLO HVP



SD

NLO HVP


E250
No window
W=SD
W=ID
W=LD
t [fm]
$\tilde{f}^{(4a\&b)}(\hat{t}) \times \Theta_W(t) \times G^{(3,3)}(t)$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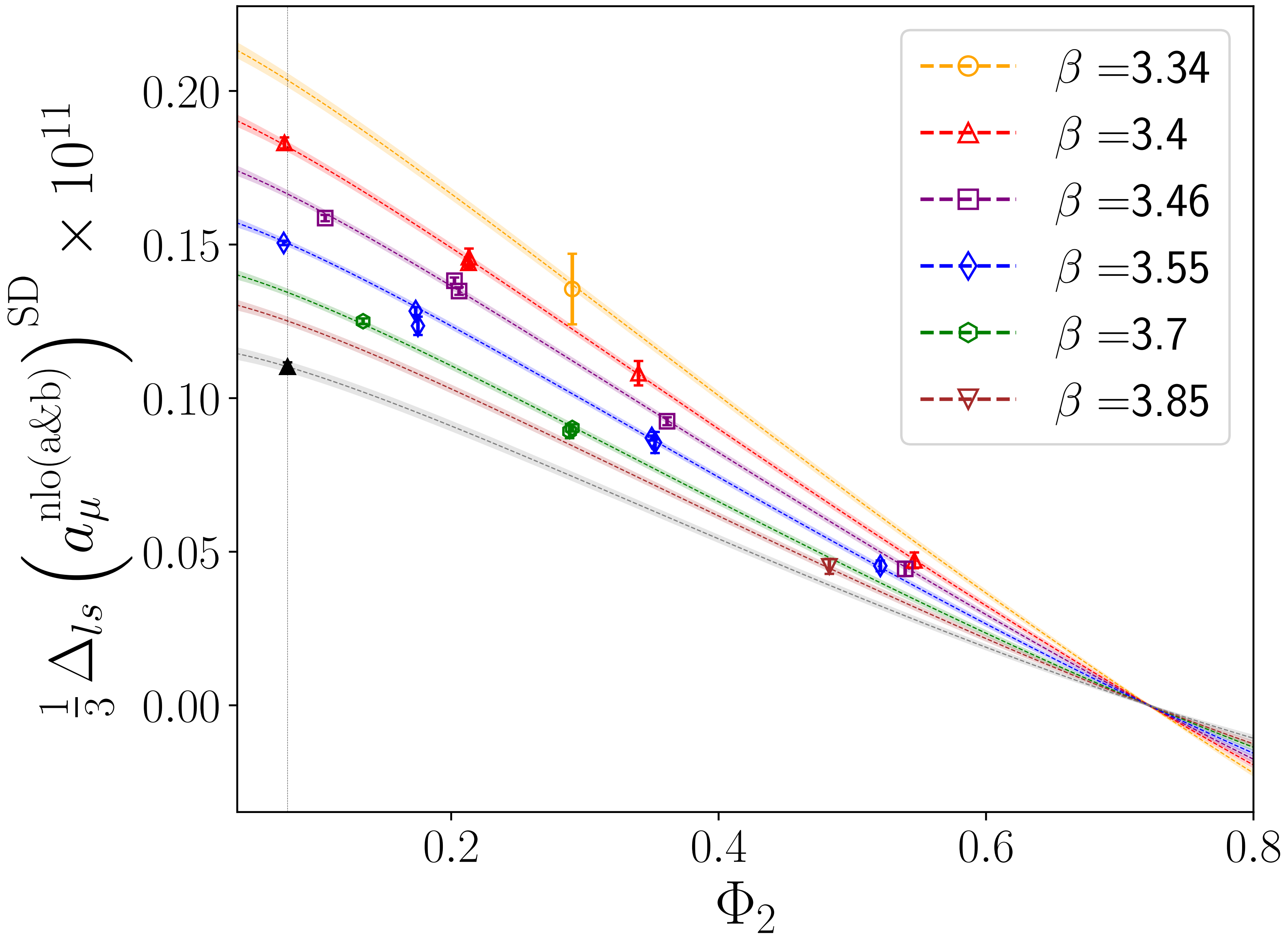
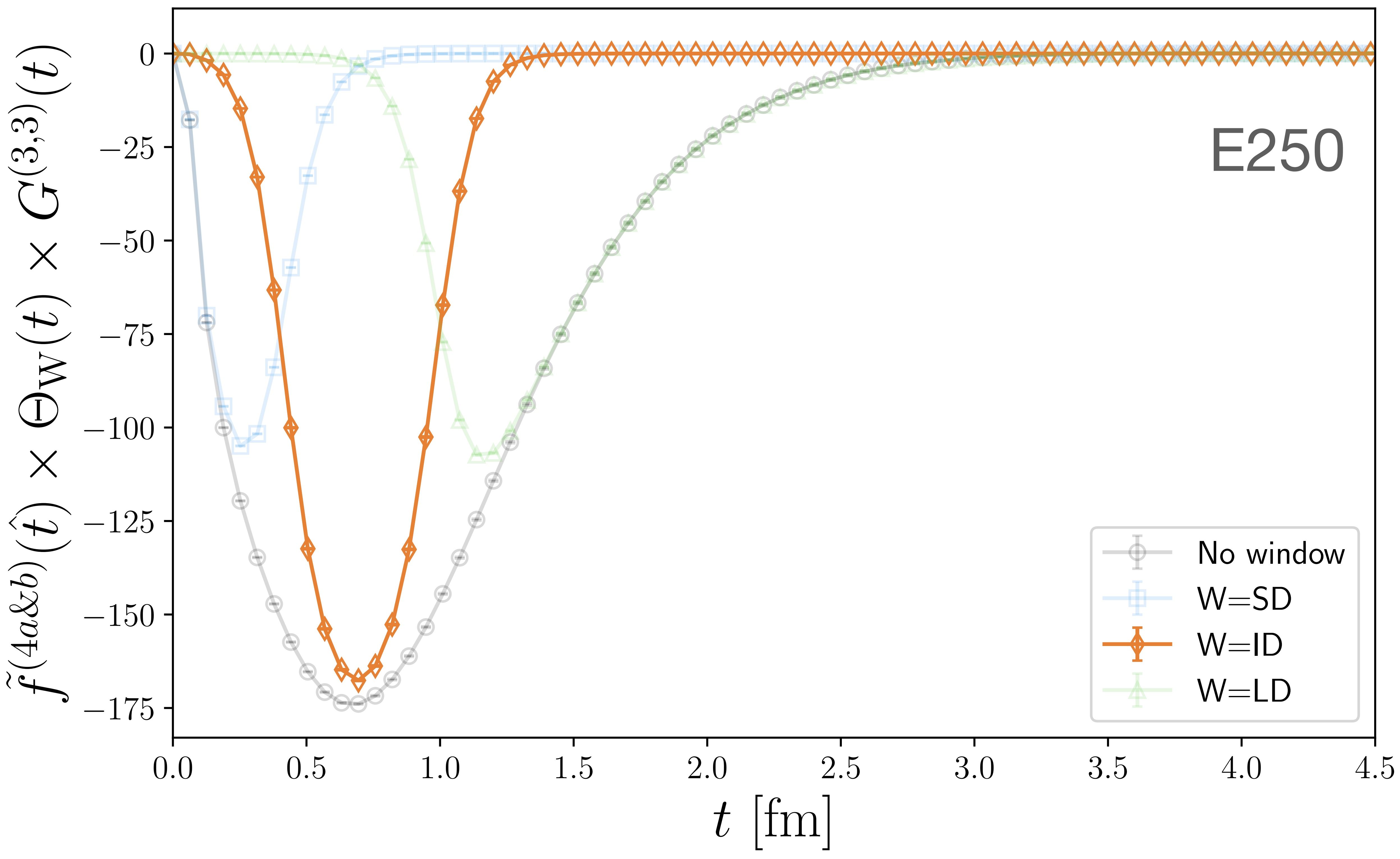
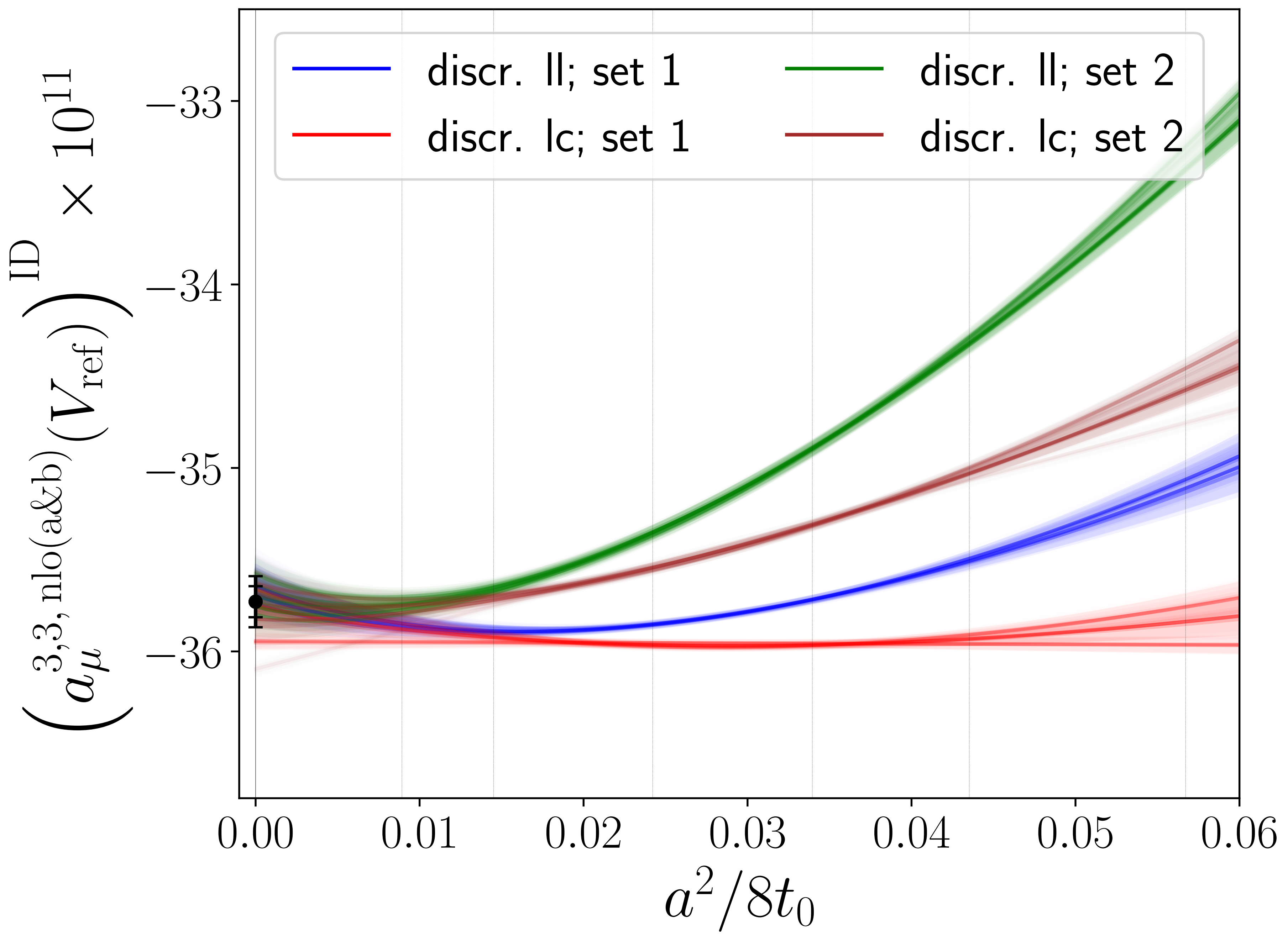
discr. lc; set 1
discr. ll; set 2
discr. lc; set 2
$a^2/8t_0$
$\left(a_\mu^{3,3,\,\mathrm{nlo(a\&b)}}(V_{\mathrm{ref}})\right)^{\mathrm{SD}}_{\mathrm{sub}}(5.0) \times 10^{11}$


β =3.34
β =3.4
β =3.46
β =3.55
β =3.7
β =3.85
Φ_2
$\frac{1}{3}\Delta_{ls}\left(a_\mu^{\mathrm{nlo(a\&b)}}\right)^{\mathrm{SD}} \times 10^{11}$


E250
No window
W=SD
W=ID
W=LD
t [fm]
$\tilde{f}^{(4a\&b)}(\hat{t}) \times \Theta_W(t) \times G^{(3,3)}(t)$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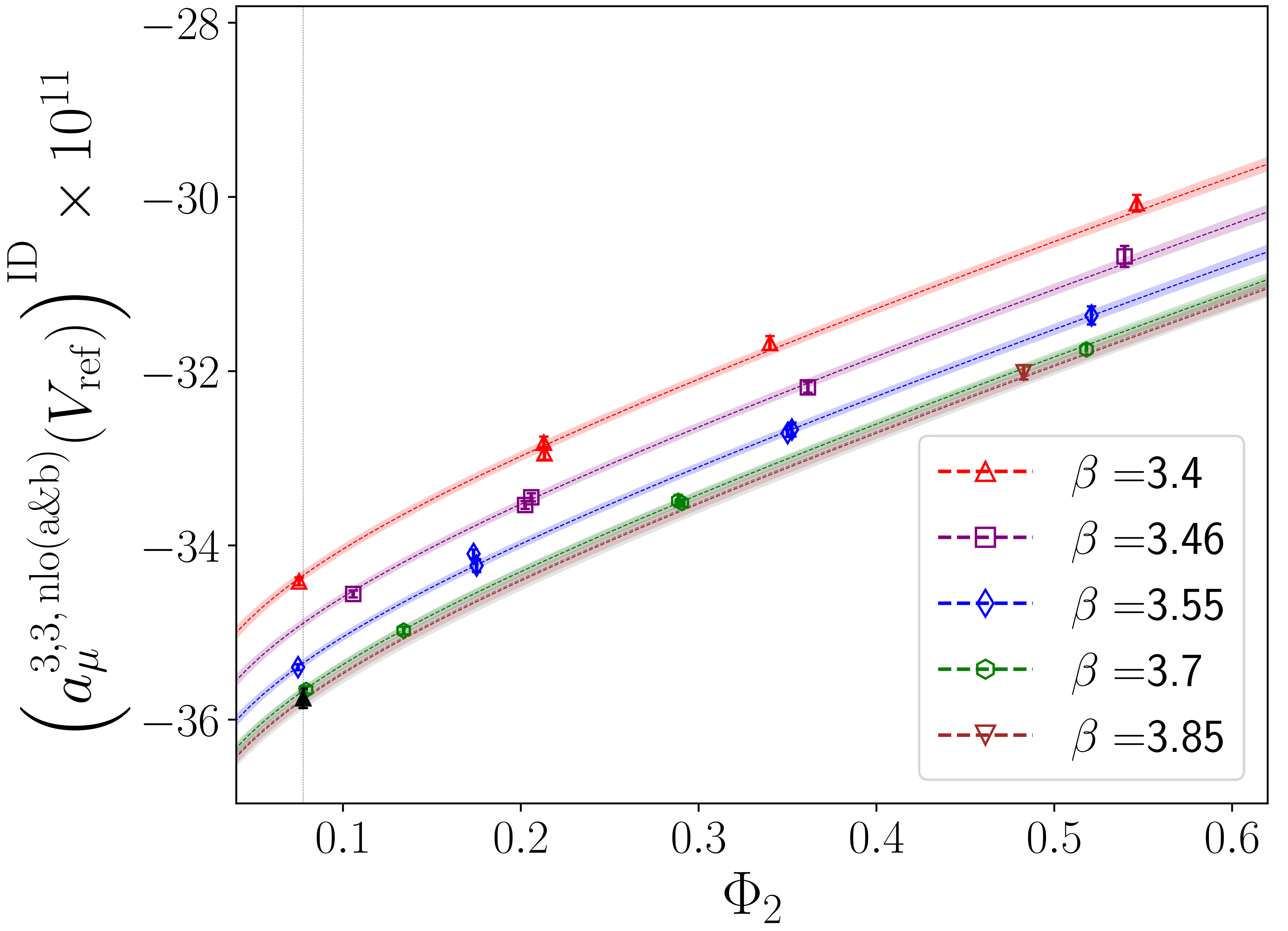
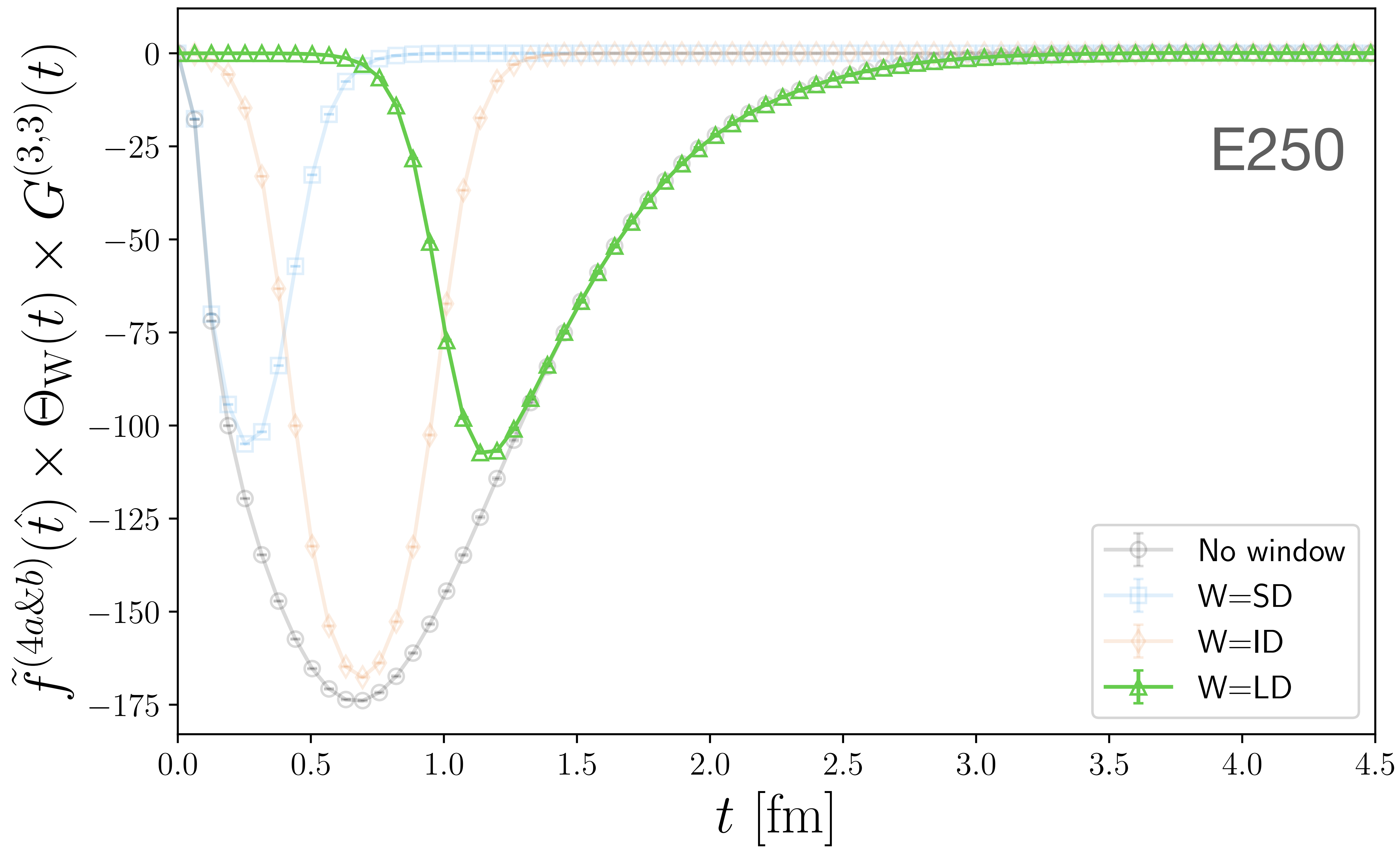
discr. lc; set 1
discr. ll; set 2
discr. lc; set 2
$a^2/8t_0$
$\left(a_\mu^{3,3,\,\mathrm{nlo(a\&b)}}(V_{\mathrm{ref}})\right)^{\mathrm{ID}} \times 10^{11}$


β =3.4
β =3.46
β =3.55
β =3.7
β =3.85
Φ_2
$\left(a_\mu^{3,3,\,\mathrm{nlo(a\&b)}}(V_{\mathrm{ref}})\right)^{\mathrm{ID}} \times 10^{11}$

LD

NLO HVP



E250
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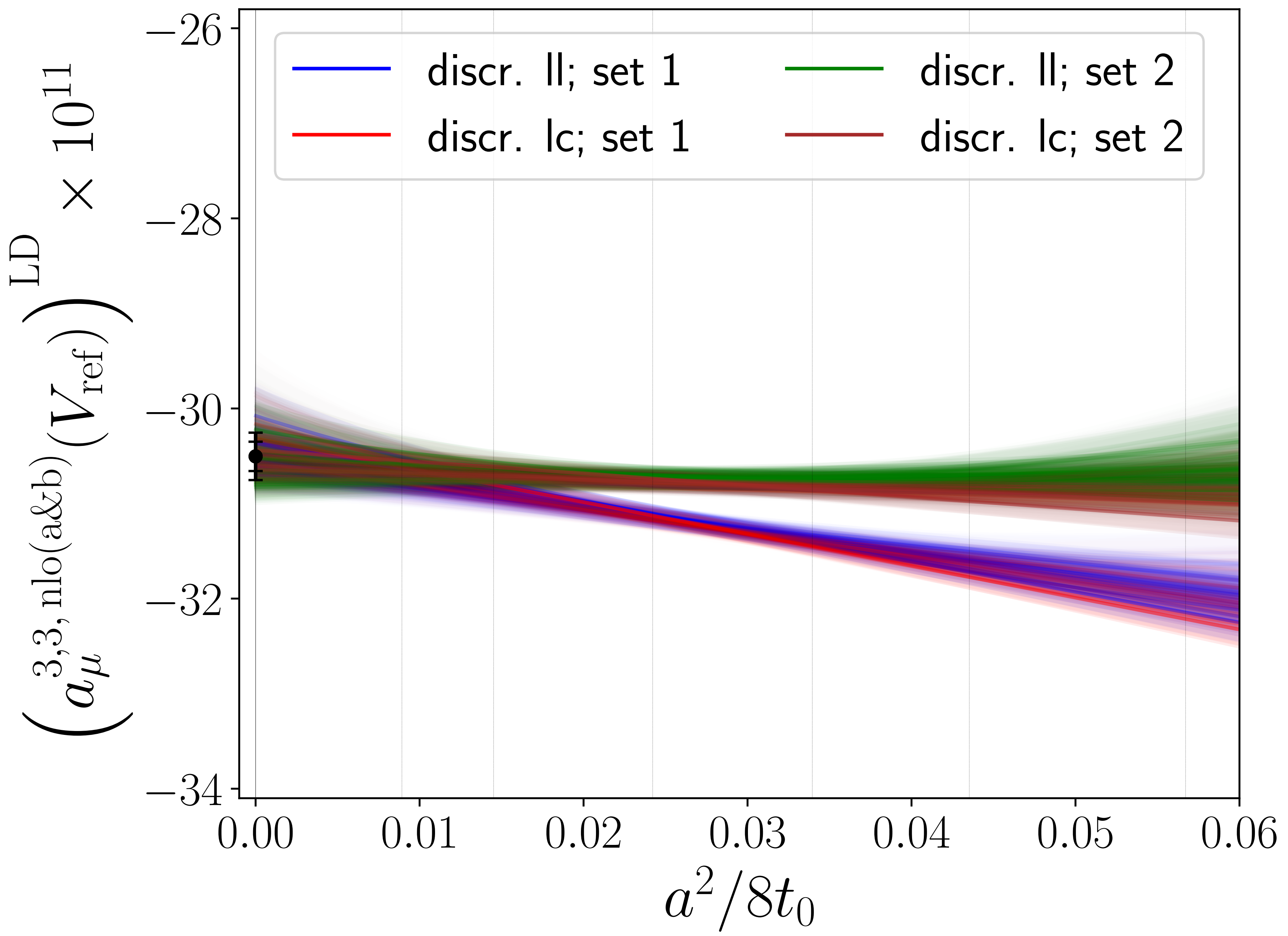
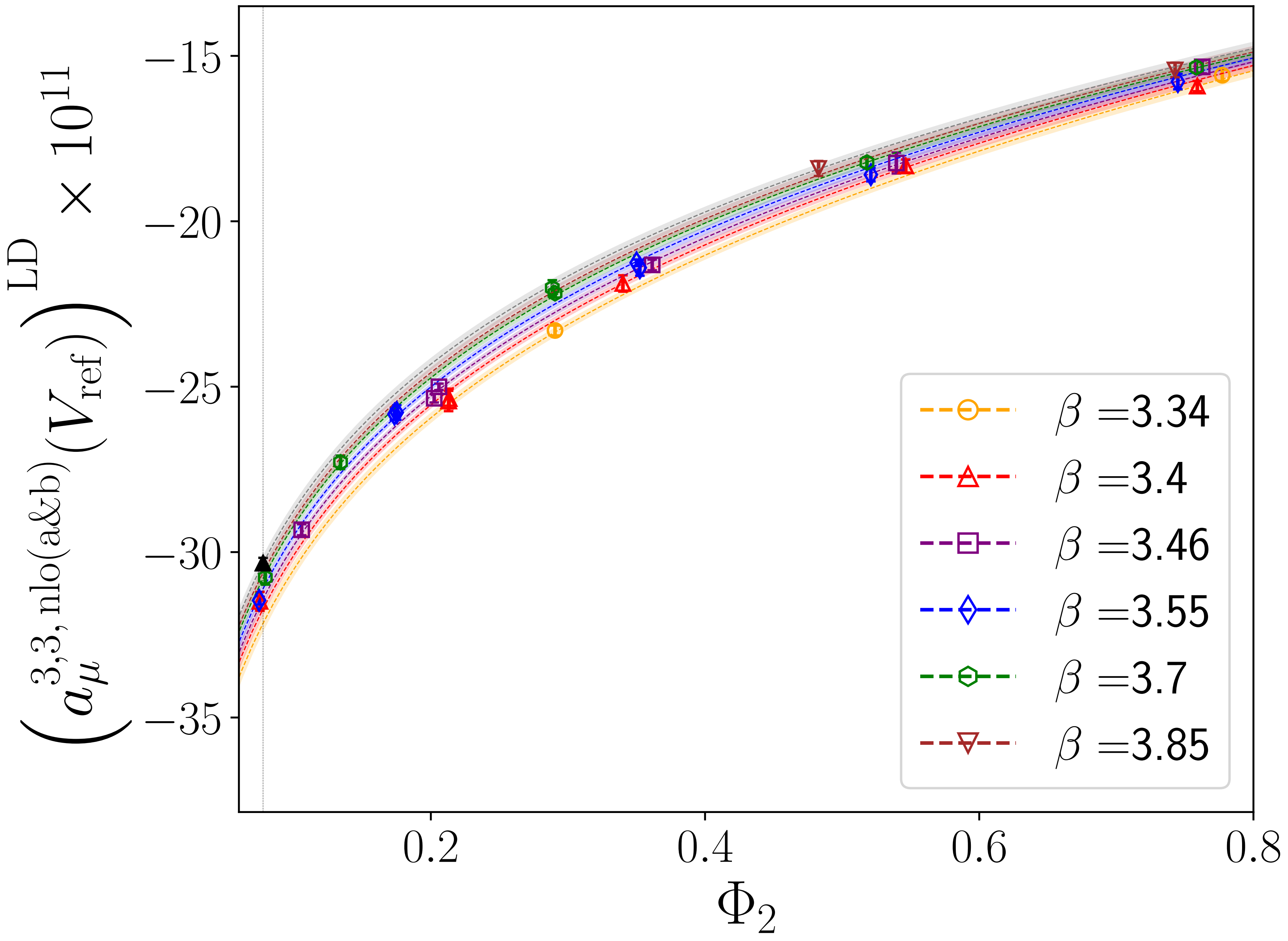
W=SD
W=ID
W=LD
$\tilde{f}^{(4a\&b)}(\hat{t}) \times \Theta_W(t) \times G^{(3,3)}(t)$
t [fm]

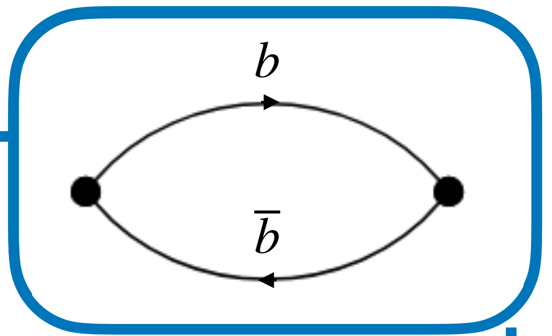


discr. ll; set 1
discr. lc; set 1
discr. ll; set 2
discr. lc; set 2
$\left(a_\mu^{3,3,\,\mathrm{nlo(a\&b)}}(V_{\mathrm{ref}})\right)^{\mathrm{LD}} \times 10^{11}$
$a^2/8t_0$



β =3.34
β =3.4
β =3.46
β =3.55
β =3.7
β =3.85
$\left(a_\mu^{3,3,\,\mathrm{nlo(a\&b)}}(V_{\mathrm{ref}})\right)^{\mathrm{LD}} \times 10^{11}$
Φ_2

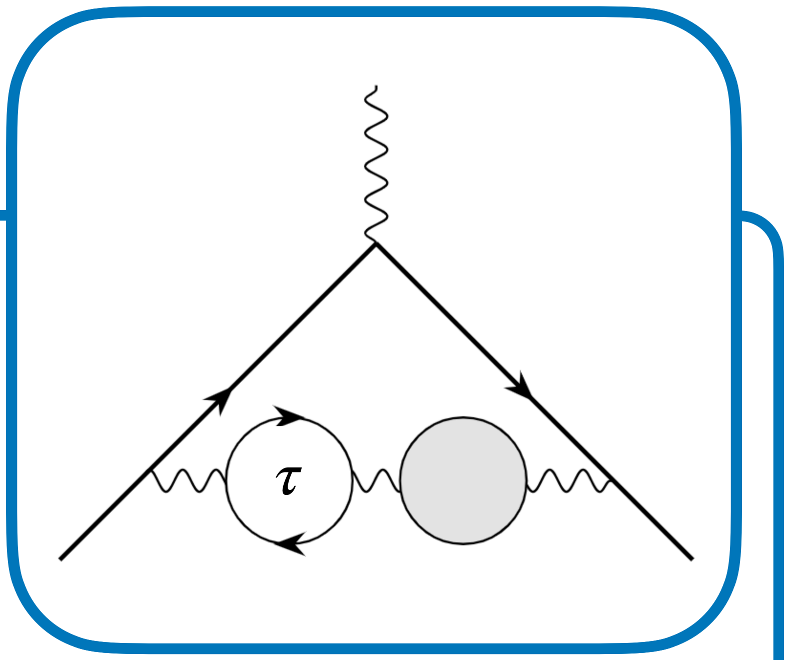
Bottom quark effects



$m_b \gg 1/a$ for all CLS ensembles, we use pQCD

$$a_\mu^{b,b,(i)} = \left(\frac{\alpha m_\mu}{6\pi m_b}\right)^2 \left(\frac{\alpha}{\pi}\right)^{N_i} \int_1^\infty \frac{dz}{z^2} K^{(i)}\left(\frac{4m_b^2}{m_\mu^2} z\right) R_b(z)$$

Tau-loop effects in NLOb



We neglected NLOb(τ)

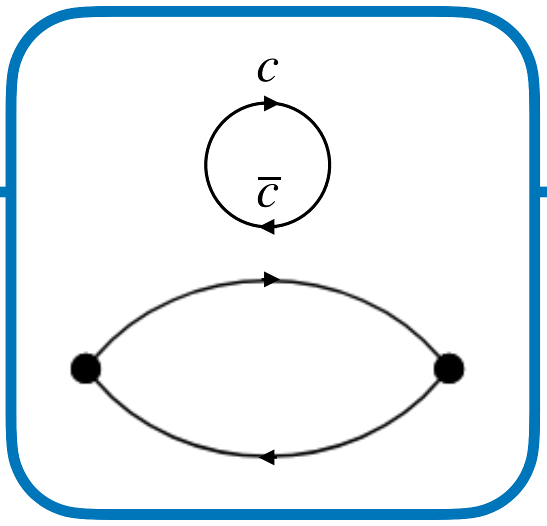
We evaluate only in E250

$$a_\mu^{\mathrm{hvp,\,nlo}(\tau)} \approx \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dt\, \tilde{f}_\tau^{(4b)}(\hat{t})\, G_{\mathrm{E250}}(t)$$

Charm sea-quark effects

$\mathscr{L}_E$ does not contain the charm quark

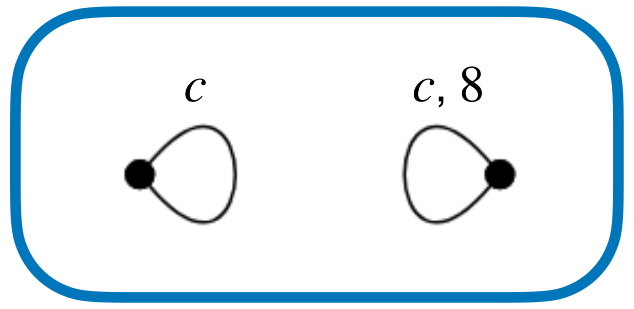
Direct and indirect effects from charm sea-quark loops are included in the error budget



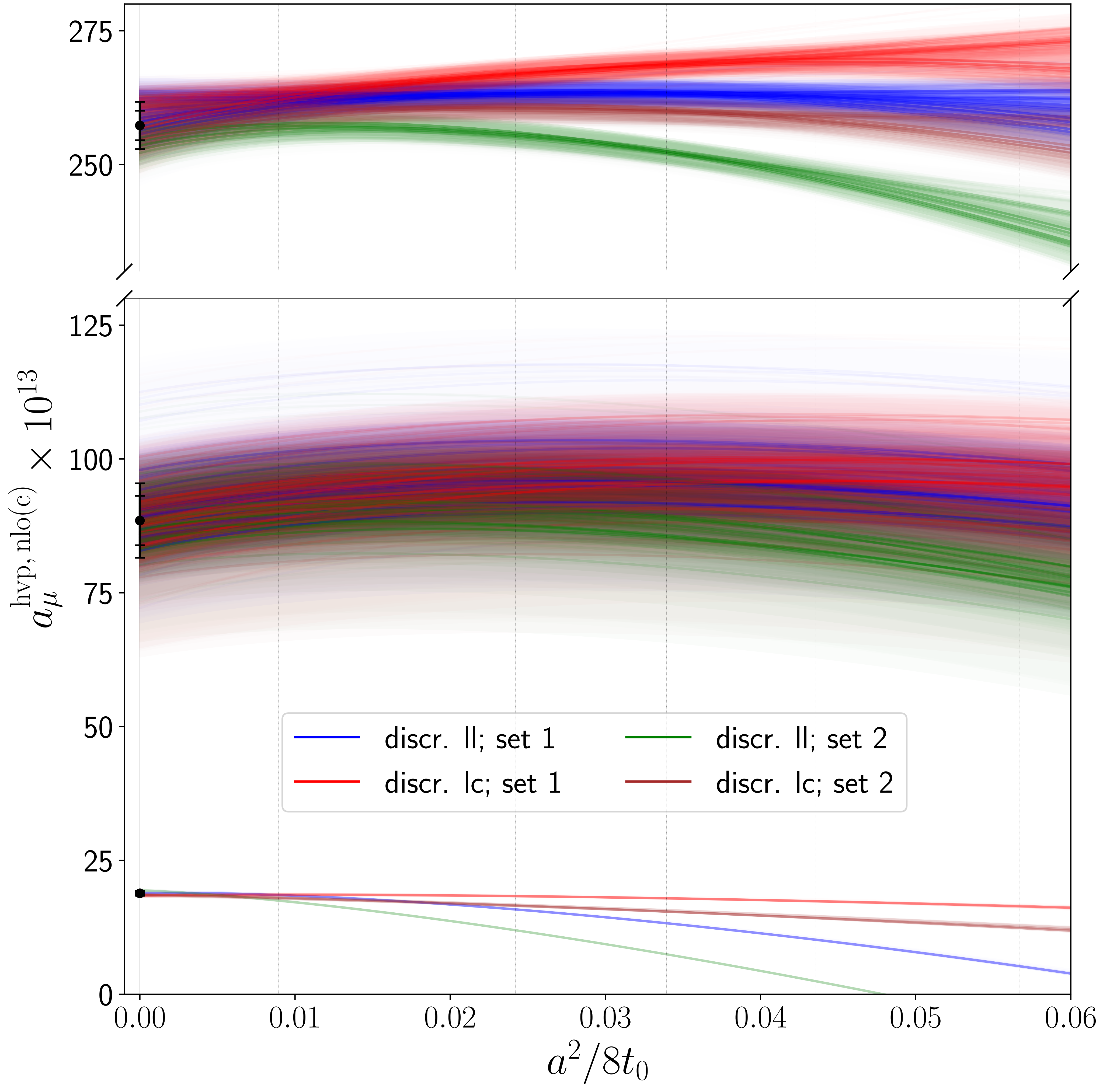
Charm disconnected contributions

Caused by $\dfrac{2}{3\sqrt{3}} G_{\mathrm{disc}}^{c8}$ and $\dfrac{4}{9} G_{\mathrm{disc}}^{cc}$

Numerically irrelevant



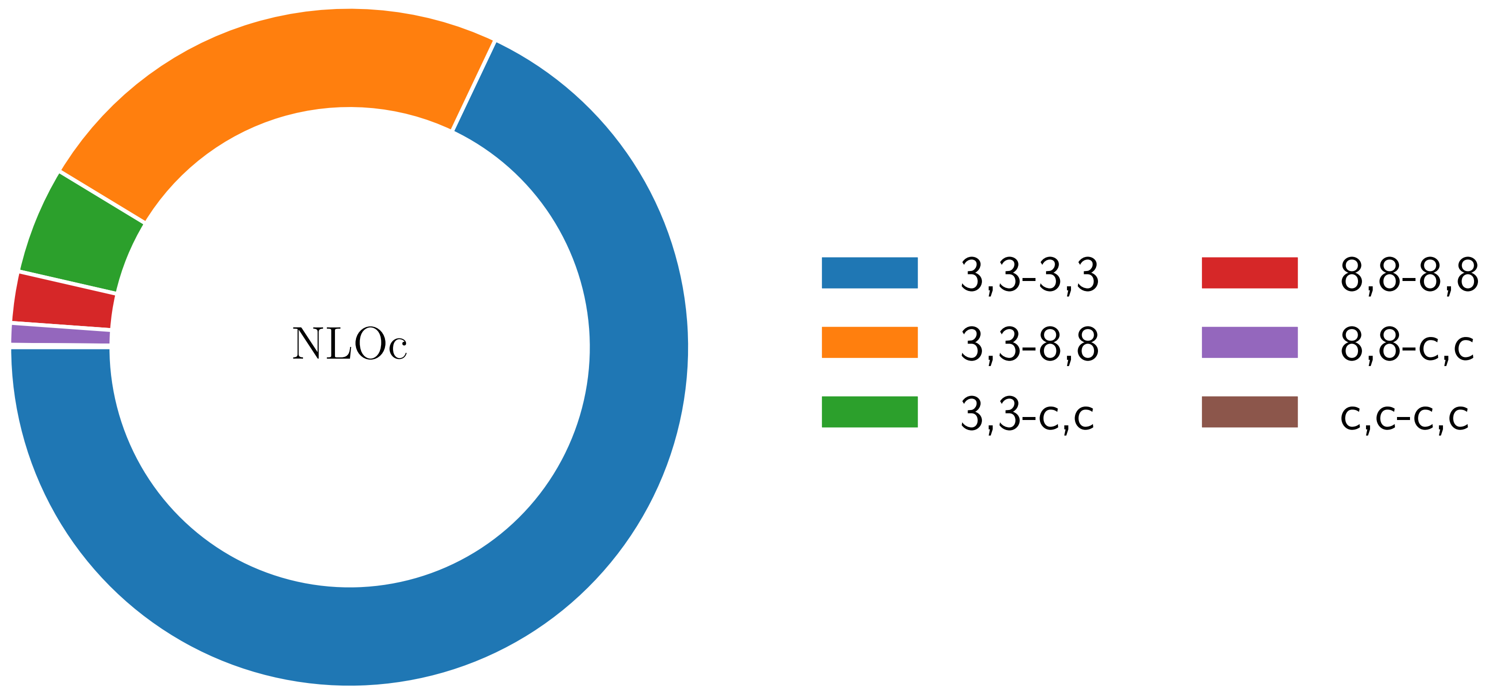
Results for NLOc

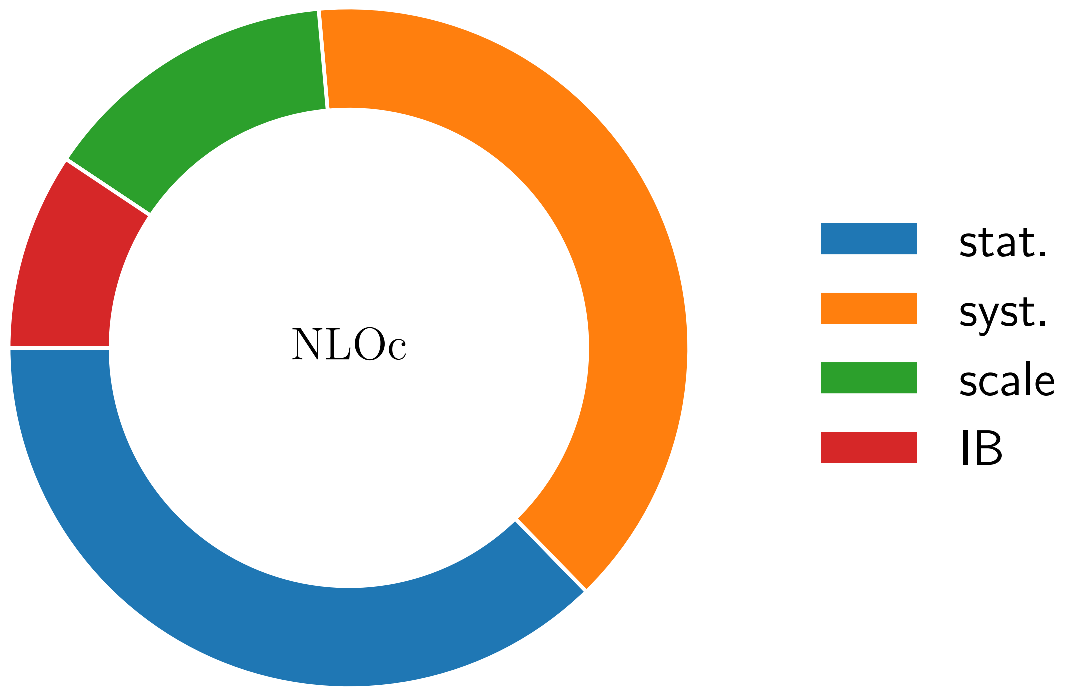


$a_\mu^{3,3-3,3}$

$\frac{2}{3}a_\mu^{3,3-8,8}$

$\frac{8}{9}a_\mu^{3,3-c,c}$





2.4 %